# AC Circuit Analysis

University of Reading

**Module**: SE1EA5 Systems and Circuits

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**Number of Lectures:** 10

**Recommended text book:**
David Irwin and Mark Nelms
Basic Engineering Circuit Analysis (8th edition)
John Wiley and Sons (2005)
ISBN: 0-471-66158-9

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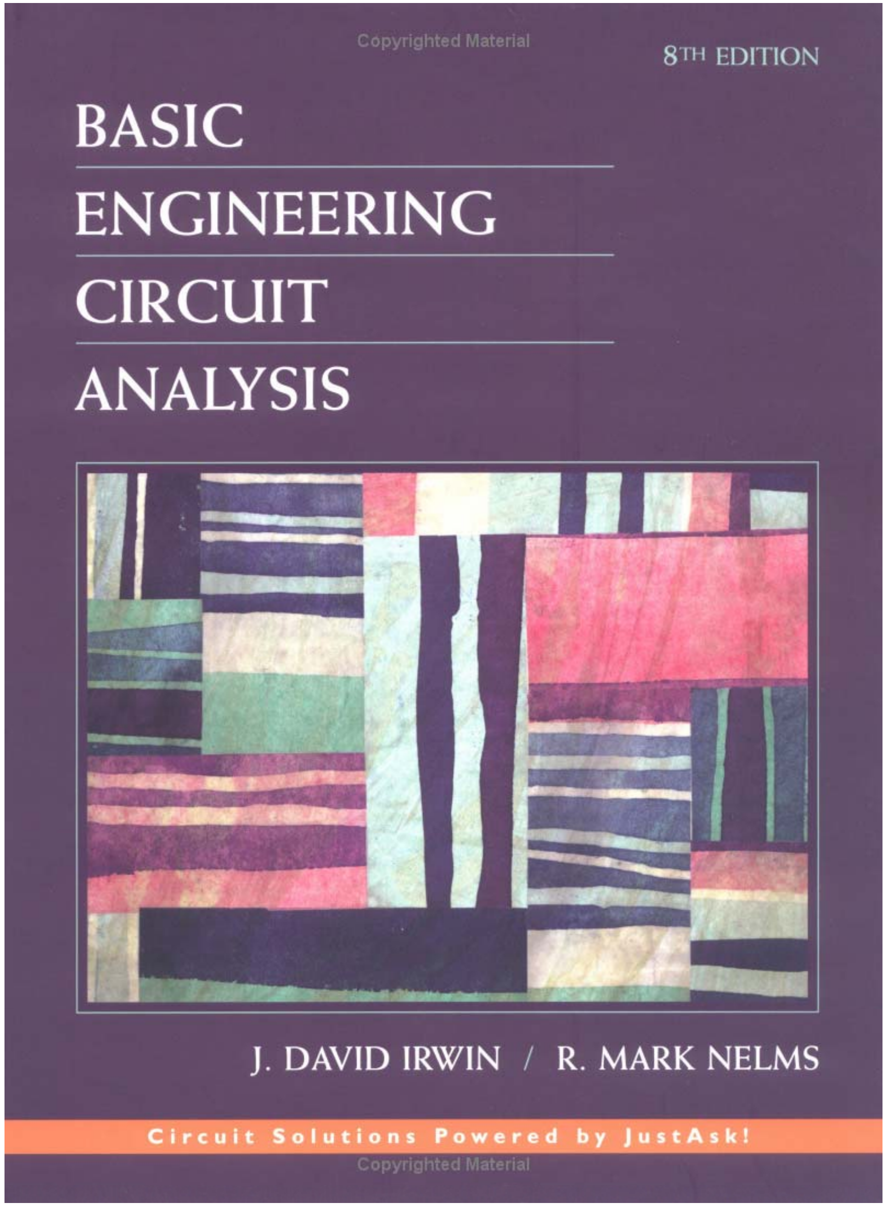
Price: £36



# AC Circuit Analysis Syllabus

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This course of lectures will extend dc circuit analysis to deal with ac circuits

The topics that will be covered include:

- AC voltages and currents
- Complex representation of sinusoids
- Phasors
- Complex impedances of inductors and capacitors
- Driving-point impedance
- Frequency response of circuits – Bode plots
- Power in ac circuits
- Energy storage in capacitors and inductors
- Three-phase power

# AC Circuit Analysis Prerequities

You should be familiar with the following topics:

**SE1EA5: Electronic Circuits**

Ohm’s Law
Series and parallel resistances
Voltage and current sources
Circuit analysis using Kirchhoff’s Laws
Thévenin and Norton's theorems
The Superposition Theorem

**SE1EC5: Engineering Mathematics**

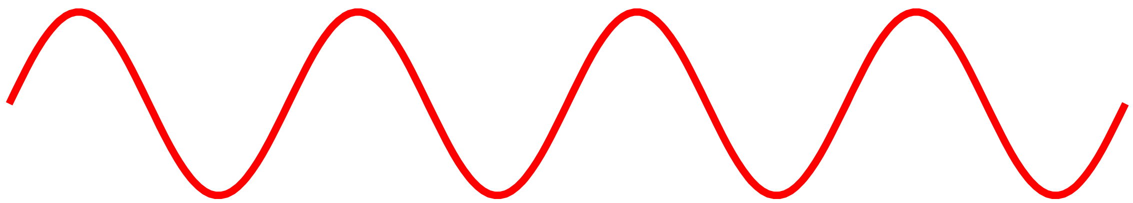
Complex numbers

# Lecture 1

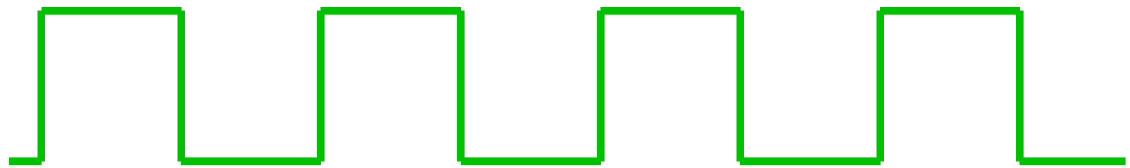
## AC Voltages and Currents
## Reactive Components

# AC Waveforms

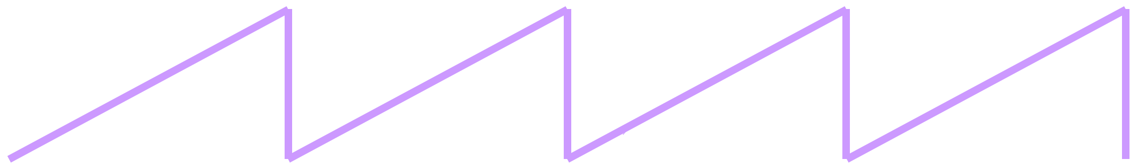
Sine waveform (sinusoid)



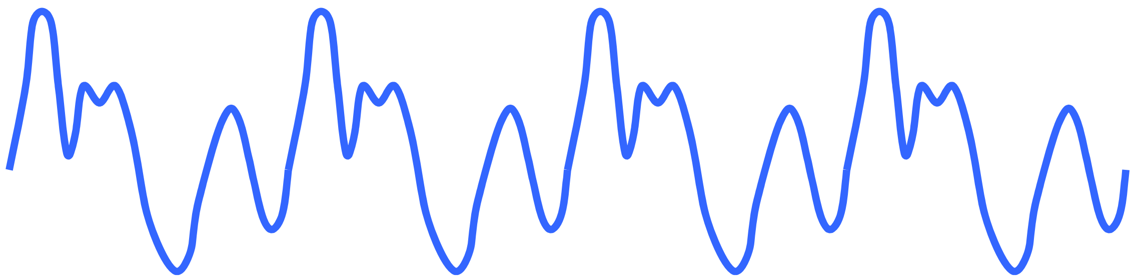
Square waveform



Sawtooth waveform

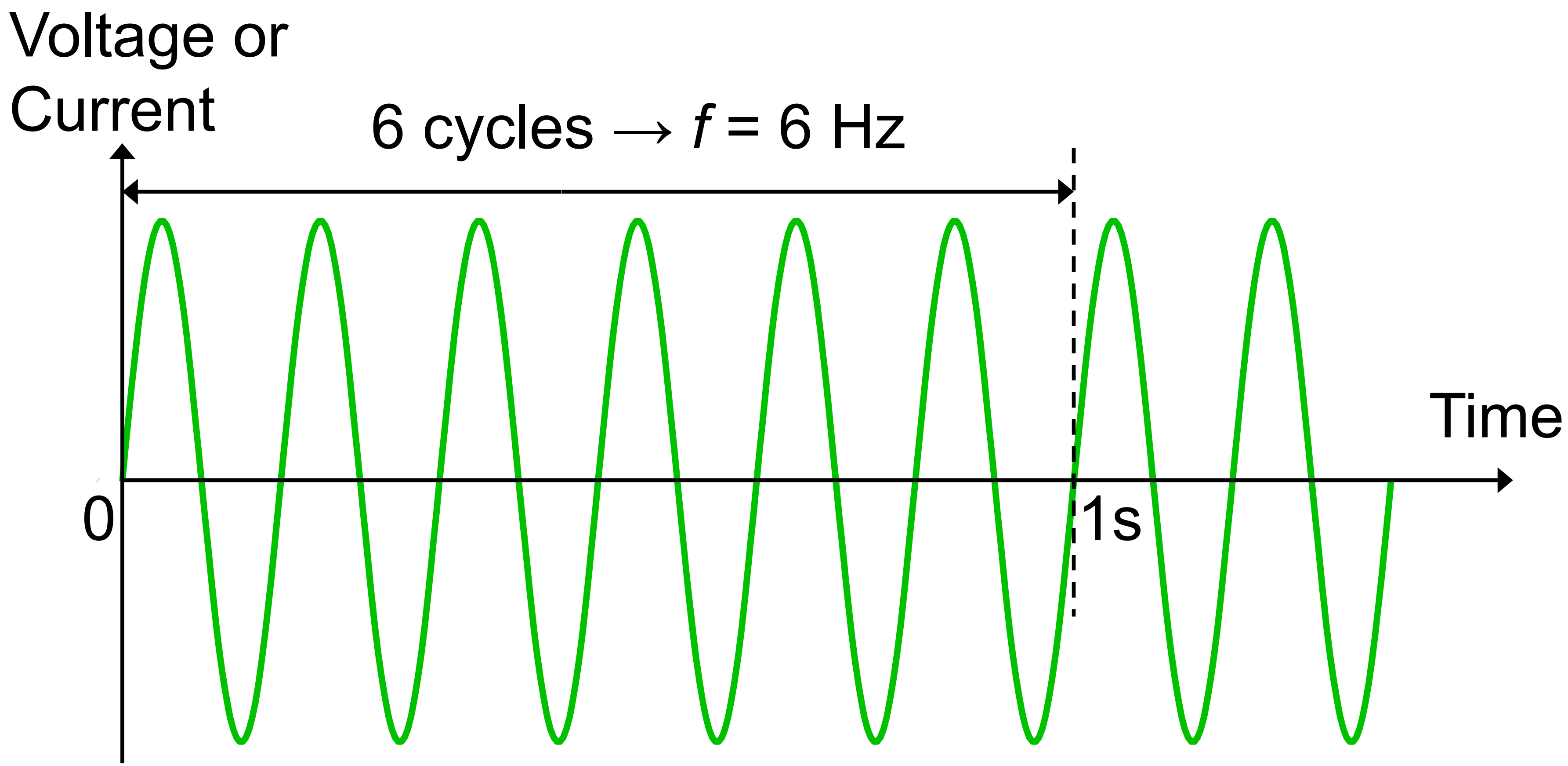


Audio waveform



# Frequency

The number of cycles per second of an ac waveform is known as the frequency *f*, and is expressed in Hertz (Hz)



# Frequency

Examples:

Electrocardiogram: 1 Hz

Mains power: 50 Hz

Aircraft power: 400 Hz

Audio frequencies: 20 Hz to 20 kHz

AM radio broadcasting: 0.5 MHz – 1.5 MHz

FM radio broadcasting: 80 MHz – 110 MHz

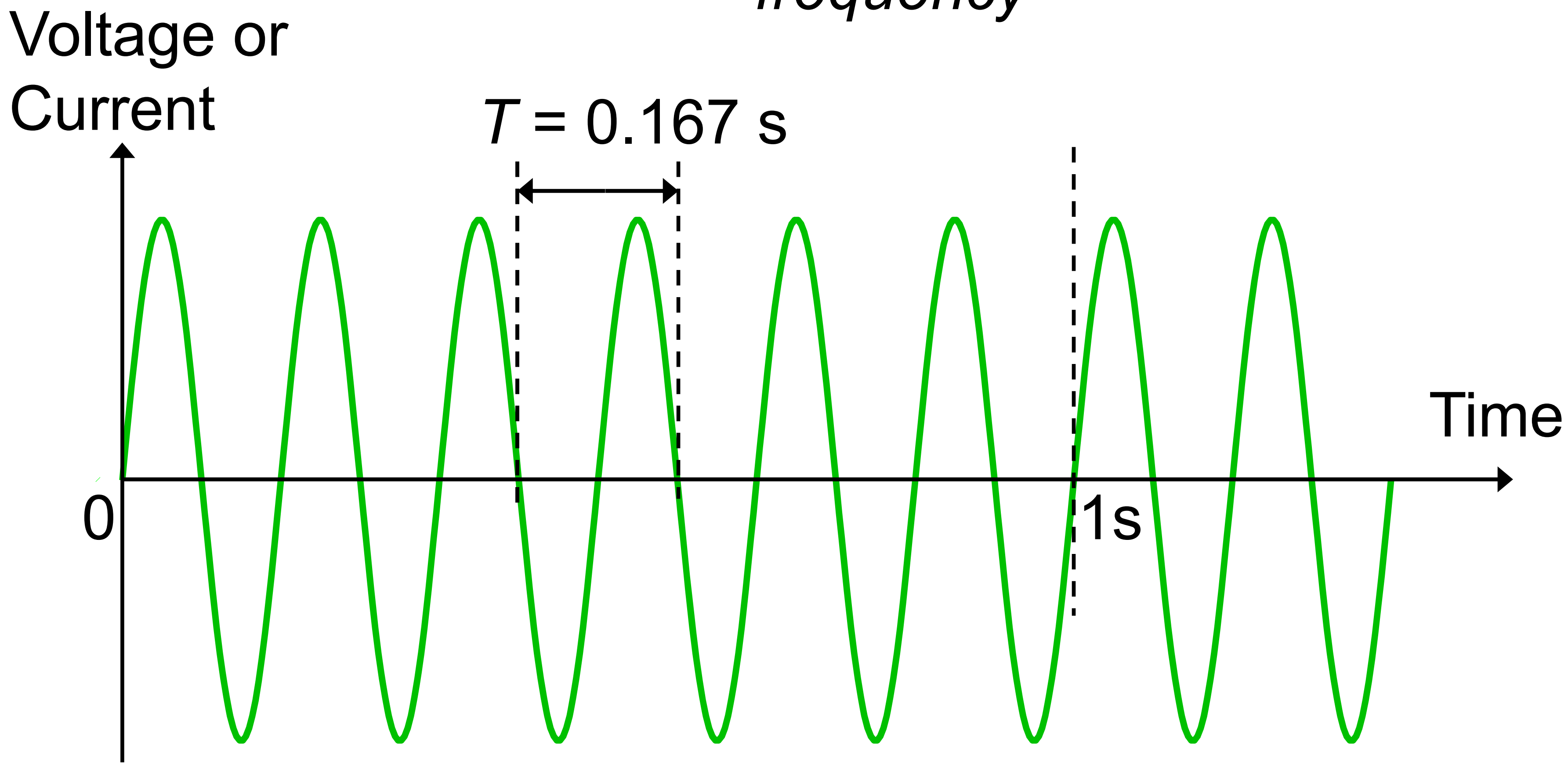
Television broadcasting: 500 MHz – 800 MHz

Mobile telephones: 1.8 GHz

# Period

The period $T$ of an ac waveform is the time taken for a complete cycle:

$$period = \frac{1}{frequency}$$



# Why Linear?

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We shall consider the steady-state response of linear ac circuits to sinusoidal inputs

Linear circuits contain linear components such as resistors, capacitors and inductors

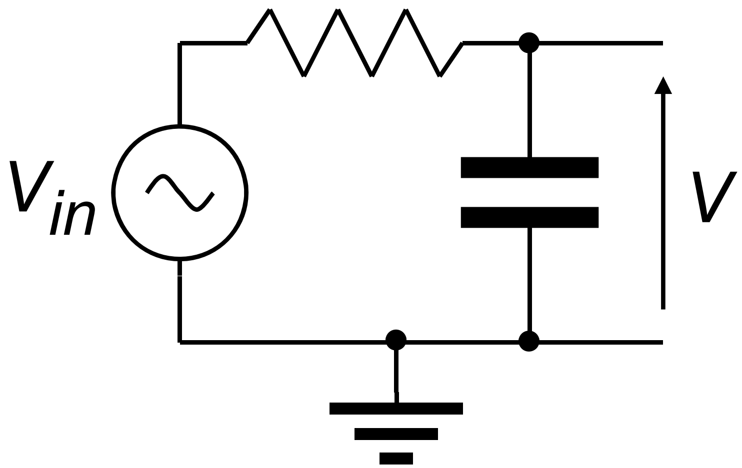
A linear component has the property that doubling the voltage across it doubles the current through it

Most circuits for processing signals are linear

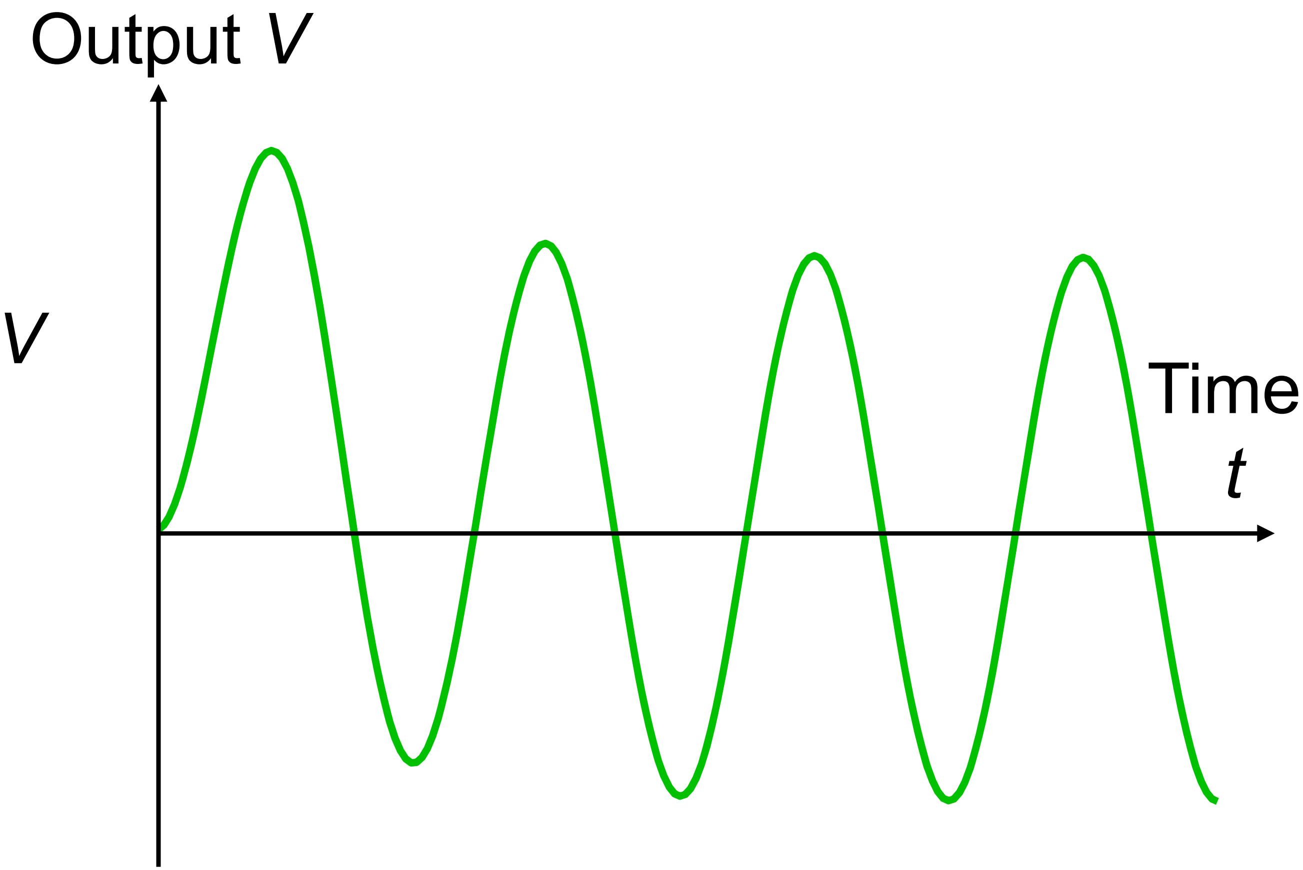
Analysis of non-linear circuits is difficult and normally requires the use of a computer.

# Why Steady-State?

Steady-state means that the input waveform has been present long enough for any transients to die away




$V_{in}$=0 for $t$≤0



# Why Sinusoidal?

A linear circuit will not change the waveform or frequency of a sinusoidal input (the amplitude and phase may be altered)

Power is generated as a sinusoid by rotating electrical machinery

Sinusoidal carrier waves are modulated to transmit information (radio broadcasts)

Any periodic waveform can be considered to be the sum of a fundamental pure sinusoid plus harmonics (Fourier Analysis)

# Fourier Analysis

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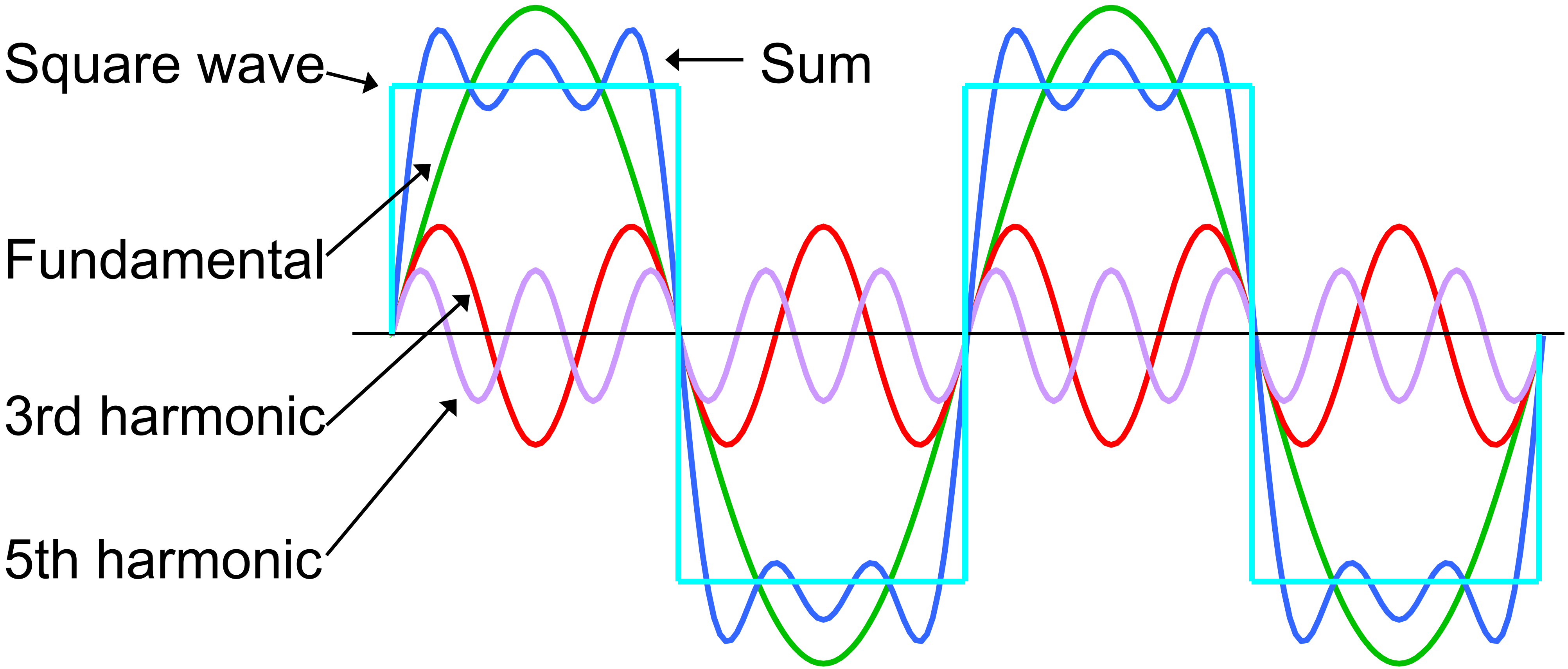
A square waveform can be considered to consist of a fundamental sinusoid together with odd harmonic sinusoids



# Representation of Sinusoids

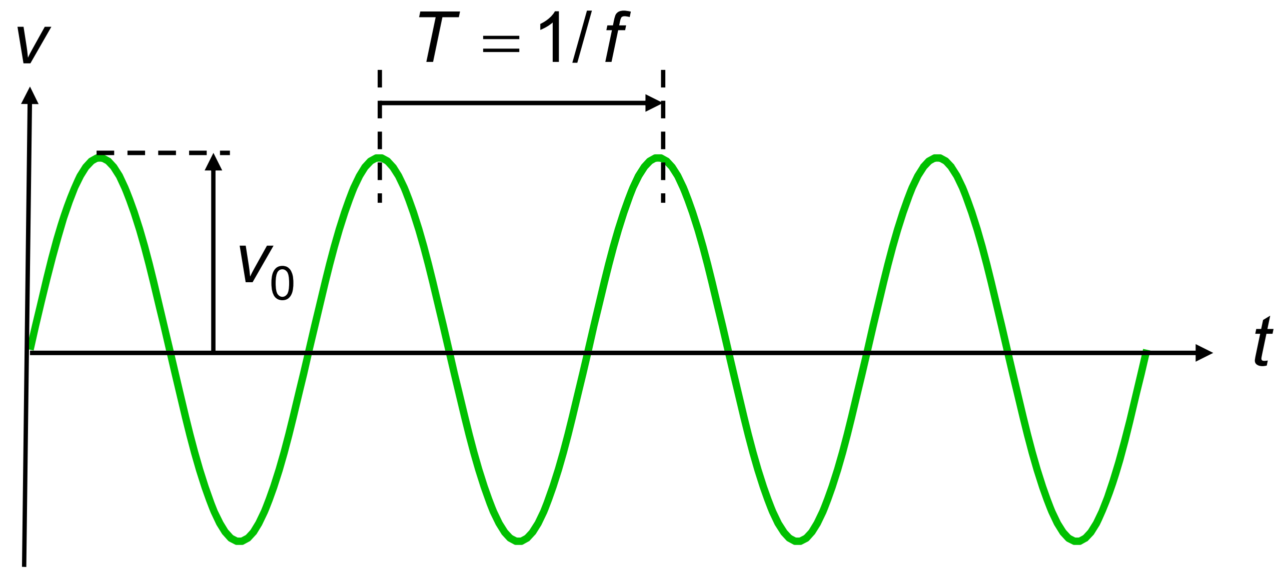
A sinusoidal voltage waveform $v(t)$ of amplitude $v_0$, and of frequency $f$ :

$$v(t) = v_0 \sin 2\pi ft = v_0 \sin \omega t$$

or:

$$v(t) = v_0 \cos 2\pi ft = v_0 \cos \omega t$$

where $\omega = 2\pi f$ is known as the angular frequency

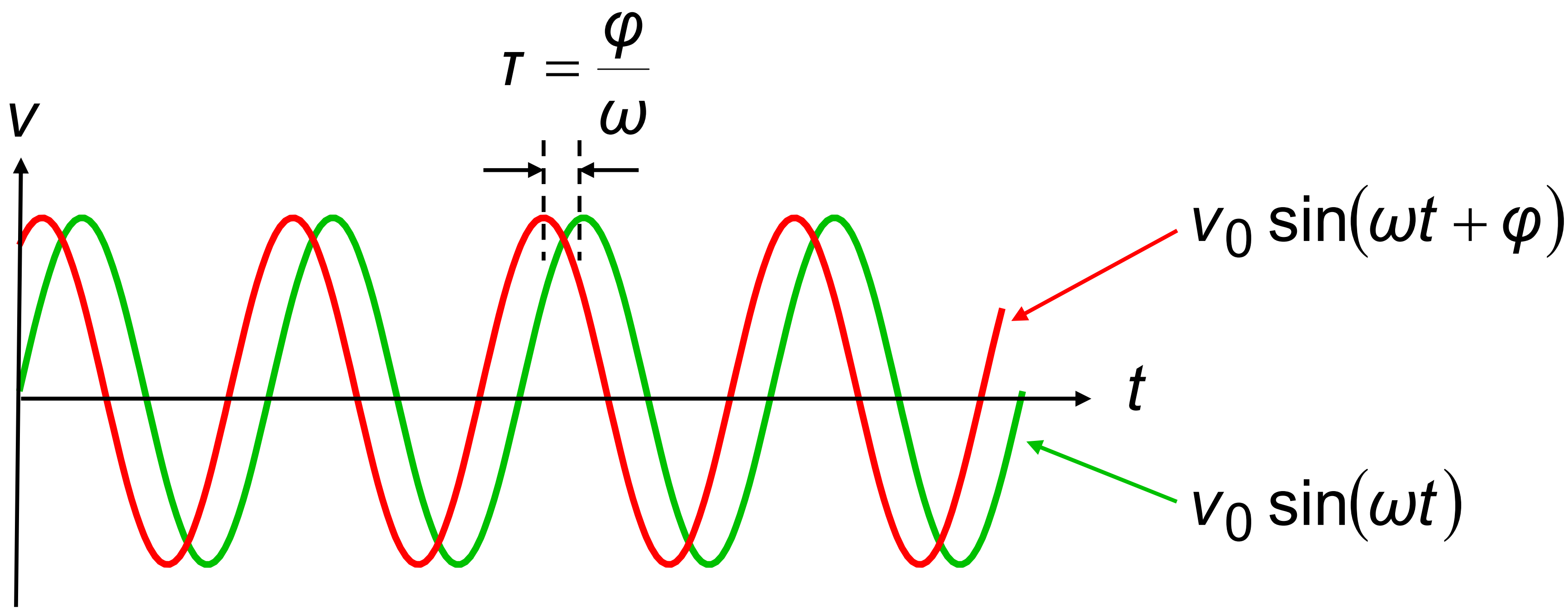


# Representation of Sinusoids

The sinusoid can have a phase term $\varphi$:

$$v(t) = v_0 \sin(\omega t + \varphi)$$

A phase shift $\varphi$ is equivalent to a time shift $-\varphi/\omega$




The phase is positive so the red trace *leads* the green trace

# Resistors

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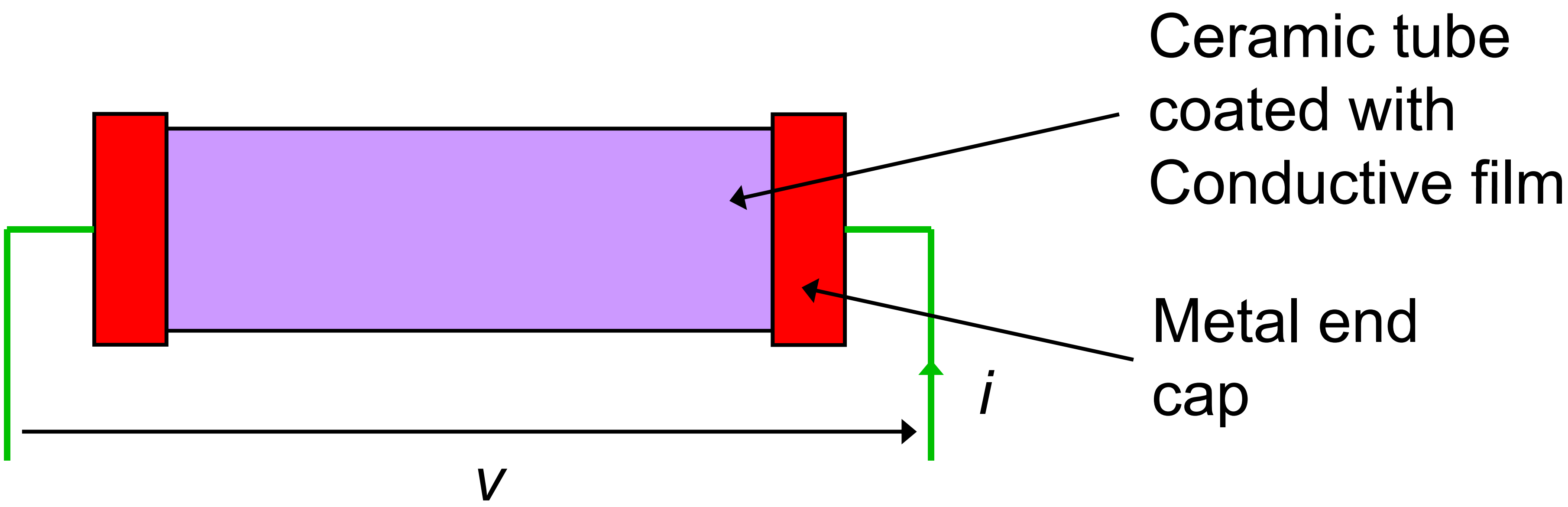


Film: carbon
metal
metal oxide

Resistance $R$

$$v = Ri$$

(Ohm's Law)

# Resistors

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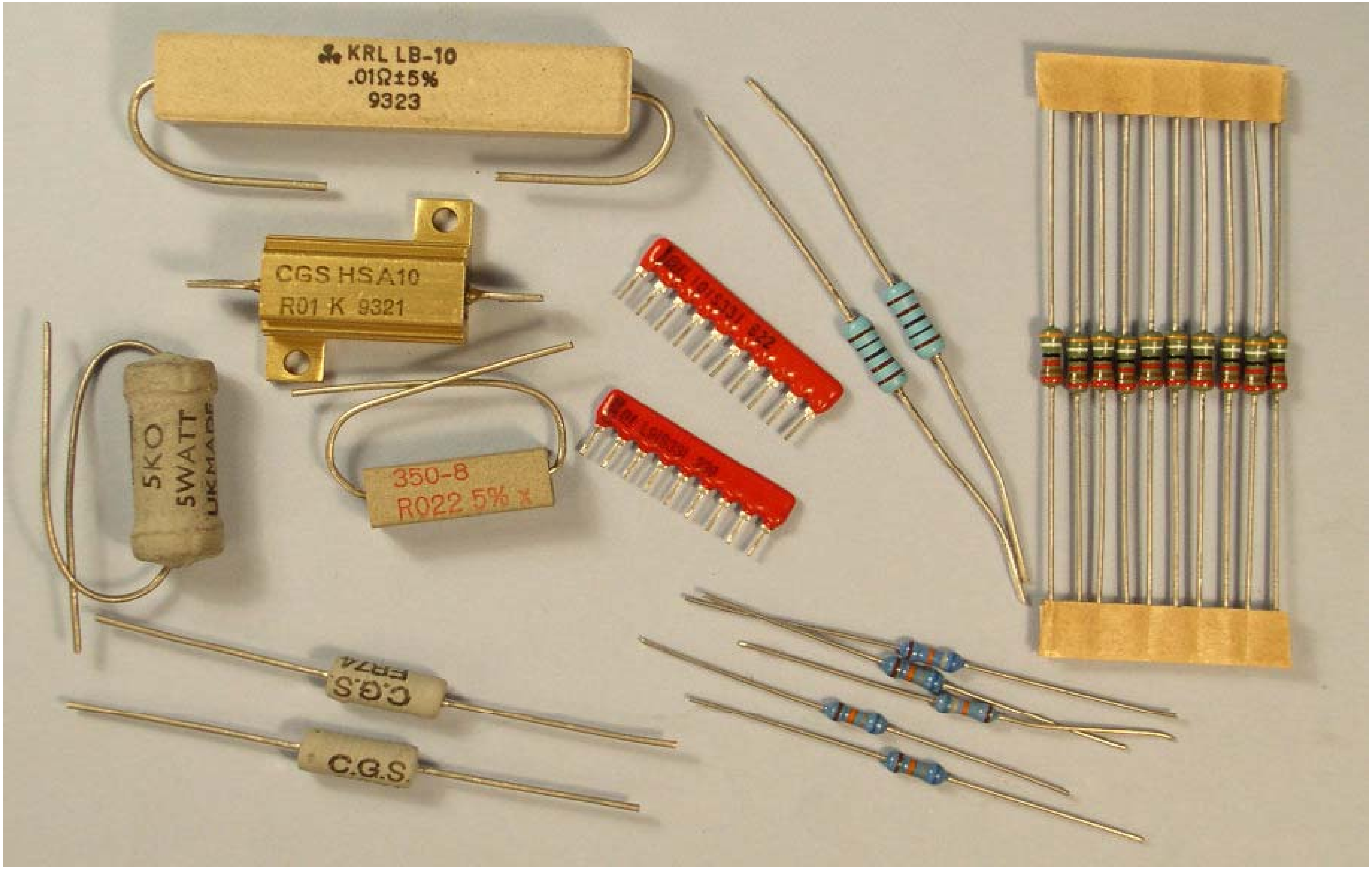
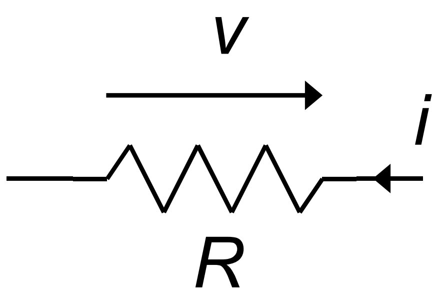


# Resistors




Ohm's Law: $v = Ri$

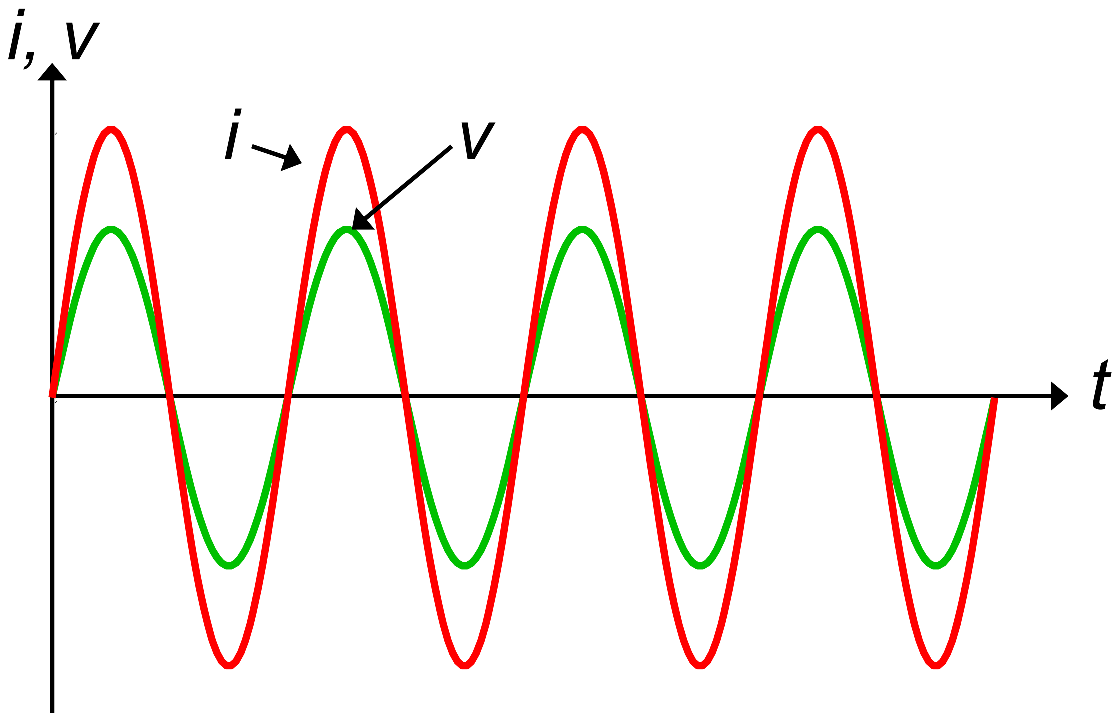
Suppose that:

$$v(t) = v_0 \sin(\omega t)$$

Then:

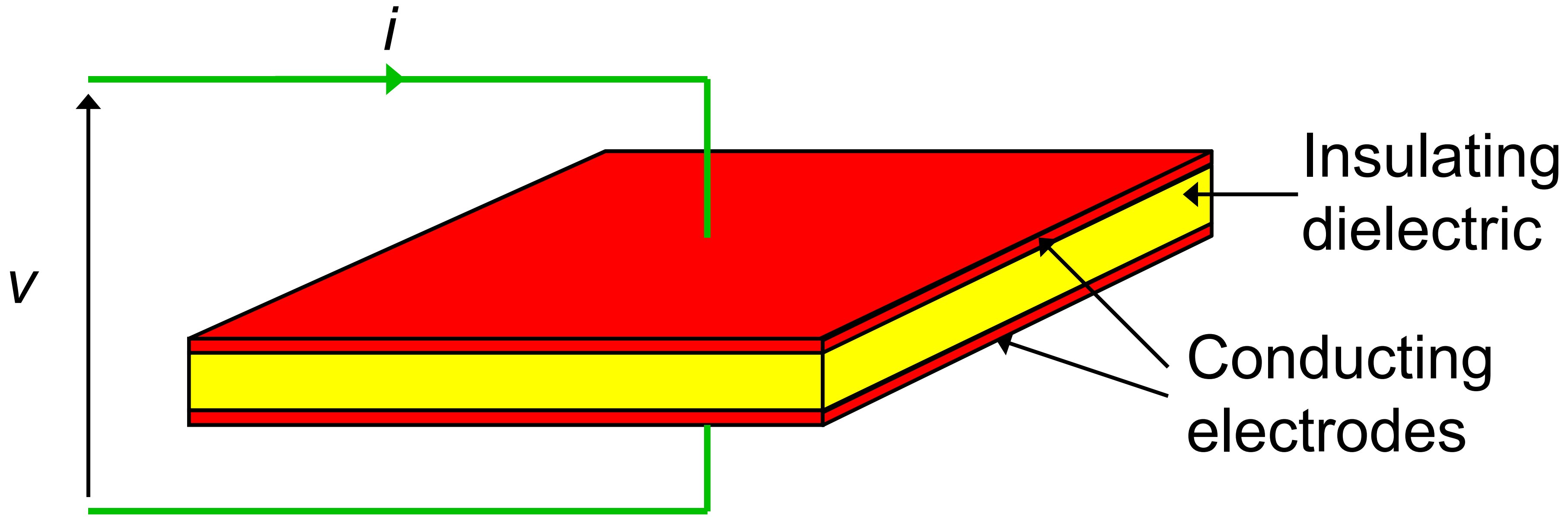
$$i(t) = \frac{v(t)}{R}$$

$$= \frac{v_0}{R} \sin(\omega t)$$




Current in phase with voltage

# Capacitors



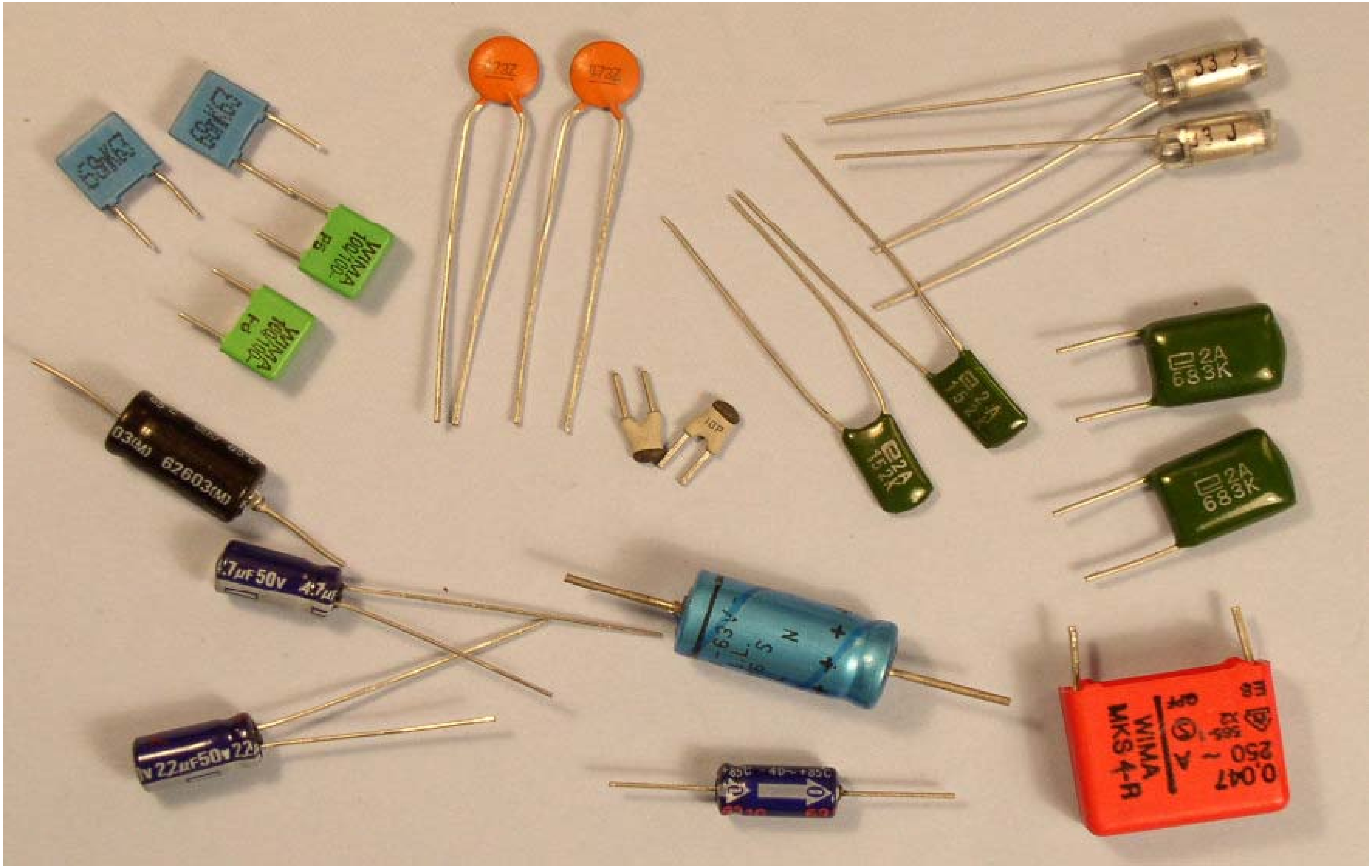

Dielectrics: air
polymer
ceramic
$Al_2 0_3$ (electrolytic)

Capacitance $C$

$$q = Cv \qquad i = C\frac{dv}{dt}$$

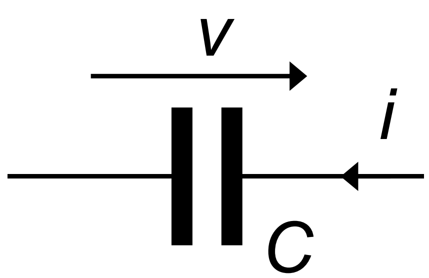
# Capacitors




WIMA
2A
683K
2A
683K
4.7µF 50v
22µF50v
62603(M)
WIMA
MKS 4-R
0,047
250 ~

# Capacitors






$$i = C\frac{dv}{dt}$$

Suppose that:

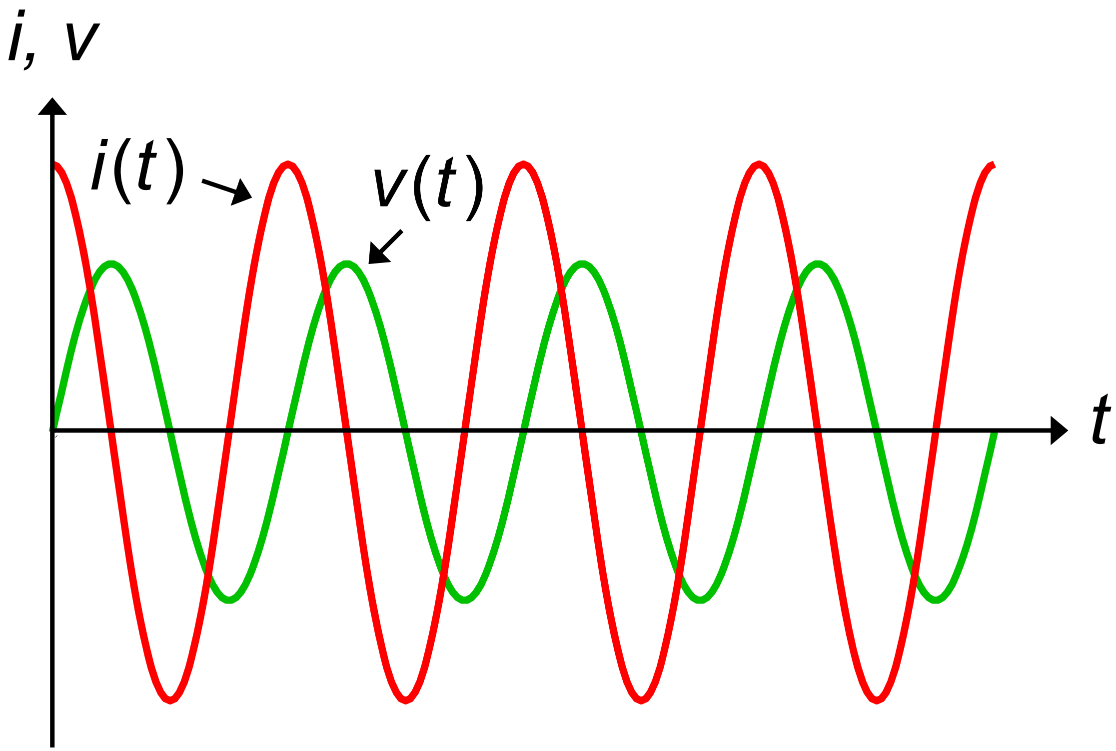
$$v(t) = v_0 \sin(\omega t)$$

Then:

$$i(t) = C\frac{d}{dt}v_0 \sin(\omega t)$$

$$= \omega C v_0 \cos(\omega t)$$

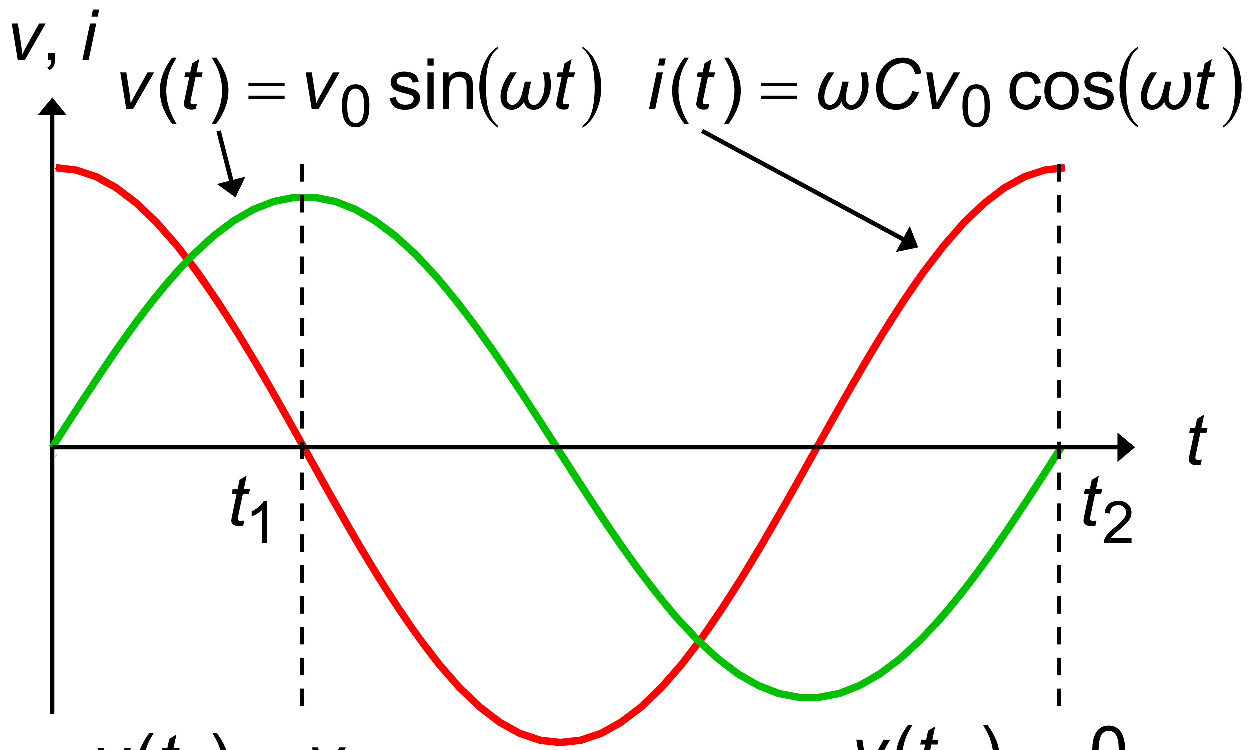
$$= \omega C v_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$




Current *leads* voltage by $\pi/2$ (90°)

# Capacitors

Does a capacitor have a "resistance"?




$$\frac{v(t_1)}{i(t_1)} = \frac{v_0}{0} = \infty \qquad \frac{v(t_2)}{i(t_2)} = \frac{0}{i_0} = 0$$

Thus "resistance" varies between ±∞: not a useful concept

# Capacitors

The *reactance* $X_C$ of a capacitor is defined:

$$X_C = \frac{v_0}{i_0}$$

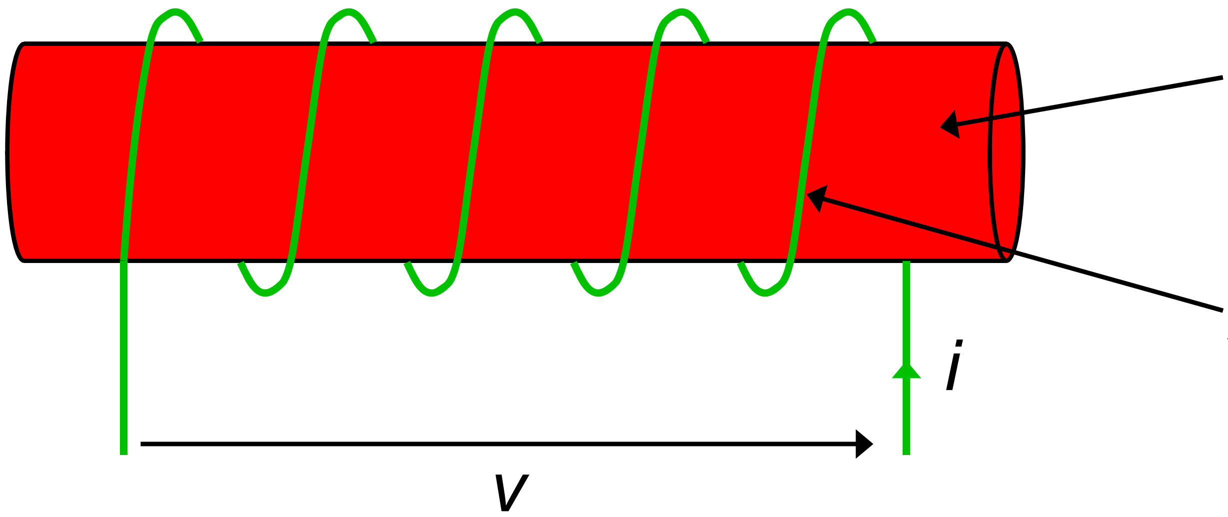
where $v_0$ is the amplitude of the voltage across the capacitor and $i_0$ is the amplitude of the current flowing through it

Thus:

$$X_C = \frac{v_0}{\omega C v_0} = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

The reactance of a capacitor is inversely proportional to its value and to frequency

# Inductors




Core: air
ferrite
iron
silicon steel

Inductance $L$

$$v = L\frac{di}{dt}$$

# Inductors

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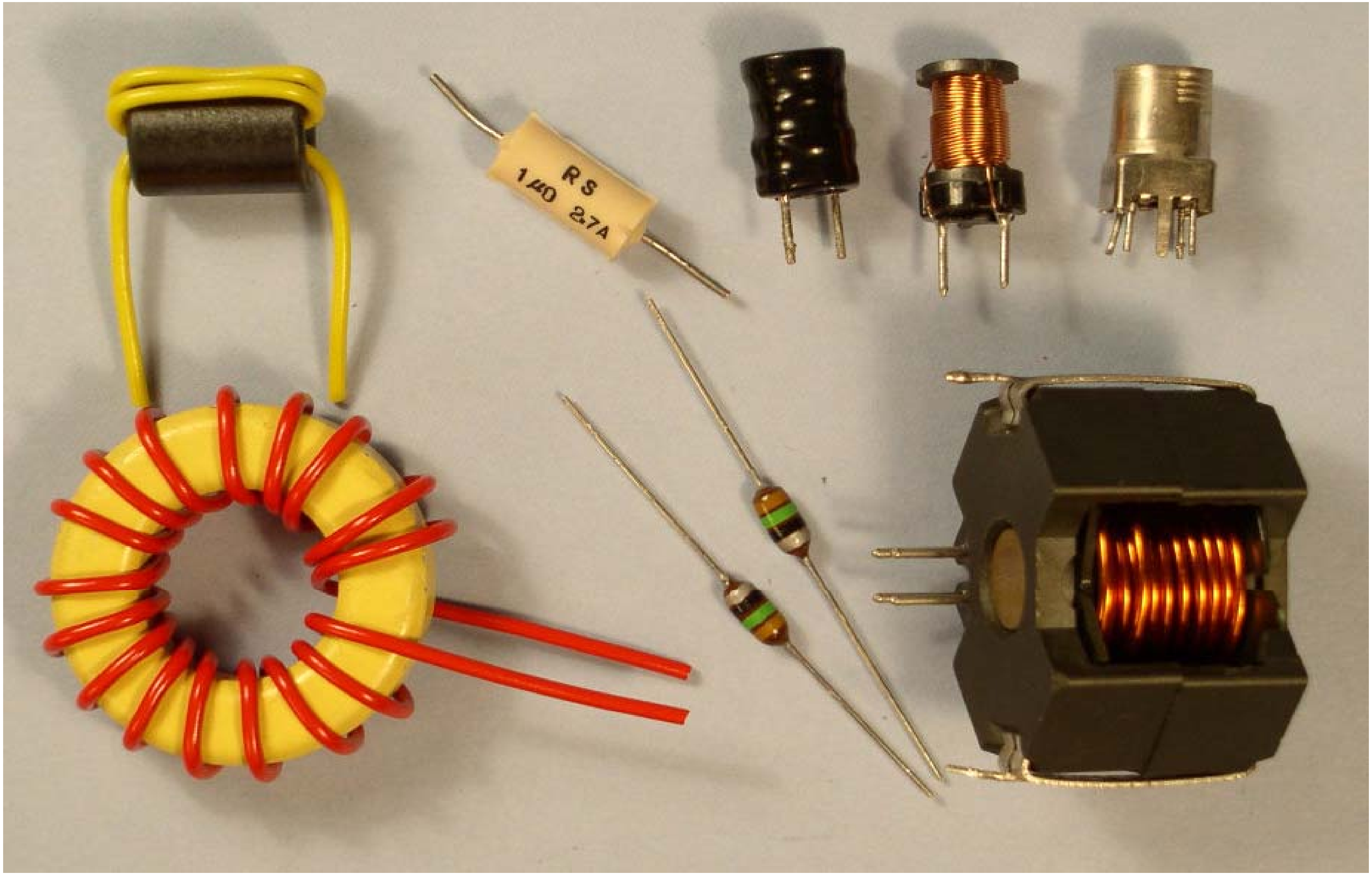
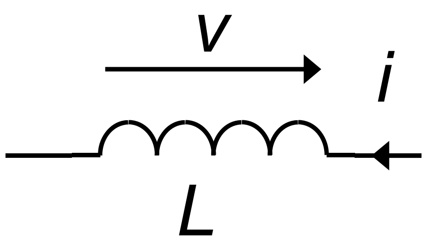


# Inductors




$$v = L\frac{di}{dt}$$

Suppose that:

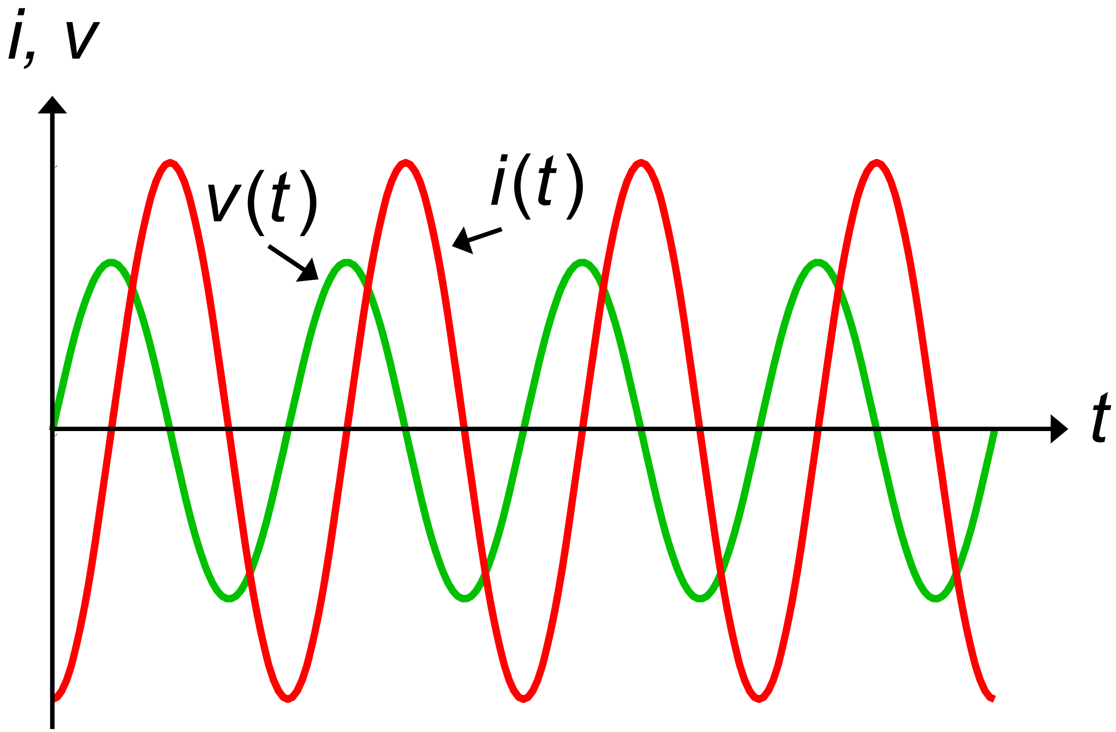
$$v(t) = v_0 \sin(\omega t)$$

Then:

$$i(t) = \frac{1}{L}\int v_0 \sin(\omega t)$$

$$= \frac{-v_0}{\omega L}\cos(\omega t)$$

$$= \frac{v_0}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right)$$




Current *lags* voltage by $\pi/2$ (90°)

# Inductors

The *reactance* $X_C$ of an inductor is defined:

$$X_C = \frac{v_0}{i_0}$$

where $v_0$ is the amplitude of the voltage across the inductor and $i_0$ is the amplitude of the current flowing through it

Thus:

$$X_c = \frac{v_0}{v_0 / \omega L} = \omega L = 2\pi f L$$

The reactance of an inductor is directly proportional to its value and to frequency

# Resistance and Reactance

| | $X = \frac{v_0}{i_0}$ | $f \to 0$ | $f \to \infty$ |
|---|---|---|---|
| Resistance $R$ | $R$ | $R$ | $R$ |
| Capacitance $C$ | $\frac{1}{\omega C}$ | open circuit | short circuit |
| Inductance $L$ | $\omega L$ | short circuit | open circuit |

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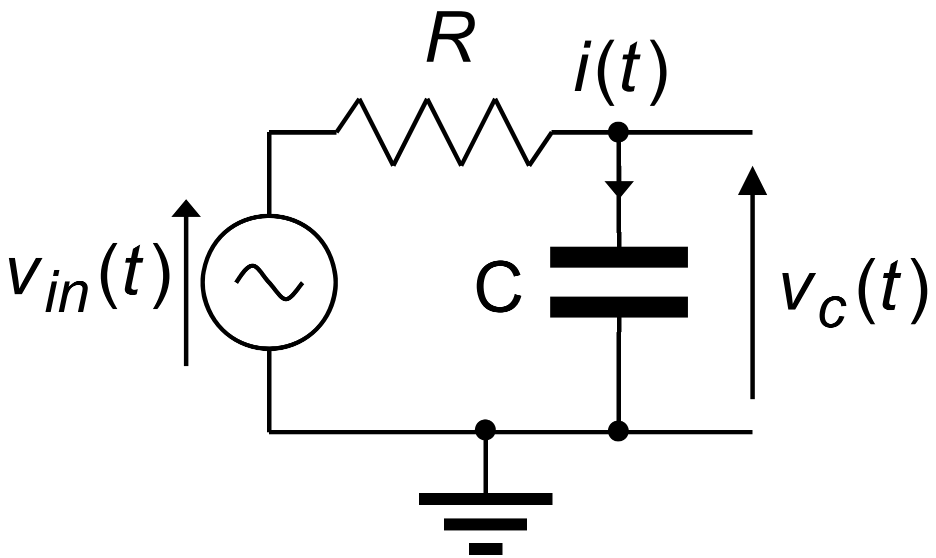
# Lecture 2

## AC Analysis using Differential Equations
## Complex Numbers
## Complex Exponential Voltages and Currents

# AC Circuit Analysis

The ac response of a circuit is determined by a differential equation:




$$v_{in}(t) = Ri(t) + v_c(t)$$

$$i(t) = C\frac{dv_c(t)}{dt}$$

$$v_{in}(t) = RC\frac{dv_c(t)}{dt} + v_c(t)$$

$$\frac{dv_c(t)}{dt} + \frac{v_c(t)}{RC} = \frac{v_{in}(t)}{RC}$$

# AC Circuit Analysis

Now suppose that the input voltage $v_{in}$ is a sinusoid of angular frequency $\omega$

The output voltage $v_c$ will be a sinusoid of the same freqeuncy, but with different amplitude and phase:

$$v_{in}(t) = v_0 \cos(\omega t)$$
$$v_c(t) = v_1 \cos(\omega t + \varphi)$$

Expanding the expression for $v_c$:

$$v_c(t) = v_1 \cos\omega t \cos\varphi - v_1 \sin\omega t \sin\varphi = A\cos\omega t + B\sin\omega t$$
$$\frac{dv_c(t)}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$$

# AC Circuit Analysis

The differential equation becomes:

$$-A\omega\sin\omega t + B\omega\cos\omega t + \frac{A}{RC}\cos\omega t + \frac{B}{RC}\sin\omega t = \frac{v_0}{RC}\cos\omega t$$

Comparing the coefficients of sin$\omega t$ and cos$\omega t$ on both sides of the equation:

$$-A\omega RC + B = 0$$

$$B\omega RC + A = v_0$$

Solving these simultaneous linear equations in $A$ and $B$:

$$A = \frac{v_0}{1+\omega^2R^2C^2} \qquad B = \frac{\omega RC v_0}{1+\omega^2R^2C^2}$$

# AC Circuit Analysis

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$$A = v_1 \cos\varphi = \frac{v_0}{1+\omega^2 R^2 C^2} \qquad B = -v_1 \sin\varphi = \frac{\omega R C v_0}{1+\omega^2 R^2 C^2}$$

Thus:

$$v_1 = v_0 \sqrt{\frac{1}{1+\omega^2 R^2 C^2}} \qquad \tan\varphi = -\omega R C$$

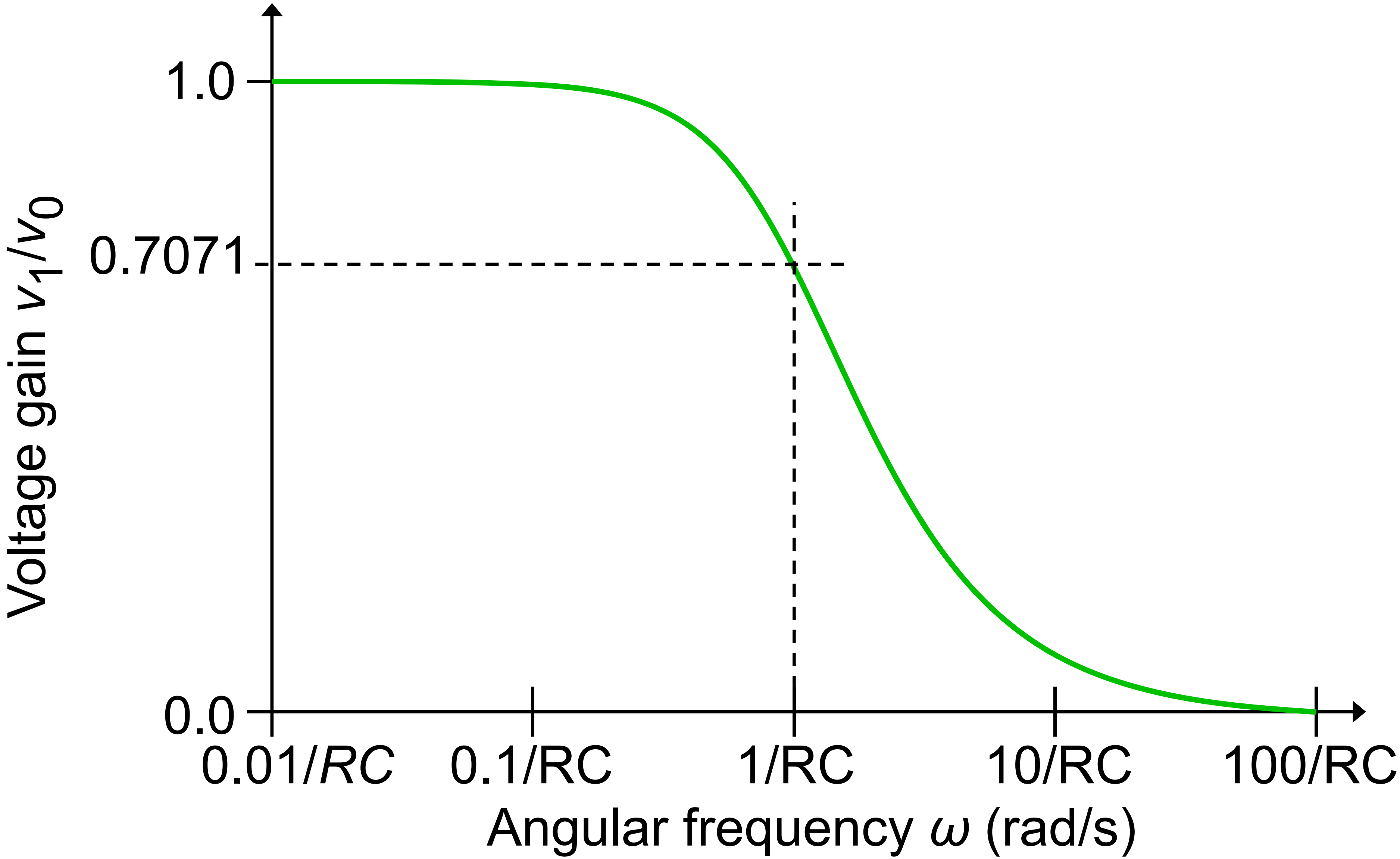
At an angular frequency $\omega$=1/$RC$:

$$v_1 = \frac{v_0}{\sqrt{2}} \qquad \varphi = -\frac{\pi}{4}$$

$$v_c(t) = \frac{v_0}{\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

The output voltage lags the input voltage by $\pi$/4 (45°)

# AC Circuit Analysis



# Complex Numbers: Rectangular Form



Complex numbers can be represented in rectangular, polar or exponential form

Rectangular form:

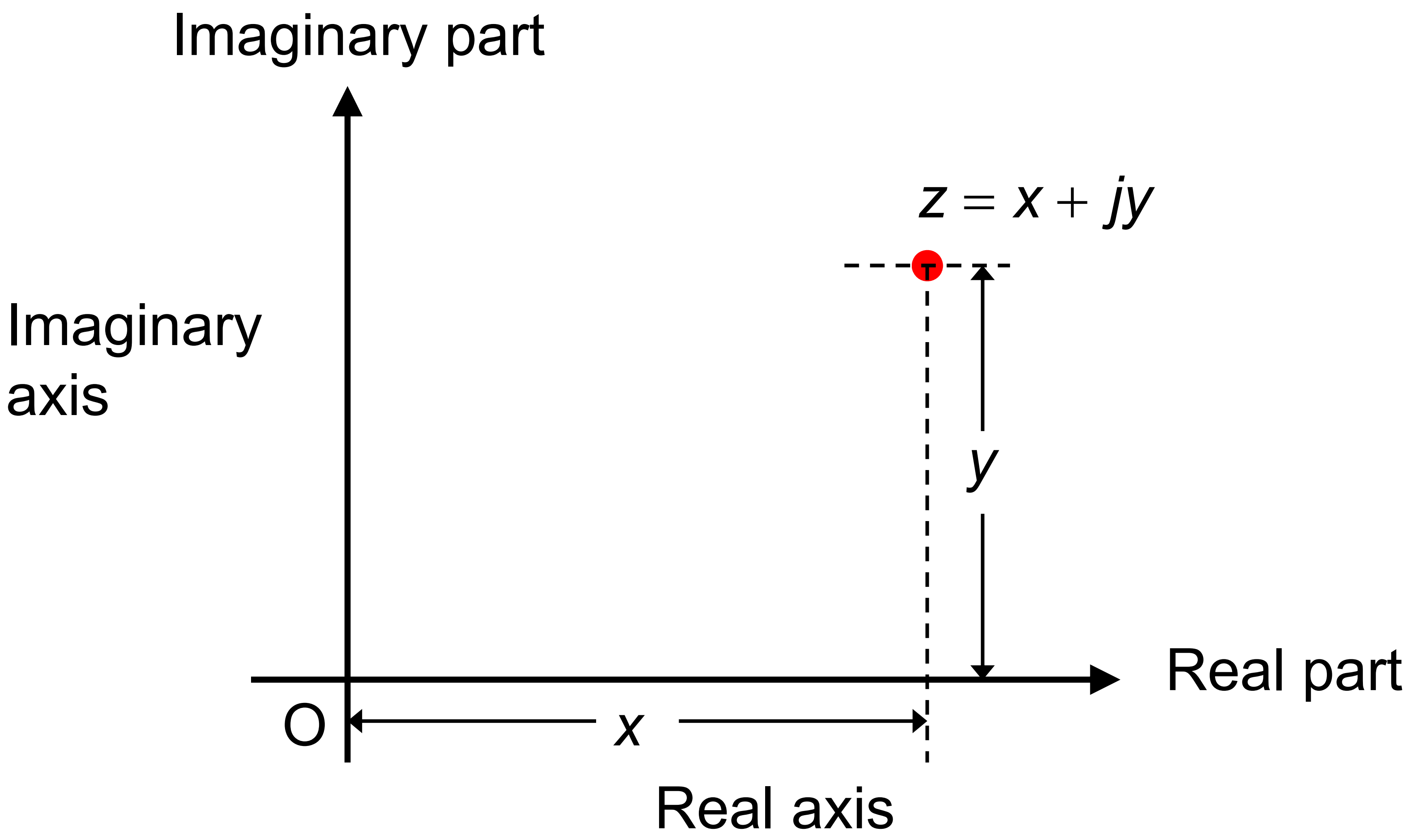
$$z = x + jy$$

where *x* is the real part, *y* is the imaginary part (*x* and *y* are both real numbers), and

$$j^2 = -1 \qquad j = \sqrt{-1}$$

Complex numbers are often the solutions of real problems, for example quadratic equations

# Complex Numbers: Argand Diagram

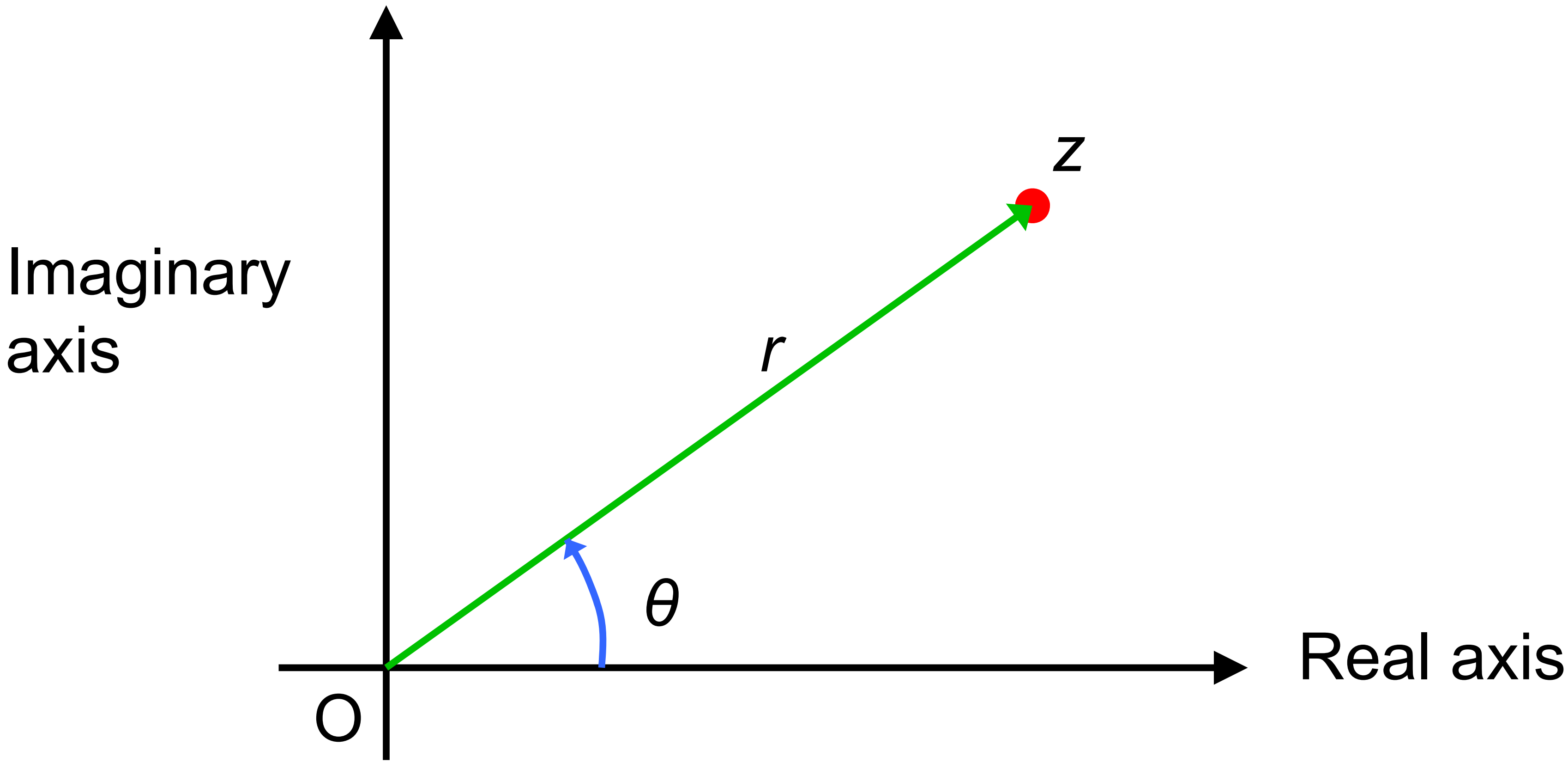


# Complex Numbers: Polar Form

Polar form:

$$z = r\angle\theta$$

where $r$ is the magnitude, and $\theta$ is the angle measured from the real axis:



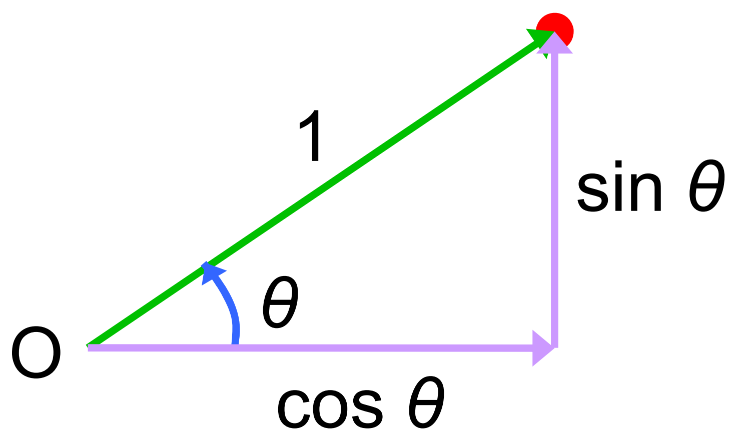
# Complex Numbers: Exponential Form

Exponential form:

$$z = re^{j\theta}$$

Euler's identity:

$$e^{j\theta} = \cos\theta + j\sin\theta$$




The polar and exponential forms are therefore equivalent

# Complex Numbers: Conversion

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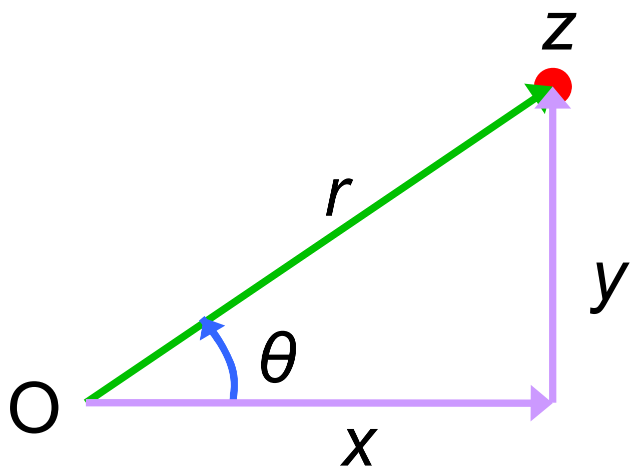


Rectangular to polar:

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \angle z \qquad \tan\theta = \frac{y}{x}$$

Polar to Rectangular:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

# Complex Numbers: Inversion

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If the complex number is in rectangular form:

$$z = \frac{1}{x + jy}$$

$$= \frac{x - jy}{(x + jy)(x - jy)}$$

$$= \frac{x - jy}{x^2 + y^2}$$

If the complex number is in polar or exponential form:

$$z = \frac{1}{Ae^{j\varphi}} = \frac{1}{A}e^{-j\varphi}$$

# Complex Numbers: Conversion

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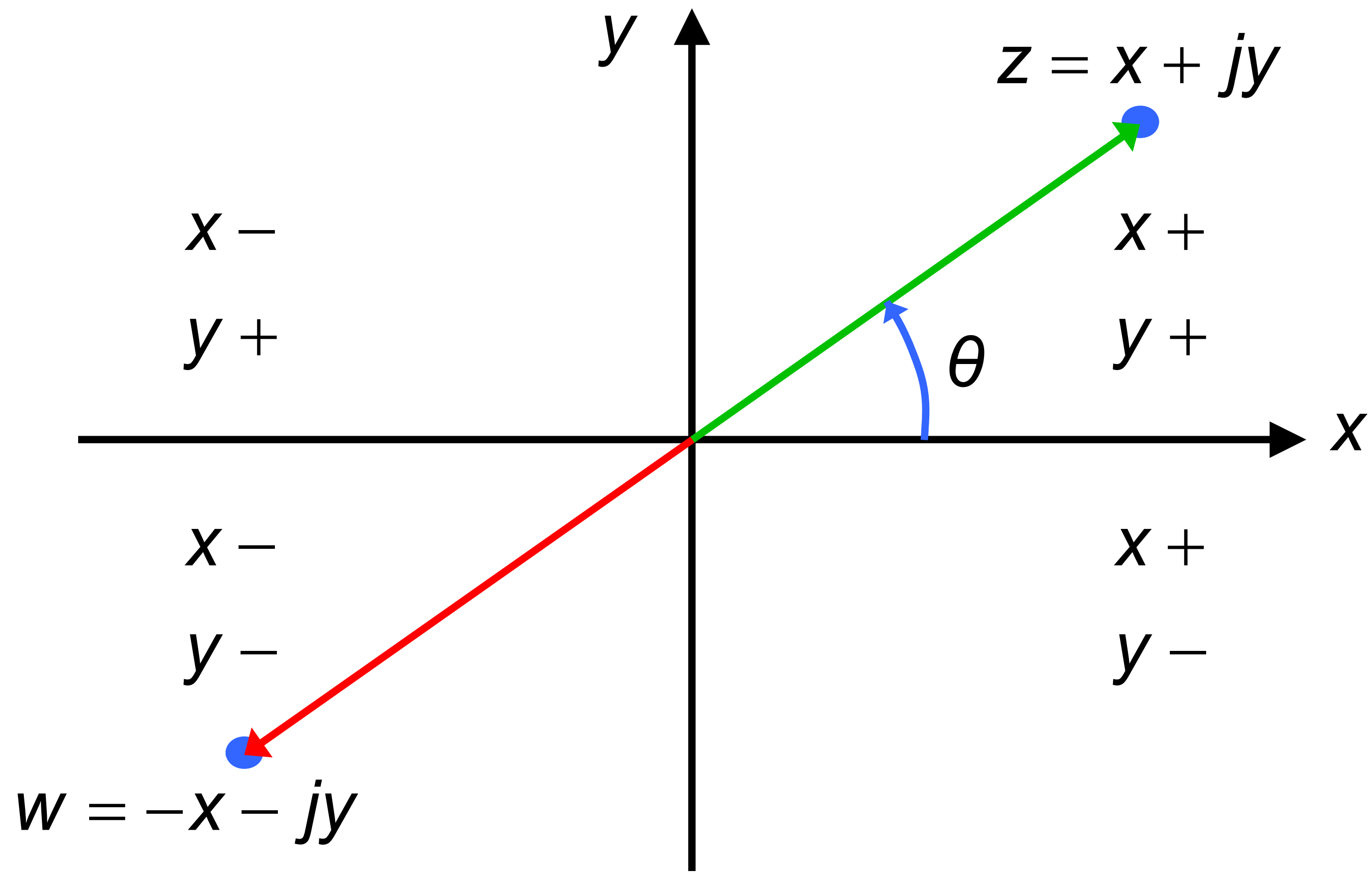
When using the inverse tangent to obtain $\theta$ from $x$ and $y$ it is necessary to resolve the ambiguity of $\pi$:

$$\tan\theta = \frac{y}{x}$$



# Complex Numbers: Conversion

When using the inverse tangent to obtain $\theta$ from $x$ and $y$ it is necessary to resolve the ambiguity of $\pi$

1. Calculate $\theta$ using inverse tangent:

$$\theta = \tan^{-1}\frac{y}{x}$$

   This should give a value in the range: $-\pi/2 \le \theta \le +\pi/2$ $(-90^\circ \le \theta \le +90^\circ)$

2. If the real part $x$ is negative then add $\pi$ (180º) :

$$\theta = \pi + \tan^{-1}\frac{y}{x}$$

# Complex Numbers: Conversion

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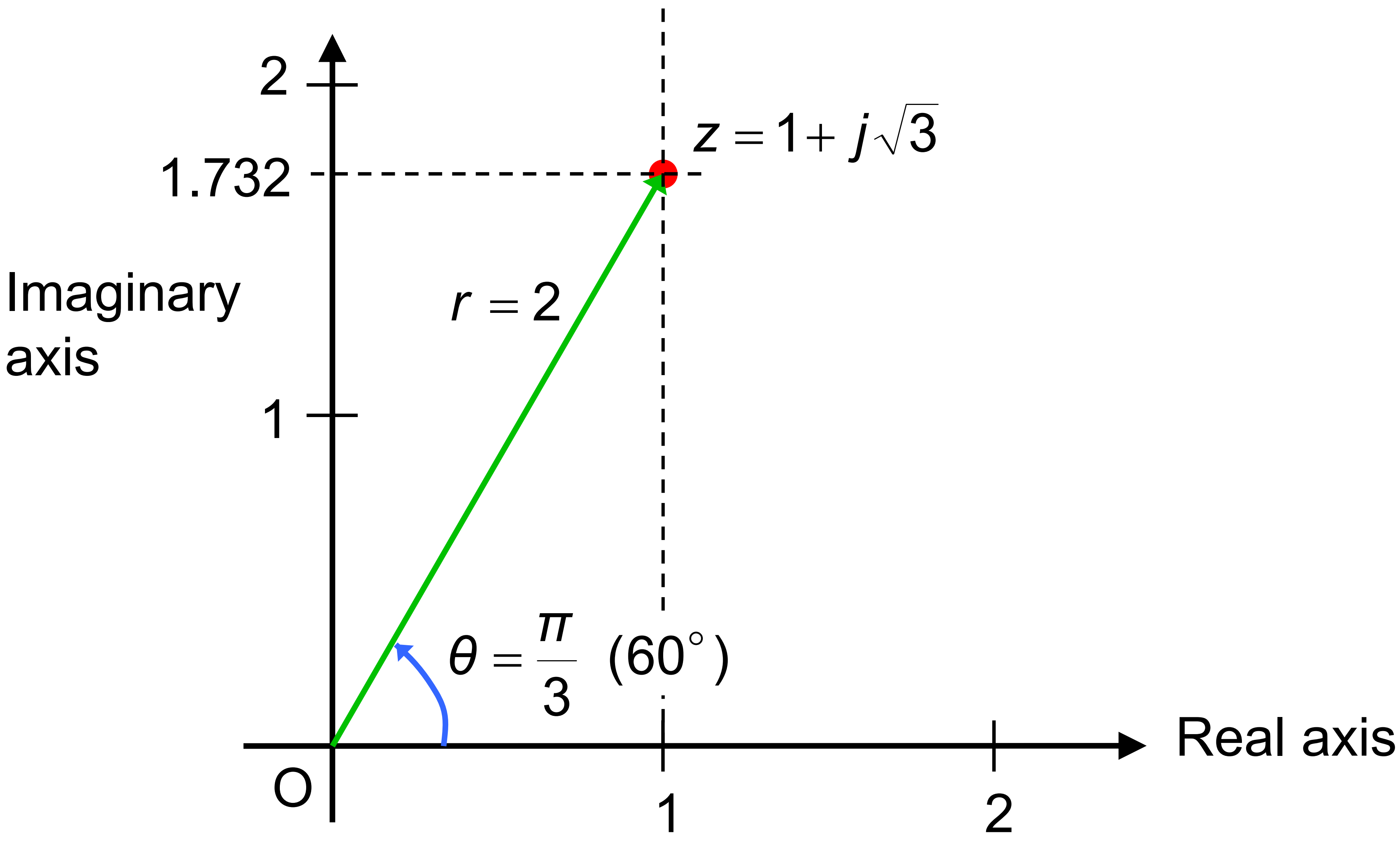
Convert $z = 2\angle\frac{\pi}{3}$ to rectangular form

Real part: $$x = 2\cos\left(\frac{\pi}{3}\right) = 2\times\frac{1}{2} = 1$$

Imaginary part: $$y = 2\sin\left(\frac{\pi}{3}\right) = 2\times\frac{\sqrt{3}}{2} = \sqrt{3}$$

Thus: $$z = 1 + j\sqrt{3}$$

# Complex Numbers: Conversion



# Complex Numbers: Conversion

Convert $z = \dfrac{1}{1+j}$ to polar or exponential form:

Magnitude: $$r = |z| = \frac{\sqrt{1^2 + 0^2}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}$$

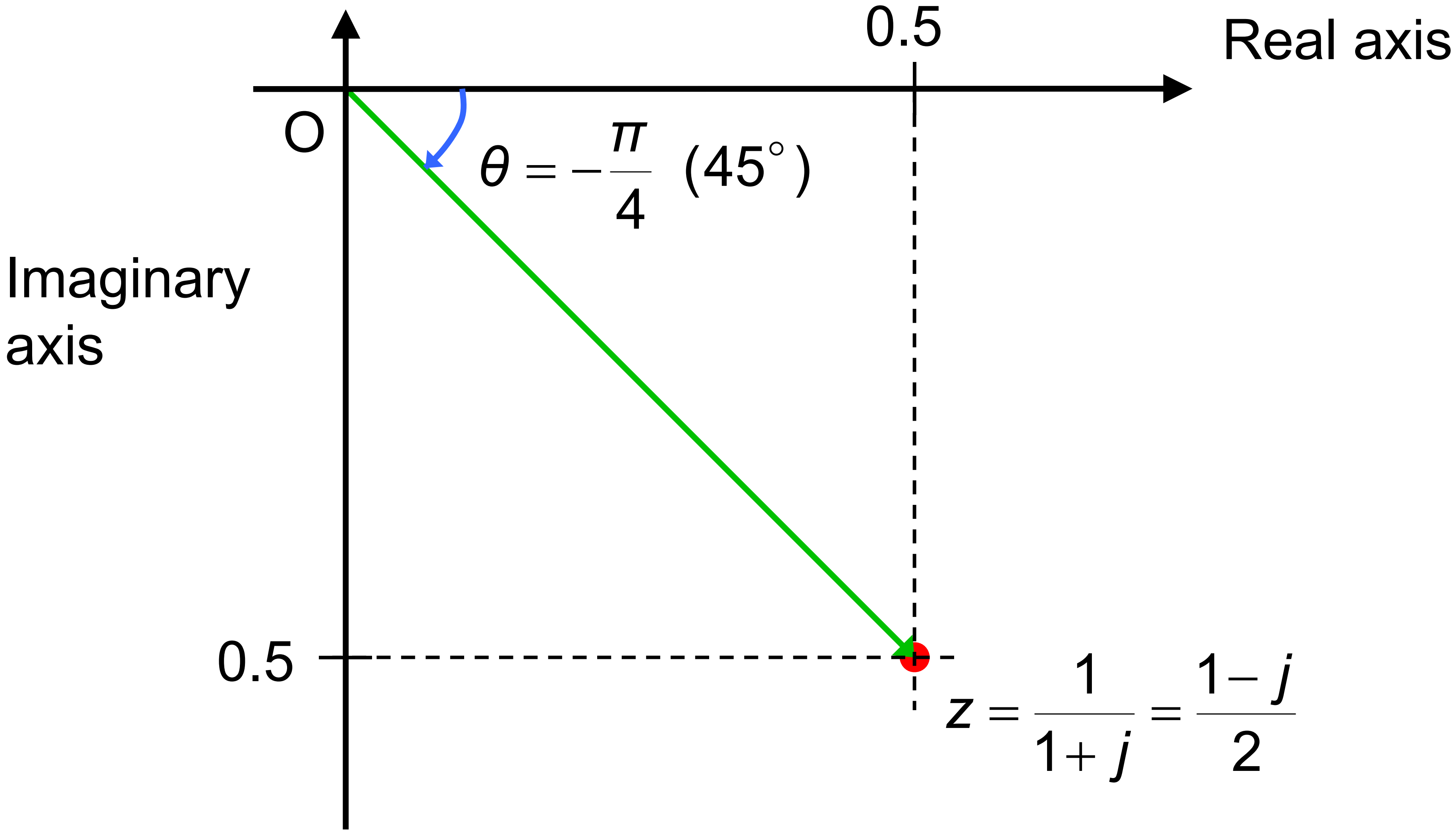
Angle: $$\theta = \angle z = \angle 1 - \angle(1+j)$$

$$= \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}\left(\frac{1}{1}\right) = 0 - \frac{\pi}{4} = -\frac{\pi}{4}$$

Thus: $z = \dfrac{1}{\sqrt{2}} \angle -\dfrac{\pi}{4}$ or $z = \dfrac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}$

# Complex Numbers: Conversion





# Complex Exponential Voltages

We shall be using complex exponential voltages and currents to analyse ac circuits:

$$v(t) = Ve^{j\omega t}$$

This is a mathematical trick for obtaining the ac response without explicitly solving the differential equations

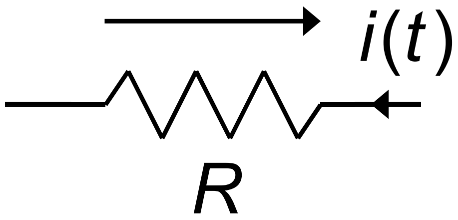
It works because differentiating a complex exponential leaves it unchanged, apart from a multiplying factor:

$$\frac{d}{dt} Ve^{j\omega t} = j\omega Ve^{j\omega t}$$

# Complex Exponential Voltages

Suppose that a complex exponential voltage is applied across a resistor:

$$v(t) = Ve^{j\omega t}$$

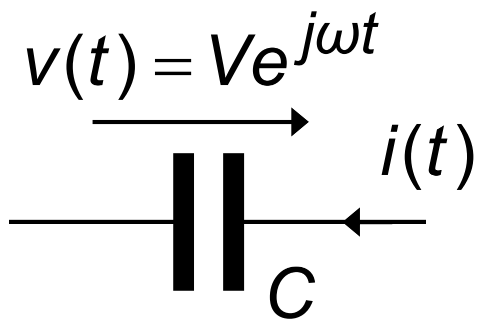



$$i(t) = \frac{v(t)}{R} = \frac{V}{R}e^{j\omega t}$$

The current through the resistor is also a complex exponential

# Complex Exponential Voltages

Suppose that a complex exponential voltage is applied across a capacitor:




$$i(t) = C\frac{dv(t)}{dt}$$

$$= C\frac{d}{dt}Ve^{j\omega t}$$

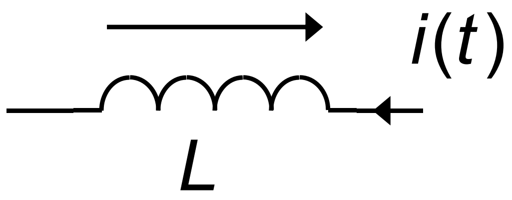
$$= j\omega CVe^{j\omega t}$$

The current through the capacitor is also a complex exponential

# Complex Exponential Voltages

Suppose that a complex exponential voltage is applied across an inductor:

$$v(t) = Ve^{j\omega t}$$




$$i(t) = \frac{1}{L}\int v(t)dt$$

$$= \frac{1}{L}\int Ve^{j\omega t}dt$$

$$= \frac{1}{j\omega L}Ve^{j\omega t}$$

The current through the inductor is also a complex exponential

# Complex Exponential Voltages

A complex exponential input to a linear ac circuits results in all voltages and currents being complex exponentials

Of course real voltages are not complex

The real voltages and currents in the circuit are simply the real parts of the complex exponentials

Complex exponential: $v_c(t) = e^{j\omega t} \quad (= \cos \omega t + j \sin \omega t)$

Real voltage: $v_r(t) = \cos \omega t$

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# Lecture 3

Phasors
Impedances
Gain and Phase Shift
Frequency Response

# Phasors

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If the input voltage to a circuit is a complex exponential:

$$v_{cin}(t) = v_0 e^{j\omega t}$$

then all other voltages and currents are also complex exponentials:

$$v_{c1}(t) = v_1 e^{j(\omega t + \varphi_1)} = v_1 e^{j\varphi_1} e^{j\omega t} = V_1 e^{j\omega t}$$

$$i_{c2}(t) = i_2 e^{j(\omega t + \varphi_2)} = i_2 e^{j\varphi_2} e^{j\omega t} = I_2 e^{j\omega t}$$

where $V_1$ and $I_2$ are time-independent voltage and current *phasors*:

$$V_1 = v_1 e^{j\varphi_1}$$

$$I_2 = i_2 e^{j\varphi_2}$$

# Phasors

The complex exponential voltages and currents can now be expressed:

$$v_{c1}(t) = V_1 e^{j\omega t}$$

$$i_{c2}(t) = I_2 e^{j\omega t}$$

Phasors are independent of time, but in general are functions of *jω* and should be written:

$$V_1(j\omega) \quad I_2(j\omega)$$

However, when there is no risk of ambiguity the dependency will be not be shown explicitly

Note that upper-case letters are used for phasor symbols

# Impedance

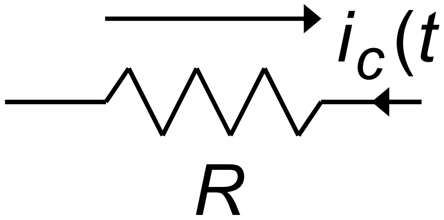
The impedance $Z$ of a circuit or component is defined to be the ratio of the voltage and current phasors:

$$Z = \frac{V}{I}$$

For a resistor:

$$v_c(t) = Ve^{j\omega t}$$

$$i_c(t) = Ie^{j\omega t}$$




$$v_c(t) = Ri_c(t)$$

$$Ve^{j\omega t} = RIe^{j\omega t}$$

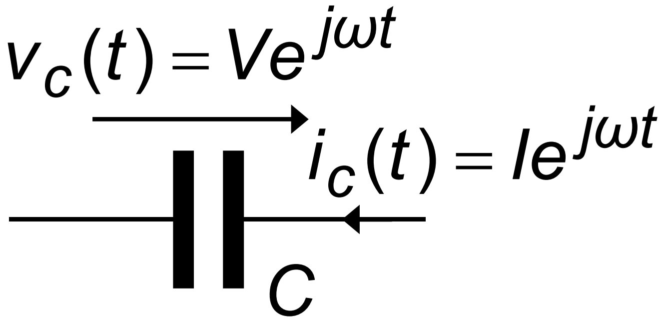
$$V = RI$$

So that:

$$Z_R = \frac{V}{I} = R$$

# Impedance

For a capacitor:




$$i_c(t) = C\frac{dv_c(t)}{dt}$$

$$Ie^{j\omega t} = C\frac{d}{dt}Ve^{j\omega t}$$

$$Ie^{j\omega t} = j\omega C V e^{j\omega t}$$

$$I = j\omega C V$$

So that:

$$Z_C = \frac{V}{I} = \frac{1}{j\omega C}$$

# Impedance

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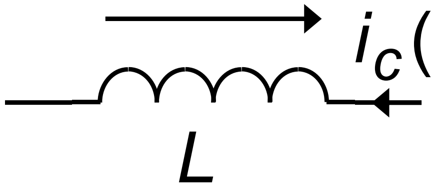
For an inductor:

$$v_c(t) = Ve^{j\omega t}$$




$$i_c(t) = Ie^{j\omega t}$$

$$v_c(t) = L\frac{di_c(t)}{dt}$$

$$Ve^{j\omega t} = L\frac{d}{dt}Ie^{j\omega t}$$

$$Ve^{j\omega t} = j\omega L I e^{j\omega t}$$

$$V = j\omega L I$$

So that:

$$Z_L = \frac{V}{I} = j\omega L$$

# Impedance

| | $Z = \dfrac{V}{I}$ | $f \to 0$ | $f \to \infty$ |
|---|---|---|---|
| Resistance $R$ | $R$ | $R$ | $R$ |
| Capacitance $C$ | $\dfrac{1}{j\omega C}$ | $Z \to \infty$ | $Z \to 0$ |
| Inductance $L$ | $j\omega L$ | $Z \to 0$ | $Z \to \infty$ |

# Impedance

All the normal circuit theory rules apply to circuits containing impedances

For example impedances in series:

$$Z = Z_1 + Z_2 + Z_3 + Z_4$$

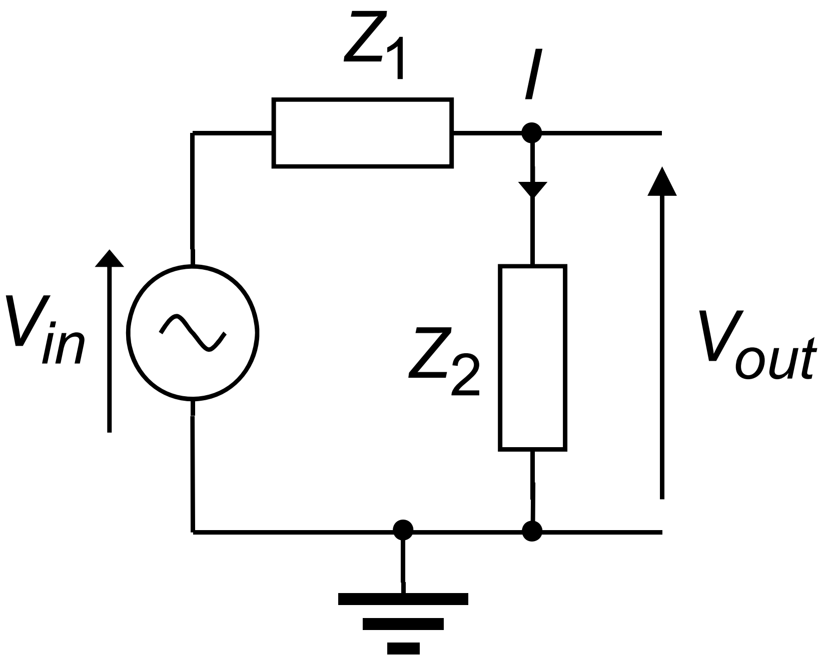
and impedances in parallel:

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4}$$

Other relevant circuit theory rules are: Kirchhoff’s laws, Thévenin and Norton's theorems, Superposition

# Impedance

Potential divider:




$$I = \frac{V_{in}}{Z_1 + Z_2}$$

$$V_{out} = IZ_2$$

$$= \frac{V_{in}Z_2}{Z_1 + Z_2}$$

$$\frac{V_{out}}{V_{in}} = \frac{Z_2}{Z_1 + Z_2}$$

# AC Circuit Analysis

Suppose that a circuit has an input $x(t)$ and an output $y(t)$, where $x$ and $y$ can be voltages or currents

The corresponding phasors are $X(j\omega)$ and $Y(j\omega)$

The real input voltage $x(t)$ is a sinusoid of amplitude $x_0$:

$$x(t) = x_0 \cos(\omega t) = re(x_0 e^{j\omega t}) = re(X e^{j\omega t})$$

and the real output voltage $y(t)$ is the real part of the complex exponential output:

$$y(t) = y_0 \cos(\omega t + \varphi) = re(y_0 e^{j\varphi} e^{j\omega t}) = re(Y e^{j\omega t})$$

# AC Circuit Analysis

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Thus:

$$\frac{y_0 e^{j\varphi}}{x_0} = \frac{Y}{X}$$

The voltage gain $g$ is the ratio of the output amplitude to the input amplitude:

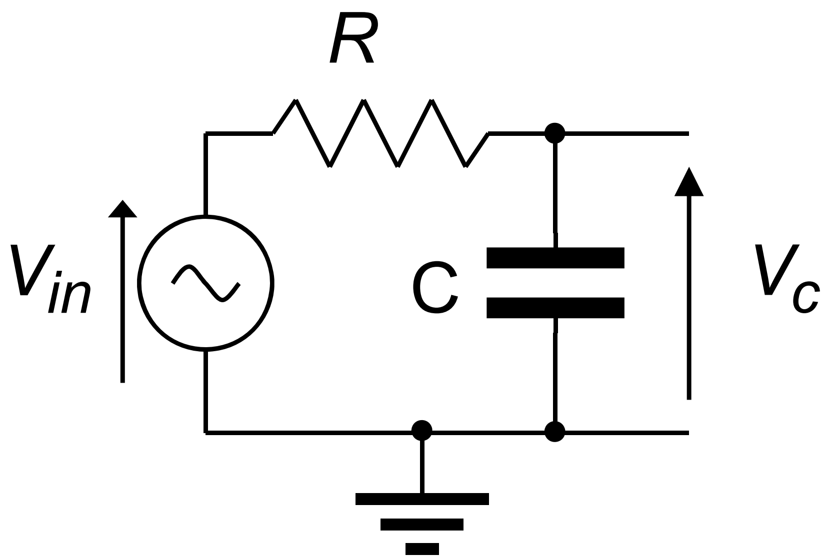
$$g = \frac{y_0}{x_0} = \left|\frac{Y}{X}\right|$$

and the phase shift is:

$$\varphi = \angle\left(\frac{Y}{X}\right)$$

# AC Circuit Analysis

Using the potential divider formula:




$$\frac{V_c}{V_{in}} = \frac{Z_C}{Z_C + Z_R}$$

$$= \frac{1/\,j\omega C}{1/\,j\omega C + R}$$

$$= \frac{1}{1 + j\omega CR}$$

# AC Circuit Analysis

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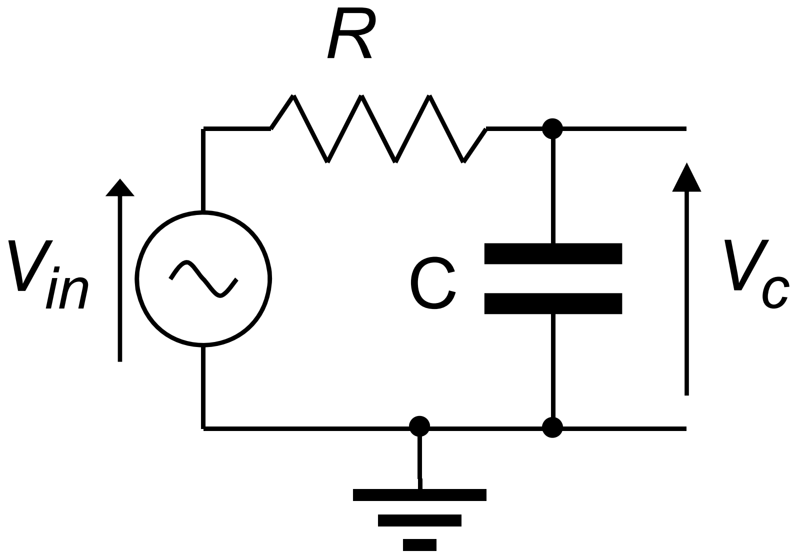


$$\frac{V_c}{V_{in}} = \frac{1}{1 + j\omega CR}$$

Voltage gain: $$g = \left|\frac{V_c}{V_{in}}\right| = \frac{1}{\sqrt{1+\omega^2 C^2 R^2}}$$

Phase shift: $$\varphi = \angle\left(\frac{V_c}{V_{in}}\right) = \tan^{-1} 0 - \tan^{-1}\omega CR$$

$$= -\tan^{-1}\omega CR$$

# Frequency Response (RC = 1)

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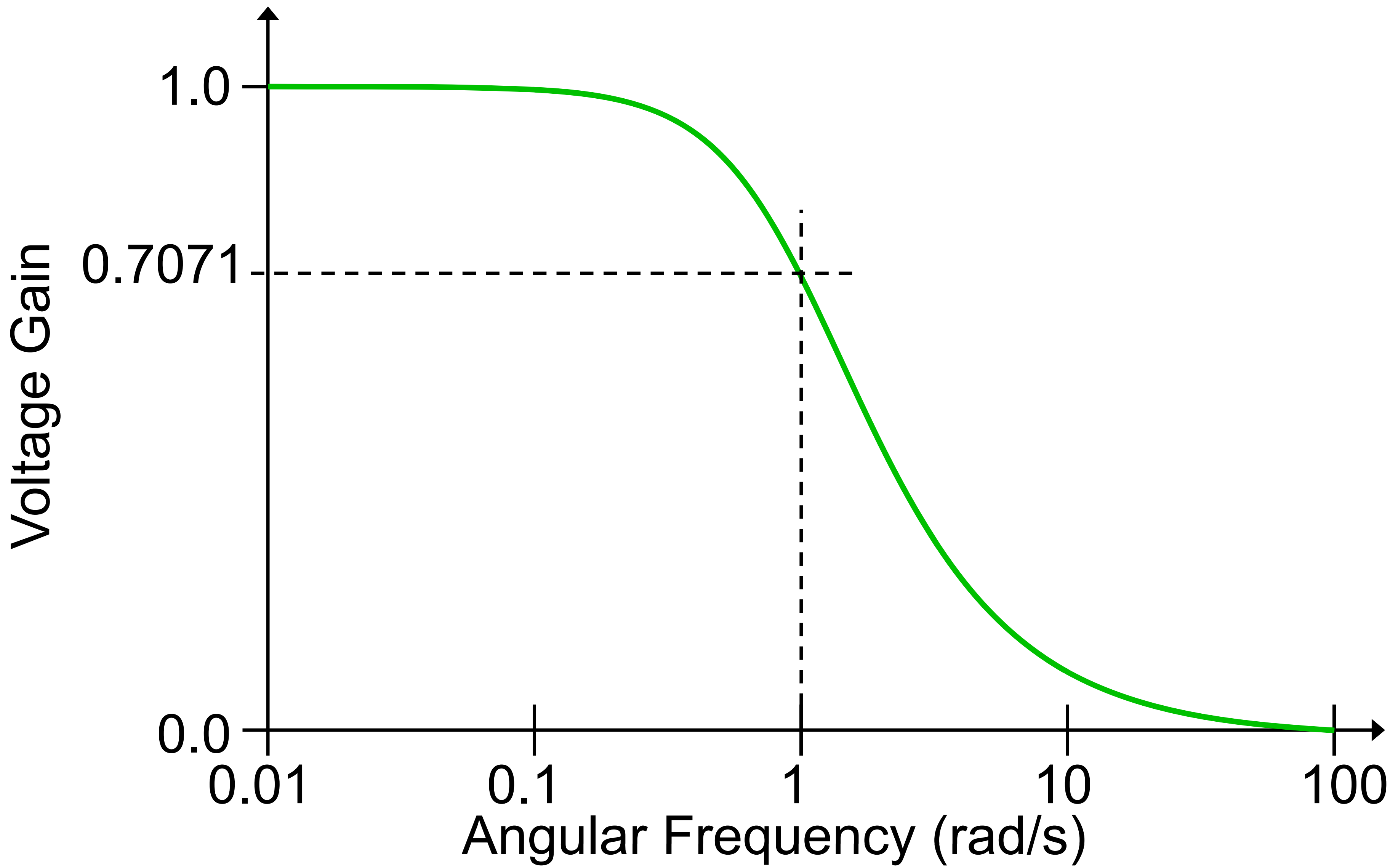
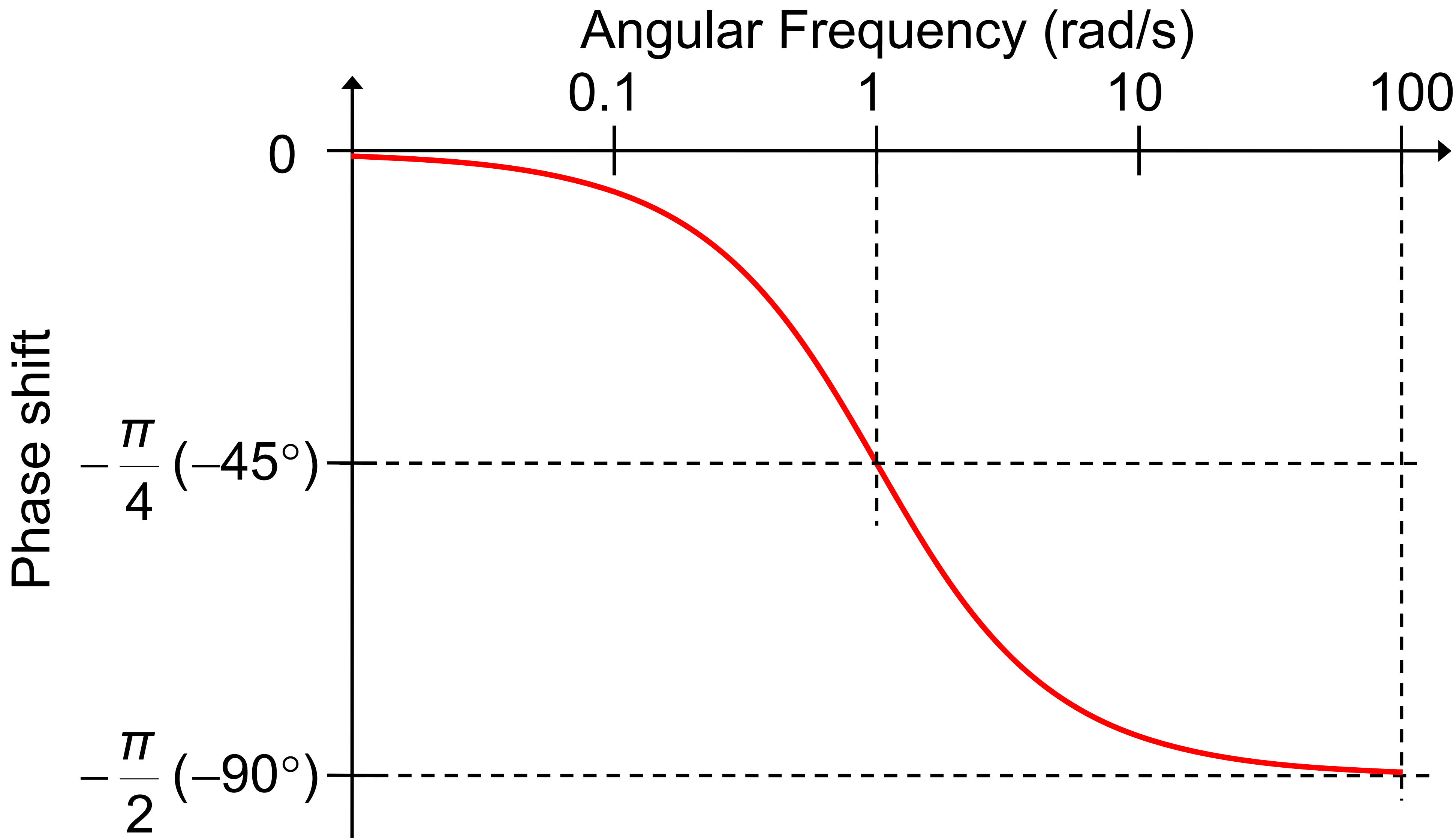


# Frequency Response (RC = 1)



# Frequency Response

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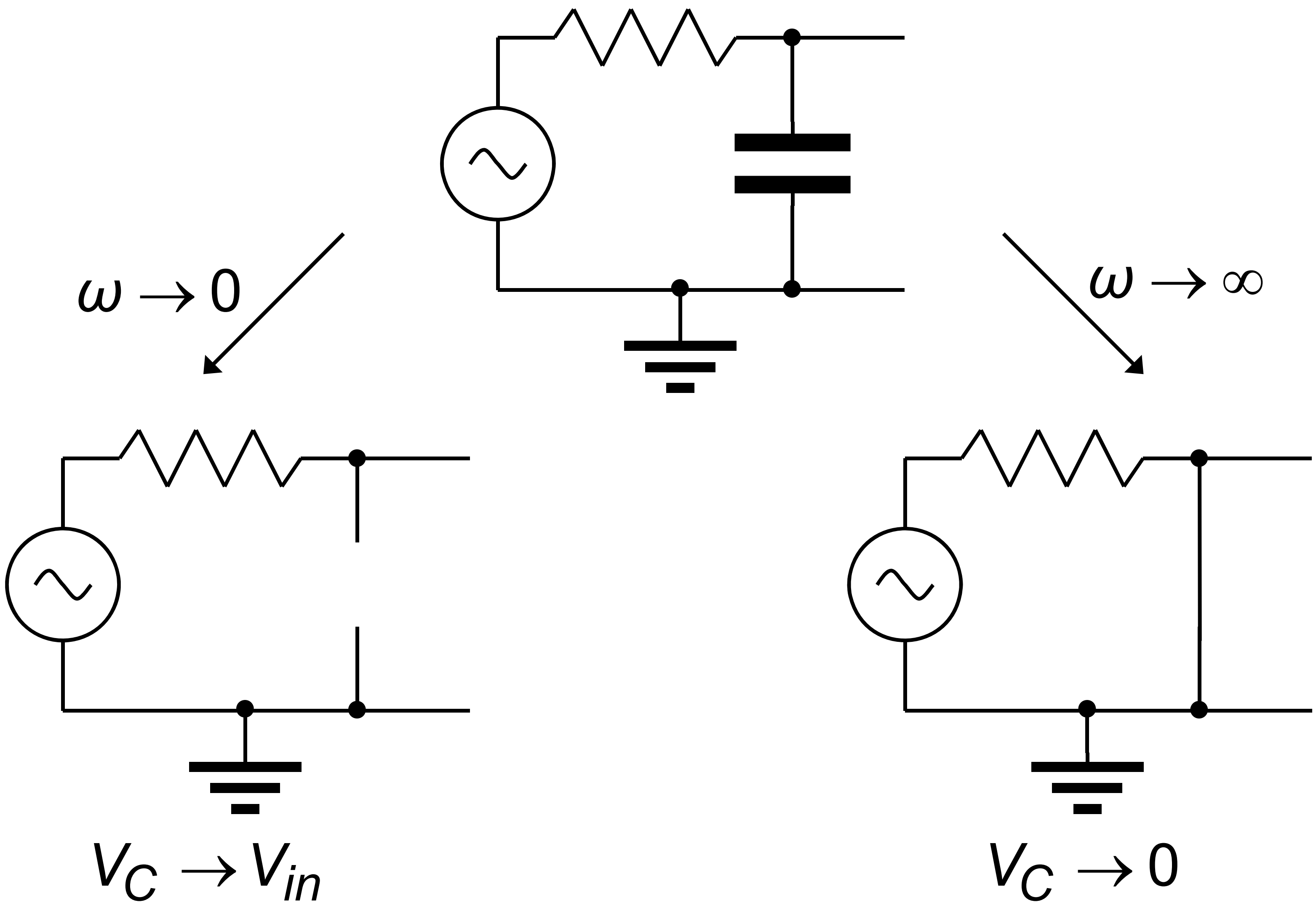
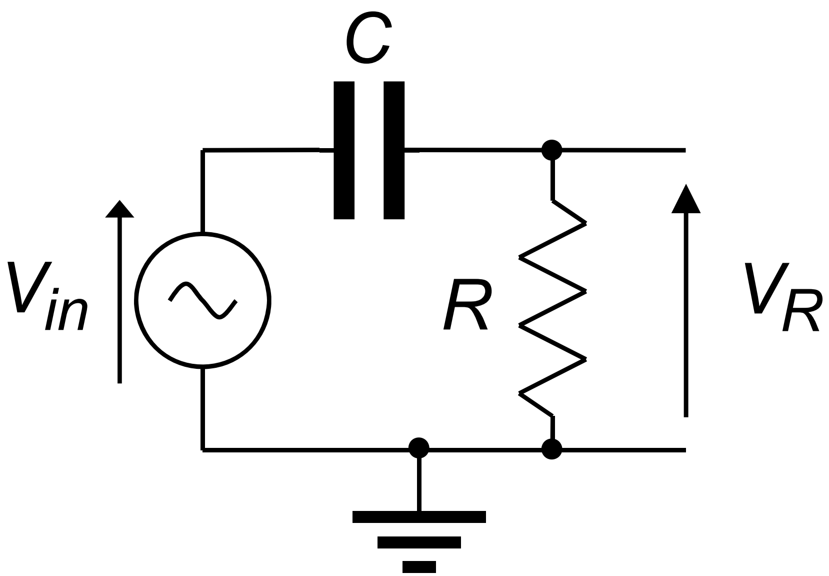
| | $g = \dfrac{1}{\sqrt{1+\omega^2 C^2 R^2}}$ | $\varphi = -\tan^{-1}\omega CR$ |
|---|---|---|
| $\omega \to 0$ | $g \to 1$ | $\varphi \to 0\ (0°)$ |
| $\omega = \dfrac{1}{CR}$ | $g = \dfrac{1}{\sqrt{2}}$ | $\varphi = -\dfrac{\pi}{4}\ (-45°)$ |
| $\omega \to \infty$ | $g \to 0$ | $\varphi \to -\dfrac{\pi}{2}\ (-90°)$ |

This is a low-pass response

# Frequency Response



# AC Circuit Analysis




$$V_R = V_{in} \frac{Z_R}{Z_R + Z_C}$$

$$= V_{in} \frac{R}{R + 1/ j\omega C}$$

$$V_R = V_{in} \frac{j\omega CR}{1 + j\omega CR}$$

$$\frac{V_R}{V_{in}} = \frac{j\omega CR}{1 + j\omega CR}$$

# AC Circuit Analysis

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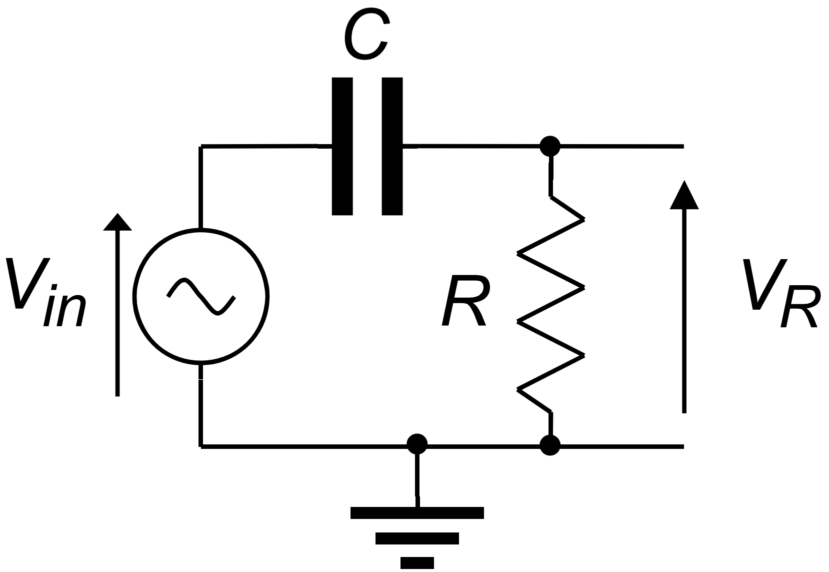


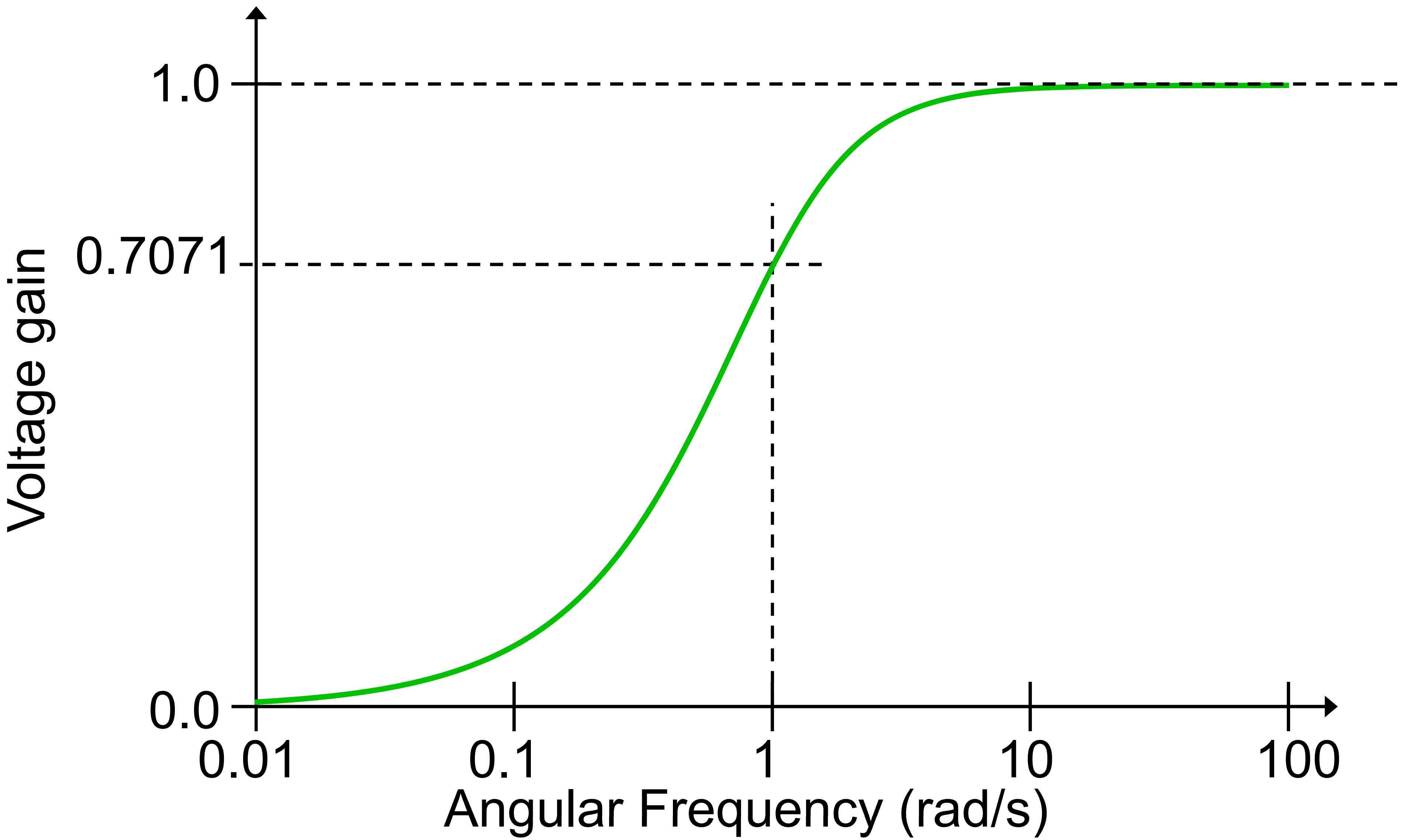

$$\frac{V_R}{V_{in}} = \frac{j\omega CR}{1 + j\omega CR}$$

Voltage gain: $$g = \left|\frac{V_R}{V_{in}}\right| = \frac{\omega CR}{\sqrt{1+\omega^2 C^2 R^2}}$$

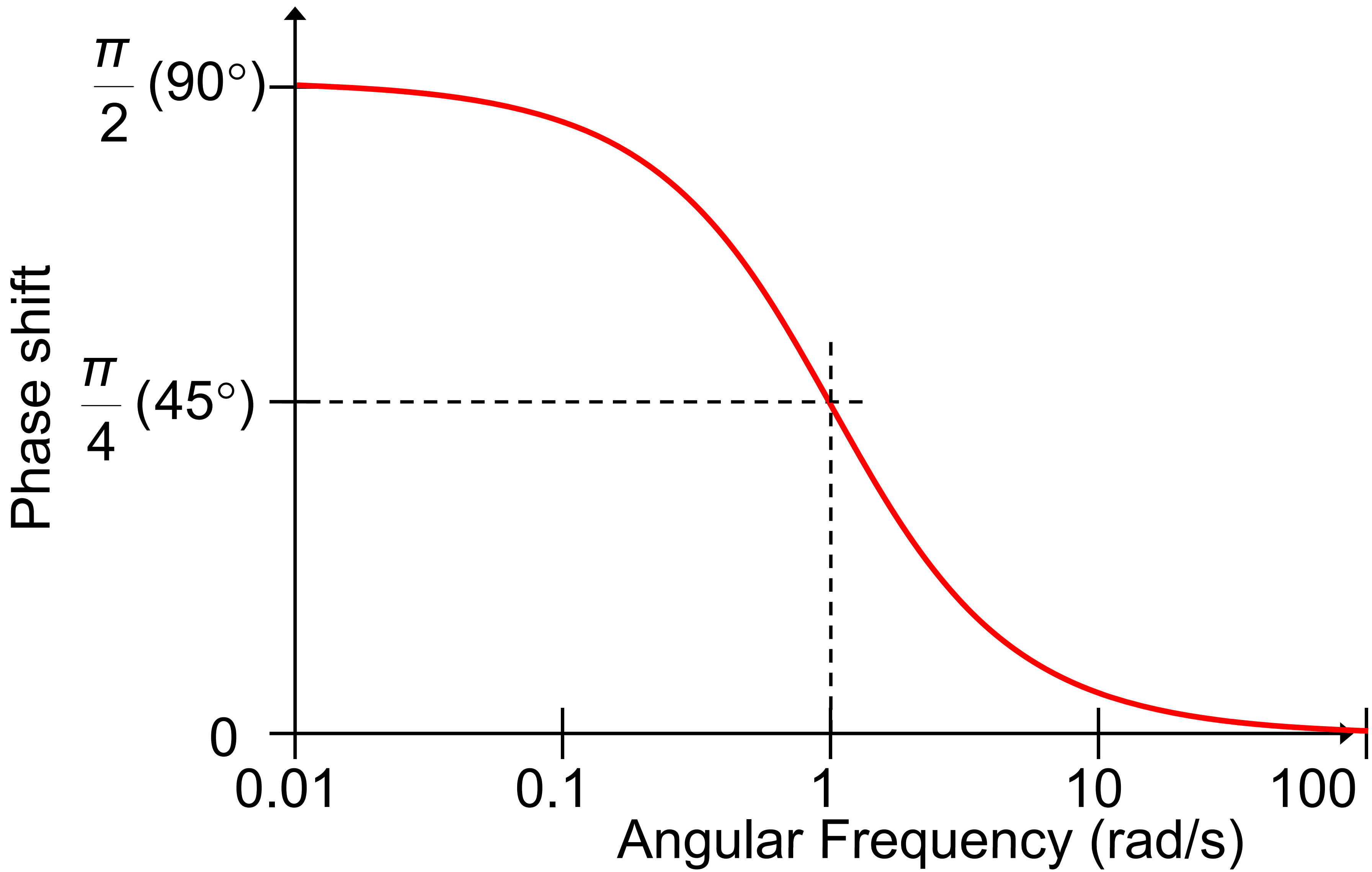
Phase shift : $$\varphi = \angle\left(\frac{V_R}{V_{in}}\right) = \tan^{-1}\infty - \tan^{-1}\omega CR$$

$$= \frac{\pi}{2} - \tan^{-1}\omega CR$$

# Frequency Response (RC = 1)



# Frequency Response (RC = 1)



# Frequency Response

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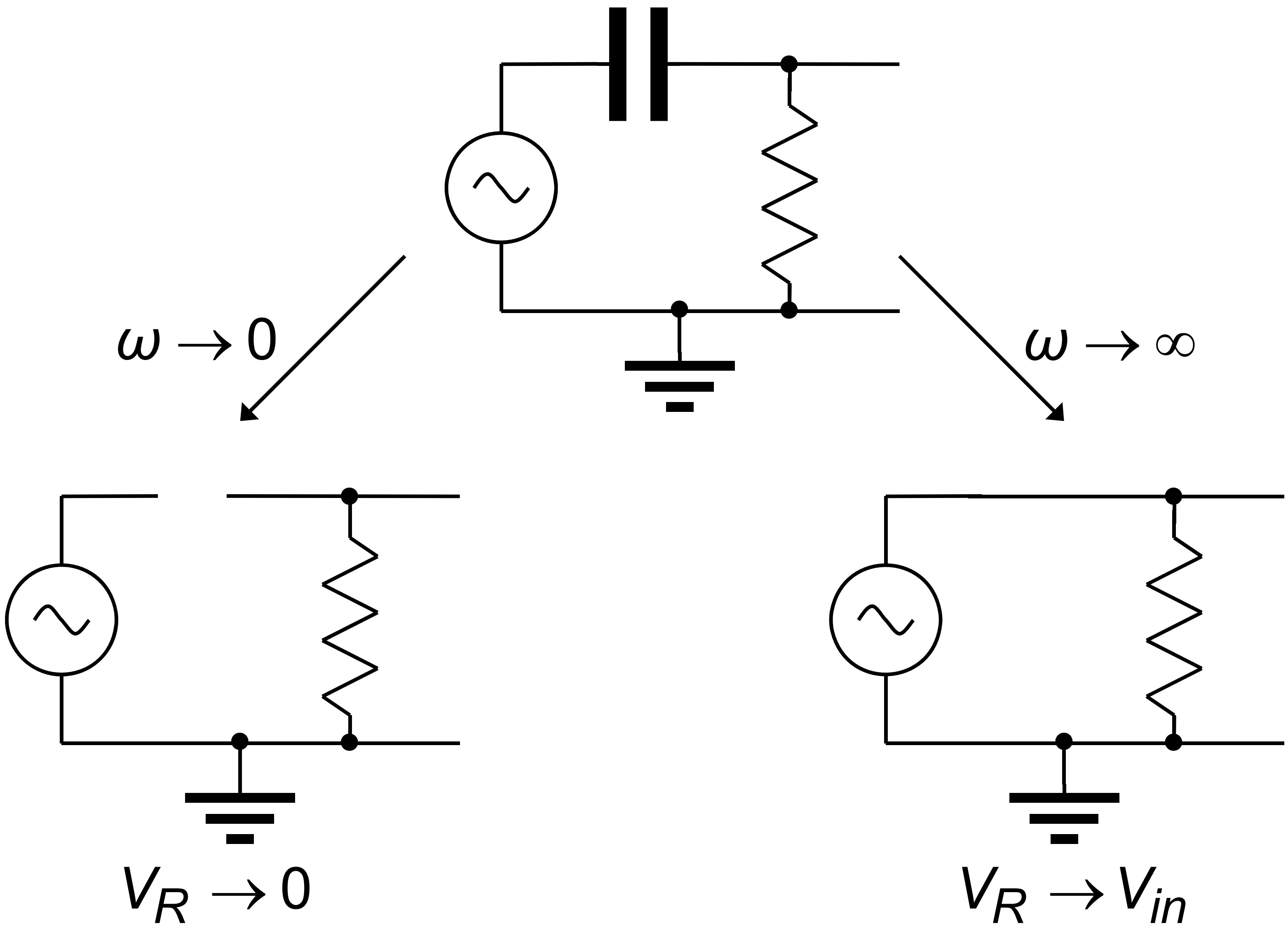
| | $g = \dfrac{\omega CR}{\sqrt{1+\omega^2 C^2 R^2}}$ | $\varphi = \dfrac{\pi}{2} - \tan^{-1}\omega CR$ |
|---|---|---|
| $\omega \to 0$ | $g \to 0$ | $\varphi \to \dfrac{\pi}{2}\ (90^\circ)$ |
| $\omega = \dfrac{1}{CR}$ | $g = \dfrac{1}{\sqrt{2}}$ | $\varphi = \dfrac{\pi}{4}\ (45^\circ)$ |
| $\omega \to \infty$ | $g \to 1$ | $\varphi \to 0\ (0^\circ)$ |

This is a high-pass response

# Frequency Response



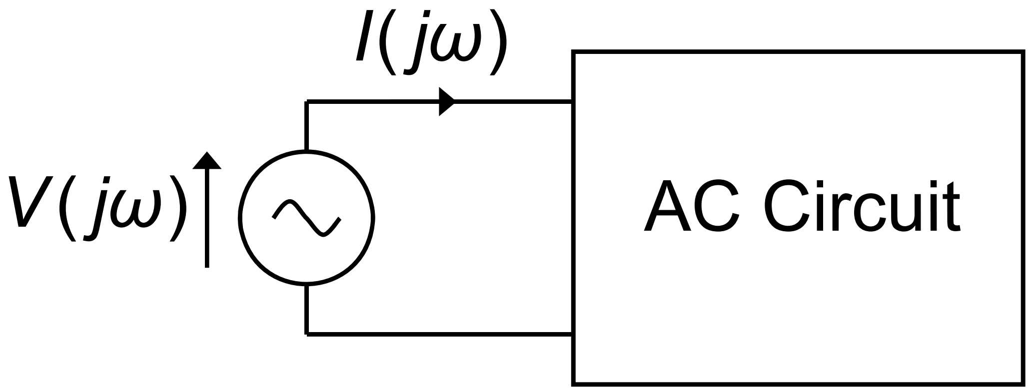
# Lecture 4

## Driving-Point Impedance

# Impedance



The impedance *Z* of a circuit or component is defined to be the ratio of the voltage and current phasors:




$$Z(j\omega) = \frac{V(j\omega)}{I(j\omega)}$$

Impedance *Z* is analogous to resistance in dc circuits and its units are ohms

When *Z* applies to a 2-terminal circuit (rather than simple component) it is known as the *driving-point impedance*

# Impedance

$Z$ can be written in rectangular form:

$$Z(j\omega) = R(j\omega) + jX(j\omega)$$

where $R$ is the *resistance* and $X$ is the *reactance*

Thus:

$$|Z| = \sqrt{R^2 + X^2}$$

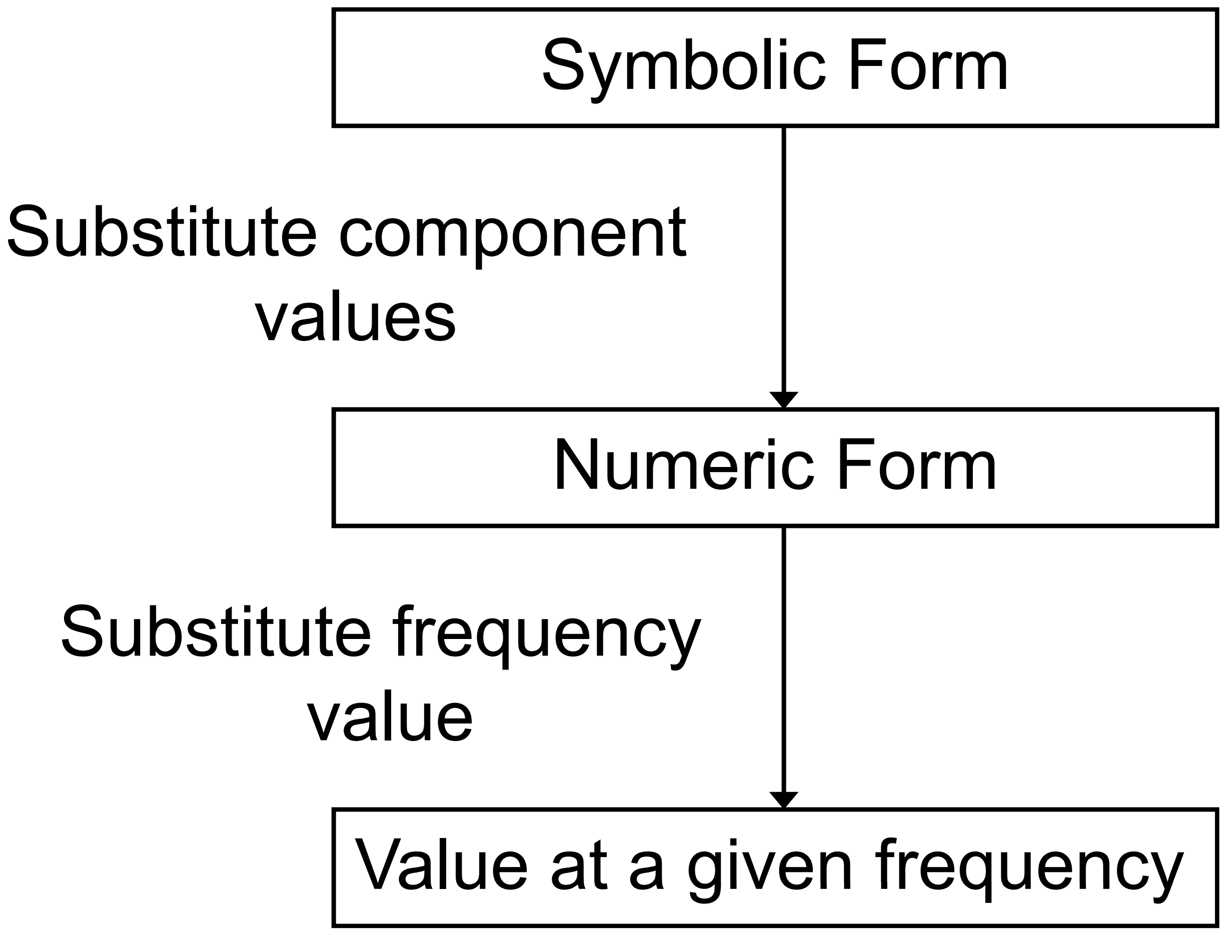
$$\angle Z = \tan^{-1}\frac{X}{R}$$

and:

$$R = |Z|\cos\angle Z$$

$$X = |Z|\sin\angle Z$$

# Symbolic and Numeric Forms




$$Z = \frac{R}{1 + j\omega CR}$$

$$Z = \frac{80}{1 + j\omega \times 8 \times 10^{-3}}$$

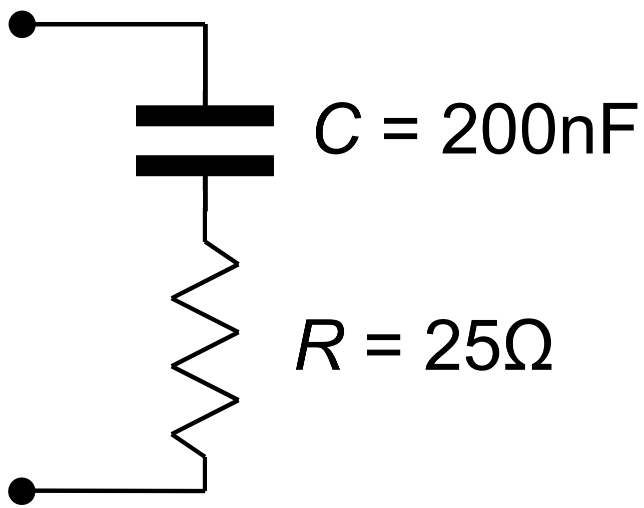
$$Z = 24 + j40$$

# Example 1



Determine the driving-point impedance of the circuit at a frequency of 40 kHz:

$$Z = Z_R + Z_C$$
$$= R + \frac{1}{j\omega C}$$
$$= 25 + \frac{1}{j2\pi \times 40 \times 10^3 \times 200 \times 10^{-9}}$$
$$= 25 + \frac{1}{j0.05027}$$
$$= 25 - j19.89\ \Omega$$

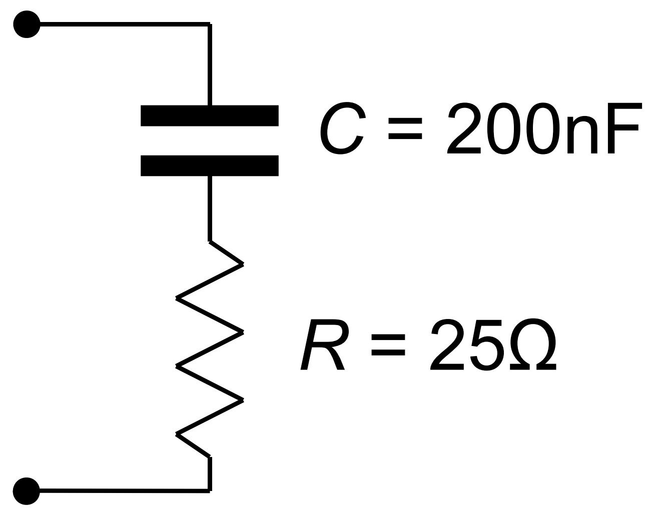


# Example 1

$$Z = 25 - j19.89\ \Omega$$

$$|Z| = \sqrt{25^2 + 19.89^2}$$
$$= 31.93\ \Omega$$

$$\angle Z = \tan^{-1}\frac{-19.89}{25}$$
$$= -0.6720\ (-38.5^\circ)$$



# Example 1

What will be the voltage across the circuit when a current of 5 A, 40 kHz flows through it?

$$
\begin{aligned}
V &= IZ \\
&= 5 \times (25 - j19.89) \\
&= 125 - j99.45 \text{ V}
\end{aligned}
$$

In polar form:

$$
\begin{aligned}
V &= IZ \\
&= (5 \times 31.93) \angle -0.6720 \ (-38.5^\circ) \\
&= 159.7\text{V} \ \angle -0.6720 \ (-38.5^\circ)
\end{aligned}
$$

# Example 2

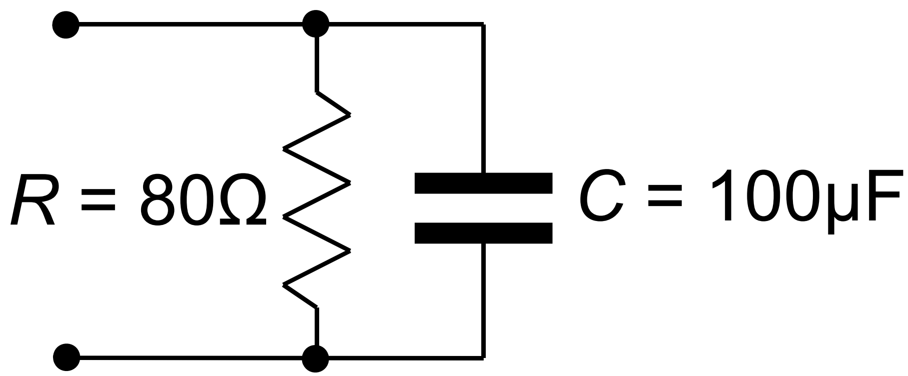
Determine the driving-point impedance of the circuit at a frequency of 20 Hz:

$$\frac{1}{Z} = \frac{1}{Z_R} + \frac{1}{Z_C}$$

$$= \frac{1}{R} + j\omega C$$

$$Z = \frac{1}{1/R + j\omega C}$$

$$= \frac{R}{1 + j\omega CR}$$



# Example 2

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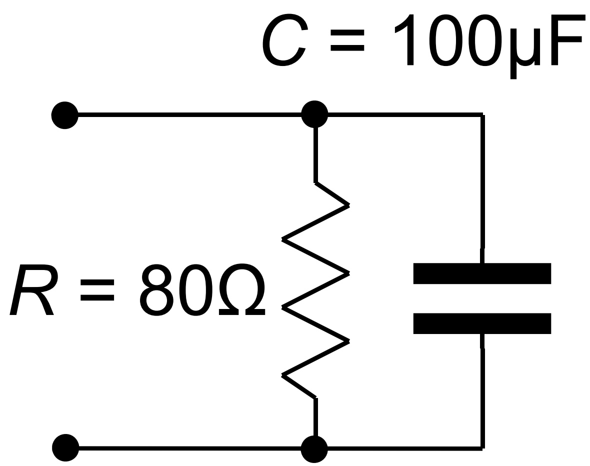
$$Z = \frac{R}{1 + j\omega CR}$$

$$= \frac{80}{1 + j2\pi \times 20 \times 100 \times 10^{-6} \times 80}$$

$$= \frac{80}{1 + j1.005}$$

$$\frac{80(1 - j1.005)}{1^2 + 1.005^2}$$

$$= 39.79 - j40.00\ \Omega$$



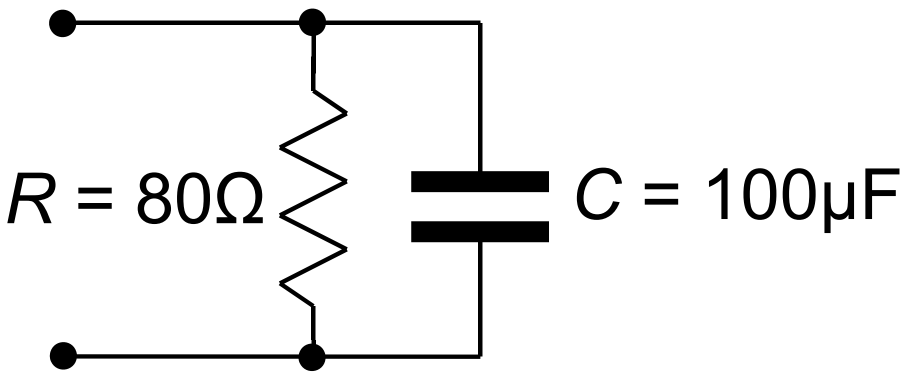
# Example 2



$$Z = 39.79 - j40.00\ \Omega$$

$$|Z| = \sqrt{39.79^2 + 40.00^2}$$
$$= 56.42\ \Omega$$

$$\angle Z = \tan^{-1}\left\{\frac{-40.00}{39.79}\right\}$$
$$= -0.7880\ (-45.2^\circ)$$



# Example 2

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What current will flow if an ac voltage of 24 V, 20 Hz is applied to the circuit?

$$I = \frac{V}{Z}$$

$$= \frac{24}{39.79 - j40.00}$$

$$= \frac{24(39.79 + j40.00)}{39.79^2 + 40.00^2}$$

$$= 0.3 + j0.3016 \text{ A}$$

$$= 0.4254\text{A} \angle 0.7880 \ (45.2^\circ)$$

$$I = \frac{V}{Z}$$

$$= \frac{24}{56.42\angle -0.7880}$$

$$= 0.4254 \text{ A} \angle 0.7880 \ (45.2^\circ)$$

$$= 0.3 + j0.3016 \text{ A}$$

# Phasor Diagrams

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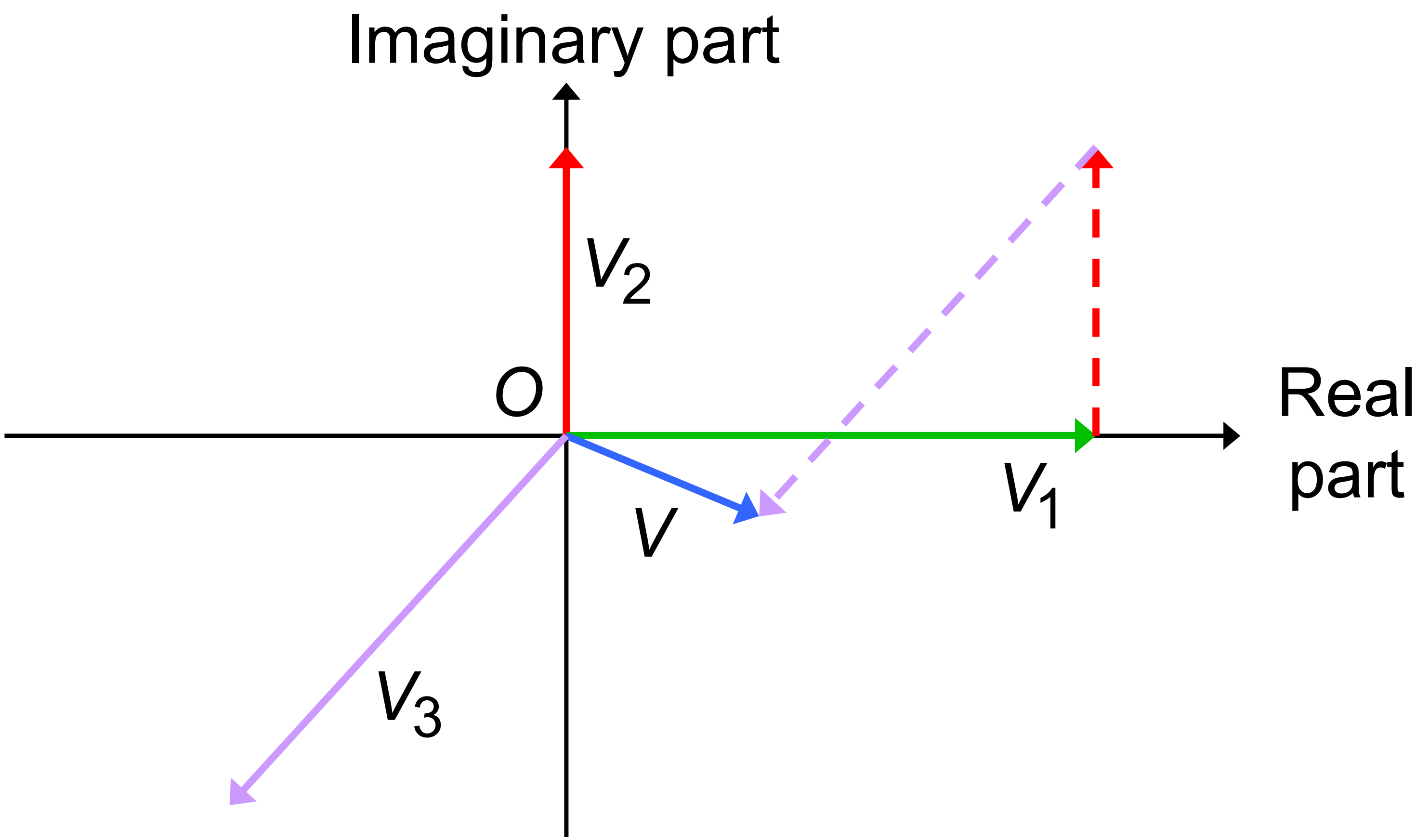
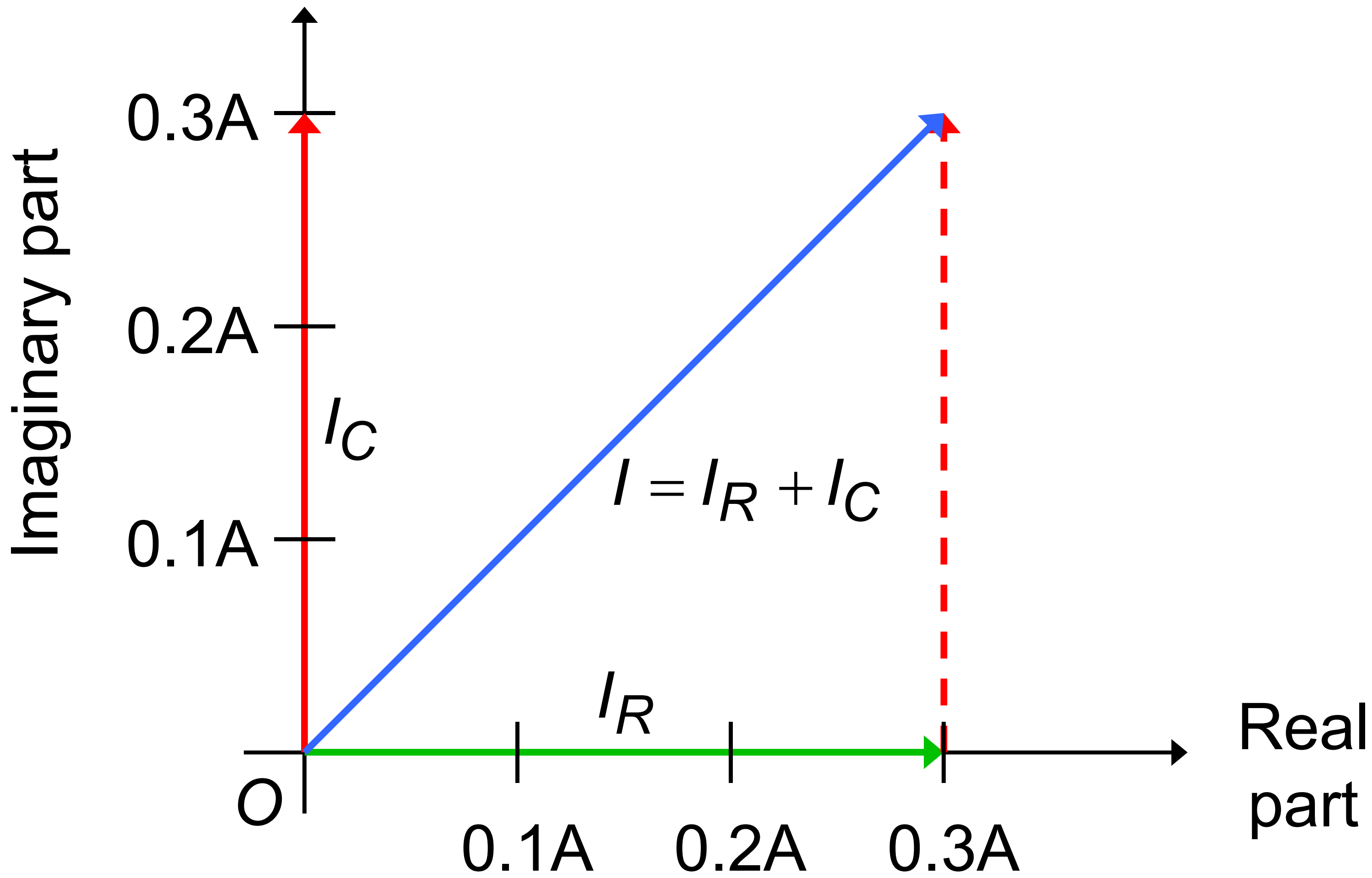
Where voltages or currents are summed the result can be represented by a phasor diagram: $V = V_1 + V_2 + V_3$



# Example 2

$$I_R = \frac{24}{80} = 0.3 \text{ A} \qquad I_C = j2\pi \times 20 \times 100 \times 10^{-6} \times 24 = j0.3016 \text{ A}$$



# Example 3

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Determine the driving-point impedance of the circuit at a frequency of 50 Hz:

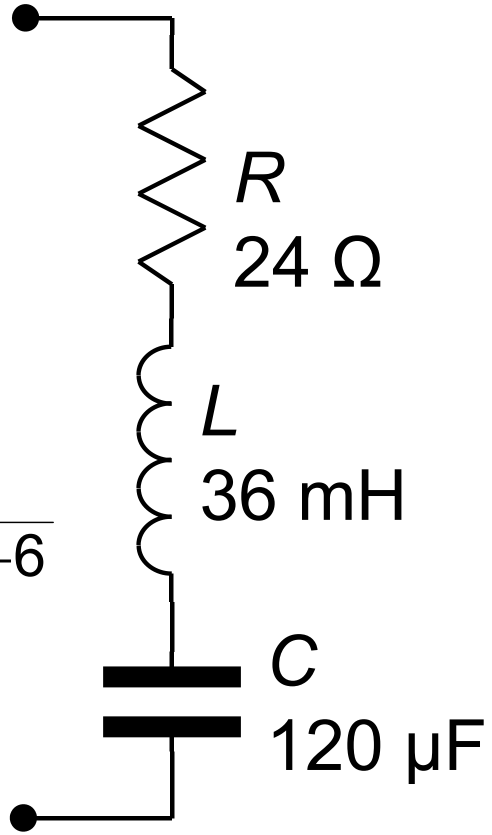
$$Z = Z_R + Z_L + Z_C$$

$$= R + j\omega L + \frac{1}{j\omega C}$$

$$= 24 + j2\pi \times 50 \times 36 \times 10^{-3} + \frac{1}{j2\pi \times 50 \times 120 \times 10^{-6}}$$

$$= 24 + j11.31 - j26.53\ \Omega$$

$$= 24 - j15.22\ \Omega$$



# Example 3

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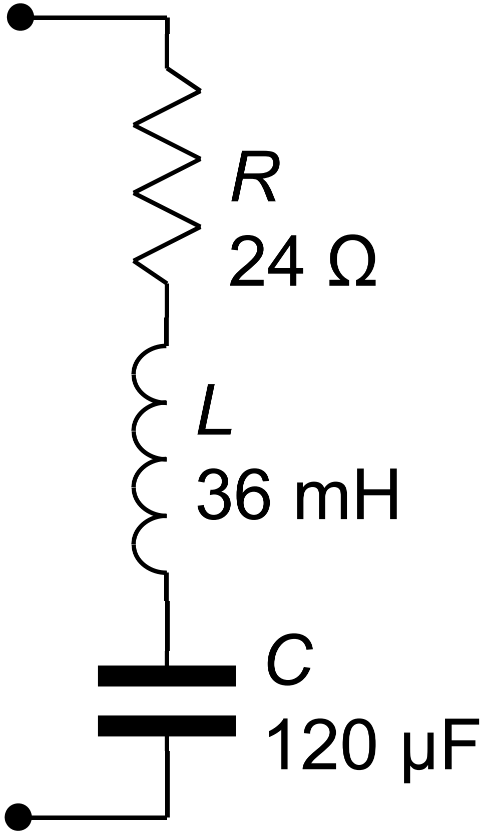
$$Z = 24 - j15.22\ \Omega$$

$$|Z| = \sqrt{24^2 - 15.22^2}$$
$$= 28.42\ \Omega$$

$$\angle Z = \tan^{-1}\left\{\frac{-15.22}{24}\right\}$$
$$= -0.5652\ (-32.4°)$$

$$Z = 28.42\Omega \angle -0.5652\ (-32.4°)$$



# Example 3

What voltage will be generated across the circuit if an ac current of 10 A, 50 Hz flows though it?

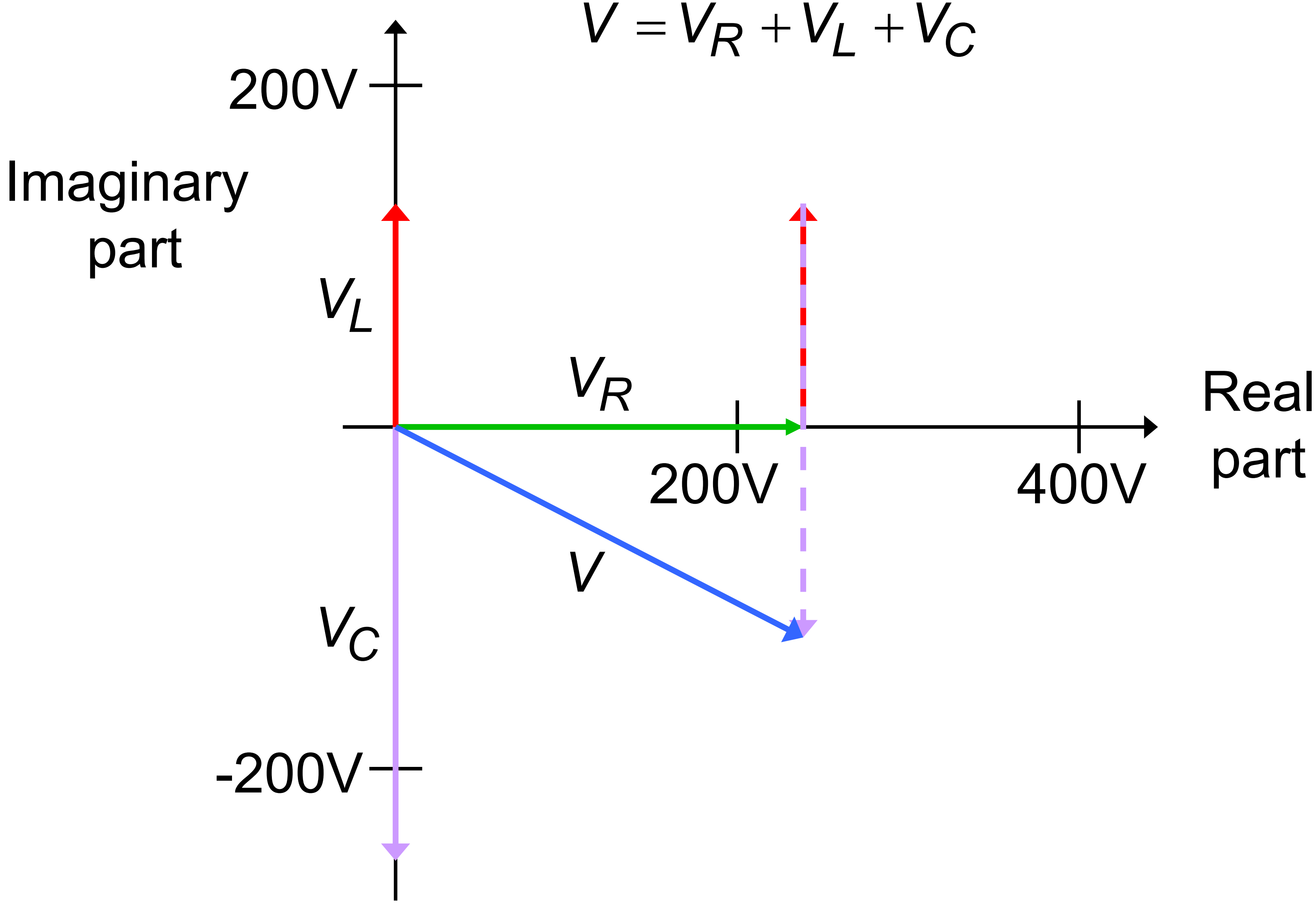
$$
\begin{aligned}
V &= ZI \\
&= 10\,\mathrm{A} \times (24 - j15.22)\,\Omega \\
&= 240 - j152.2\ \mathrm{V}
\end{aligned}
$$

In polar form:

$$
\begin{aligned}
V &= ZI \\
&= (10 \times 28.42) \angle -0.5652\ (-32.4^\circ) \\
&= 284.2\ \mathrm{V}\ \angle -0.5652\ (-32.4^\circ)
\end{aligned}
$$

# Example 3

$$V = V_R + V_L + V_C$$



# Example 4

Determine the driving-point impedance of the circuit at a frequency of 400 Hz:

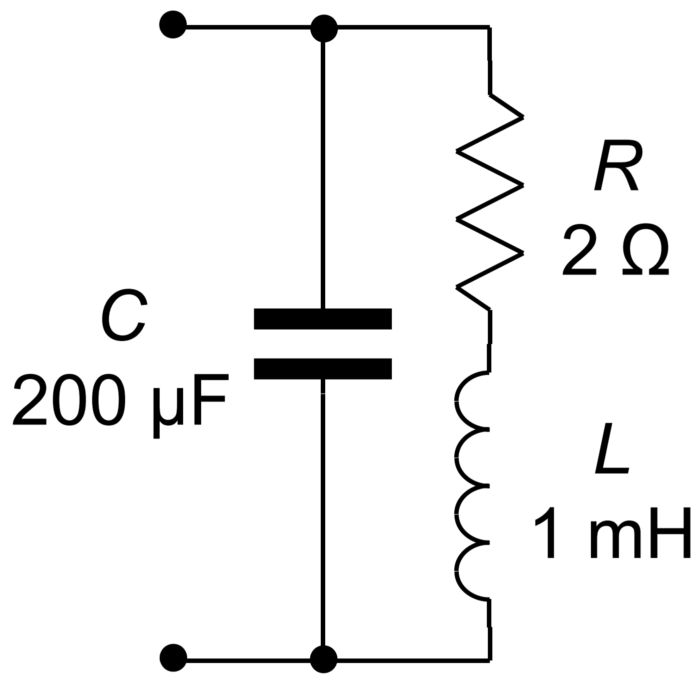
$$\frac{1}{Z} = \frac{1}{Z_R + Z_L} + \frac{1}{Z_C}$$

$$= \frac{1}{R + j\omega L} + j\omega C$$

$$Z = \frac{1}{1/(R + j\omega L) + j\omega C}$$

$$= \frac{R + j\omega L}{1 + j\omega C(R + j\omega L)}$$

$$= \frac{R + j\omega L}{1 + j\omega CR - \omega^2 LC}$$



# Example 4

$$Z = \frac{R + j\omega L}{1 + j\omega CR - \omega^2 LC}$$

$$\frac{2 + j2\pi \times 400 \times 10^{-3}}{1 + j2\pi \times 400 \times 200 \times 10^{-6} \times 2 - (2\pi \times 400)^2 \times 10^{-3} \times 200 \times 10^{-6}}$$

$$= \frac{2 + j2.513}{1 + j1.005 - 1.263}$$

$$= \frac{2 + j2.513}{-0.2633 + j1.005}$$

$$= \frac{(2 + j2.513) \times (-0.2633 - j1.005)}{0.2633^2 + j1.005^2}$$

$$= 1.852 - j2.474$$

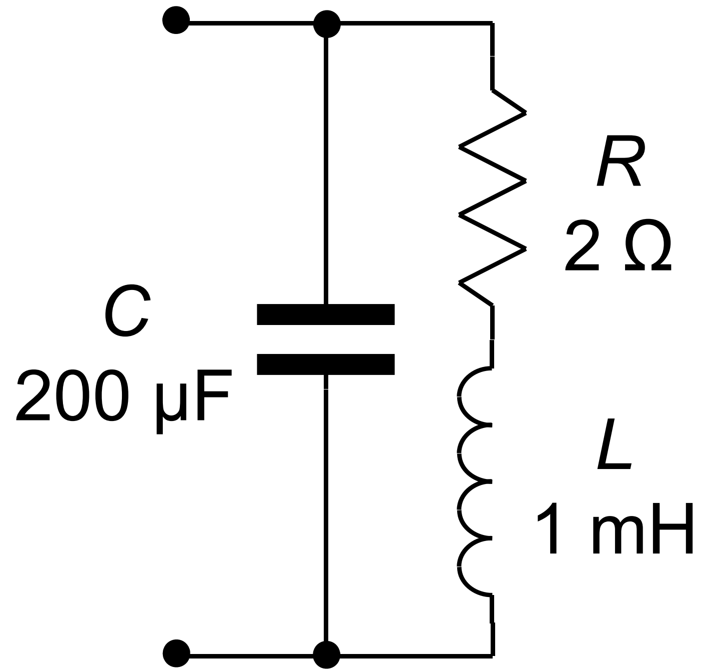
# Example 4

$$Z = 1.852 - j2.474$$

$$|Z| = \sqrt{1.852^2 + 2.474^2}$$
$$= 3.091$$

$$\angle Z = \tan^{-1}\frac{2.474}{1.852}$$
$$= -0.9282\ (-53.2°)$$

$$Z = 3.091\,\Omega\ \angle -0.9282\ (-53.2°)$$



# Example 4



What current will flow if an ac voltage of 120 V, 400 Hz is applied to the circuit?

$$I = \frac{V}{Z}$$

$$= \frac{120}{1.852 - j2.474}$$

$$= \frac{120 \times (1.852 + j2.474)}{1.852^2 + 2.474^2}$$

$$= \frac{222.4 + j297.0}{9.556}$$

$$= 23.26 + j31.08$$

$$= 38.82\text{A} \angle 0.9282 \; (53.2^\circ)$$

$$I = \frac{V}{Z}$$

$$= \frac{120}{3.091 \angle -0.9282}$$

$$= 38.82 \text{ A } \angle 0.9282$$

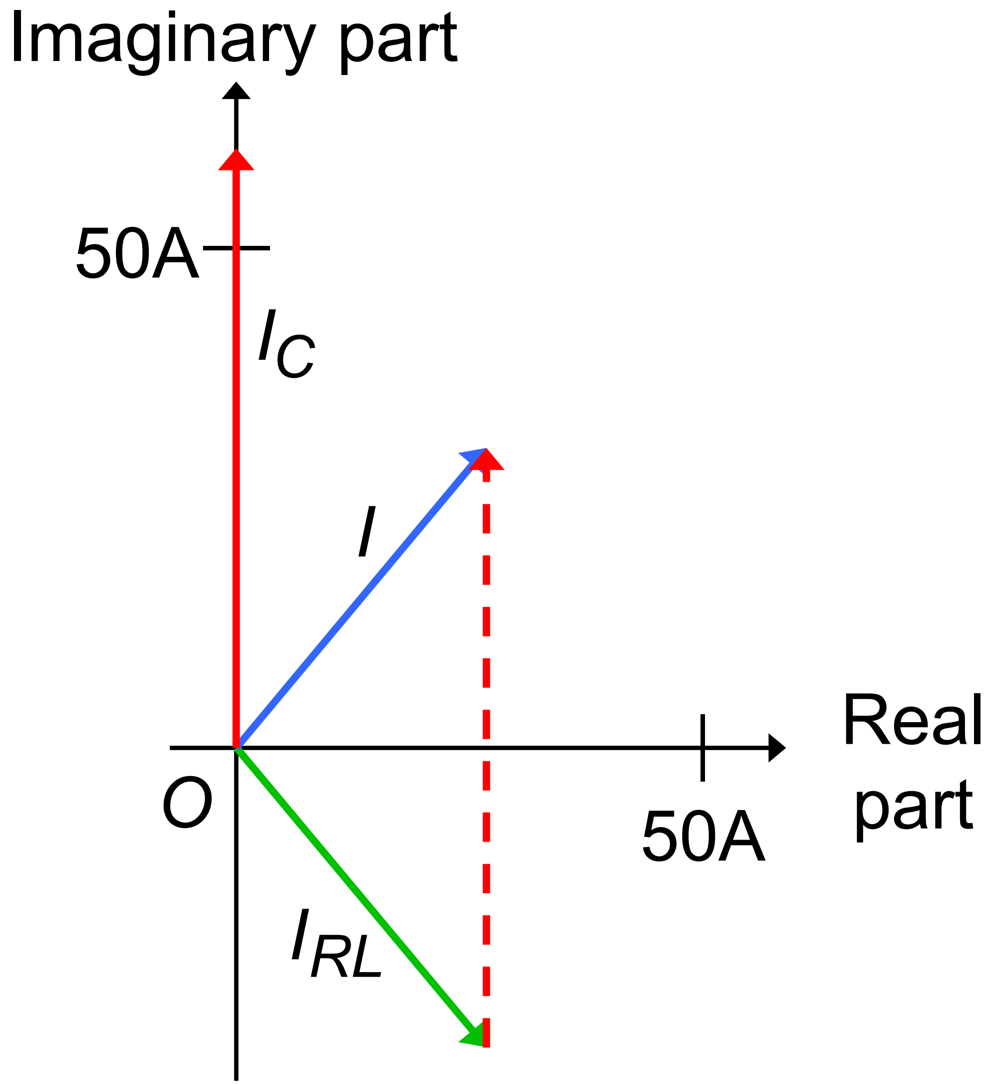
$$= 23.26 + j31.08$$

# Example 4

$$I = I_{RL} + I_C$$

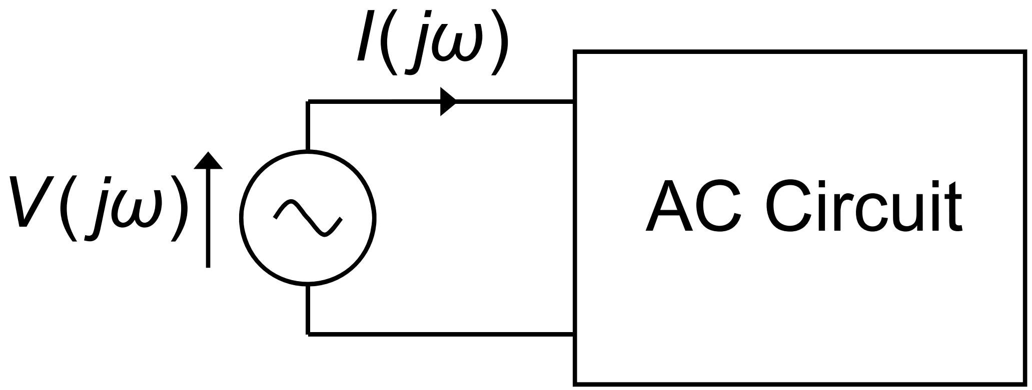
$$I_{RL} = \frac{120}{Z_{RL}}$$
$$= 23.27 - j29.23$$

$$I_C = \frac{120}{Z_C}$$
$$= j60.32$$



# Admittance

The admittance *Y* of a circuit or component is defined to be the ratio of the current and voltage phasors:




$$Y(j\omega) = \frac{I(j\omega)}{V(j\omega)} = \frac{1}{Z(j\omega)}$$

Admittance *Y* is analogous to conductance in dc circuits and its unit is Siemens

$$Y(j\omega) = G(j\omega) + jB(j\omega)$$

where *G* is the *conductance* and *B* is the *susceptance*

# Admittance

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| | $Y = \frac{I}{V}$ | $f \to 0$ | $f \to \infty$ |
|---|---|---|---|
| Resistance $R$ | $\frac{1}{R}$ | $\frac{1}{R}$ | $\frac{1}{R}$ |
| Capacitance $C$ | $j\omega C$ | $Y \to 0$ | $Y \to \infty$ |
| Inductance $L$ | $\frac{1}{j\omega L}$ | $Y \to \infty$ | $Y \to 0$ |

# Admittance

All the normal circuit theory rules apply to circuits containing admittances

For example admittances in series:

$$\frac{1}{Y} = \frac{1}{Y_1} + \frac{1}{Y_2} + \frac{1}{Y_3} + \frac{1}{Y_4}$$

and admittances in parallel:

$$Y = Y_1 + Y_2 + Y_3 + Y_4$$

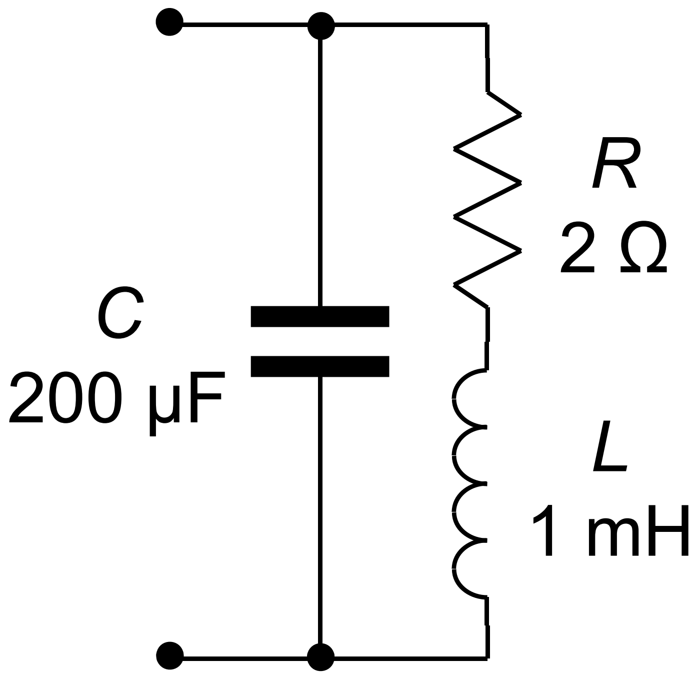
Other relevant circuit theory rules are: Kirchhoff's laws, Thévenin and Norton's theorems, Superposition

# Example 5

Determine the driving-point admittance of the circuit at a frequency of 400 Hz:

$$Y = Y_C + \frac{1}{1/Y_R + 1/Y_L}$$

$$= j\omega C + \frac{1}{R + j\omega L}$$



# Example 5

Determine the driving-point admittance of the circuit at a frequency of 400 Hz:

$$Y = j2\pi \times 400 \times 200 \times 10^{-6} + \frac{j2\pi \times 400 \times 10^{-3} \times 2}{2 + j2\pi \times 400 \times 10^{-3}}$$

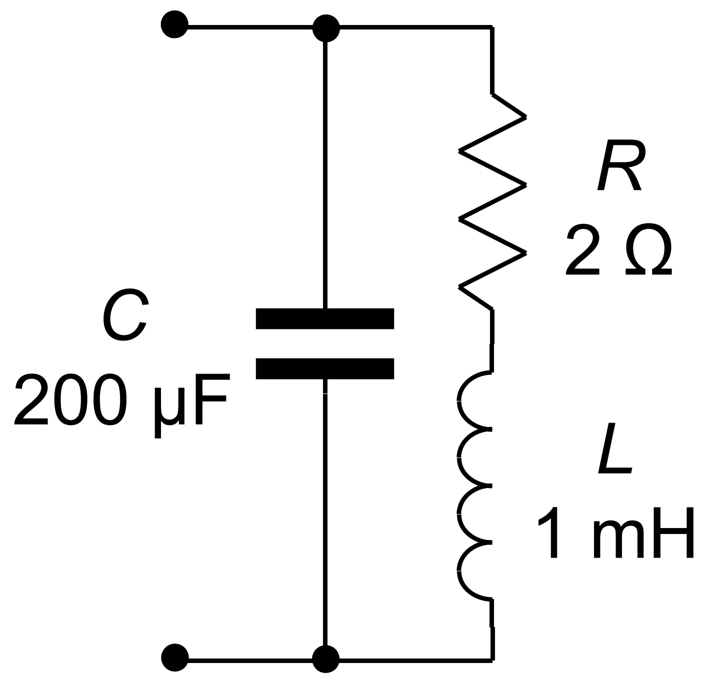
$$= j0.5027 + \frac{1}{2 + j2.513}$$

$$= j0.5027 + \frac{2 - j2.513}{2^2 + 2.513^2}$$

$$= j0.5027 + \frac{2 - j2.513}{10.32}$$

$$= j0.5027 + 0.1939 - j0.2436$$

$$= 0.1939 + j0.2590 \text{ S}$$



# Lecture 5

## Resonant Circuits

# Resonant Circuits

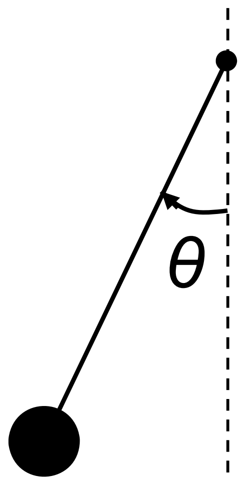
Passive resonant circuits must contain a resistor, capacitor and an inductor

The behaviour of resonant circuits changes rapidly around a particular frequency (the resonance frequency)

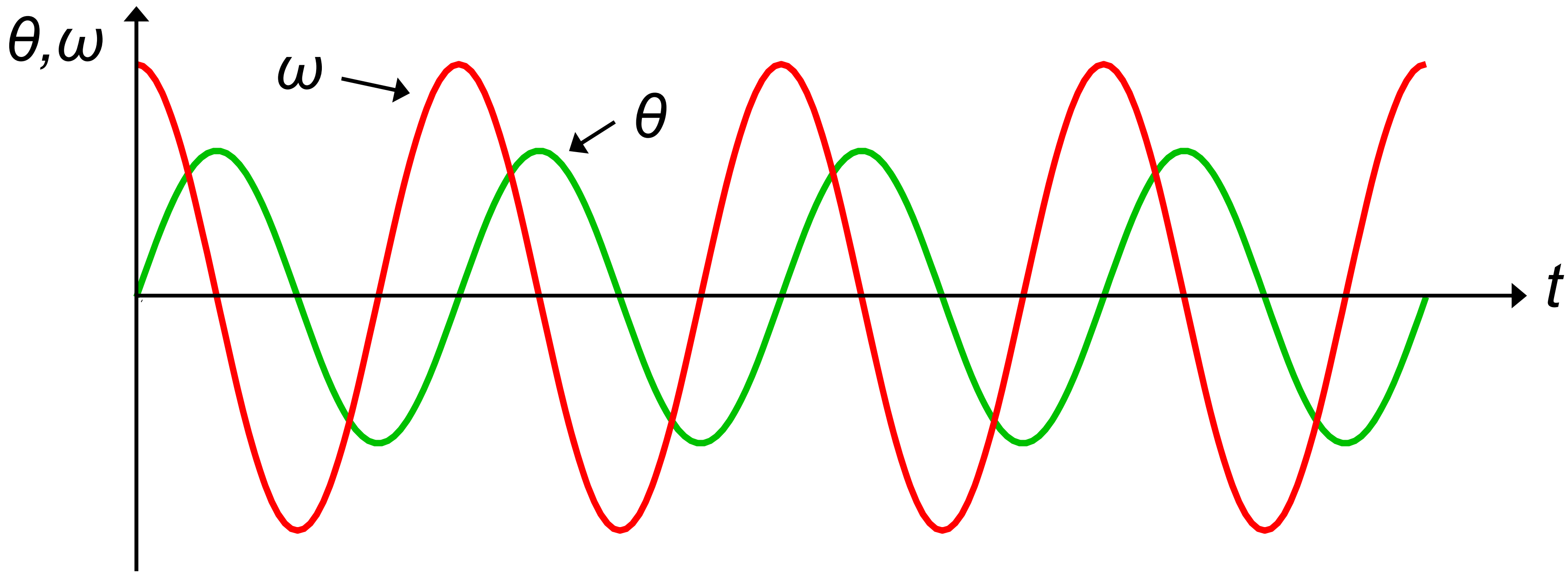
Resonant circuits can be characterised by two parameters: the resonance frequency and the Q-factor

There are two basic resonant circuit configurations: series and parallel

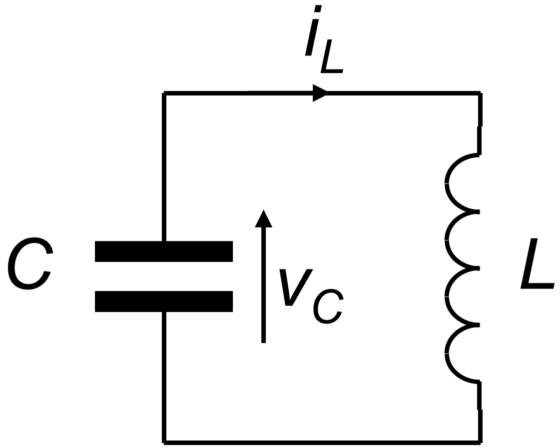
# Resonant Circuits



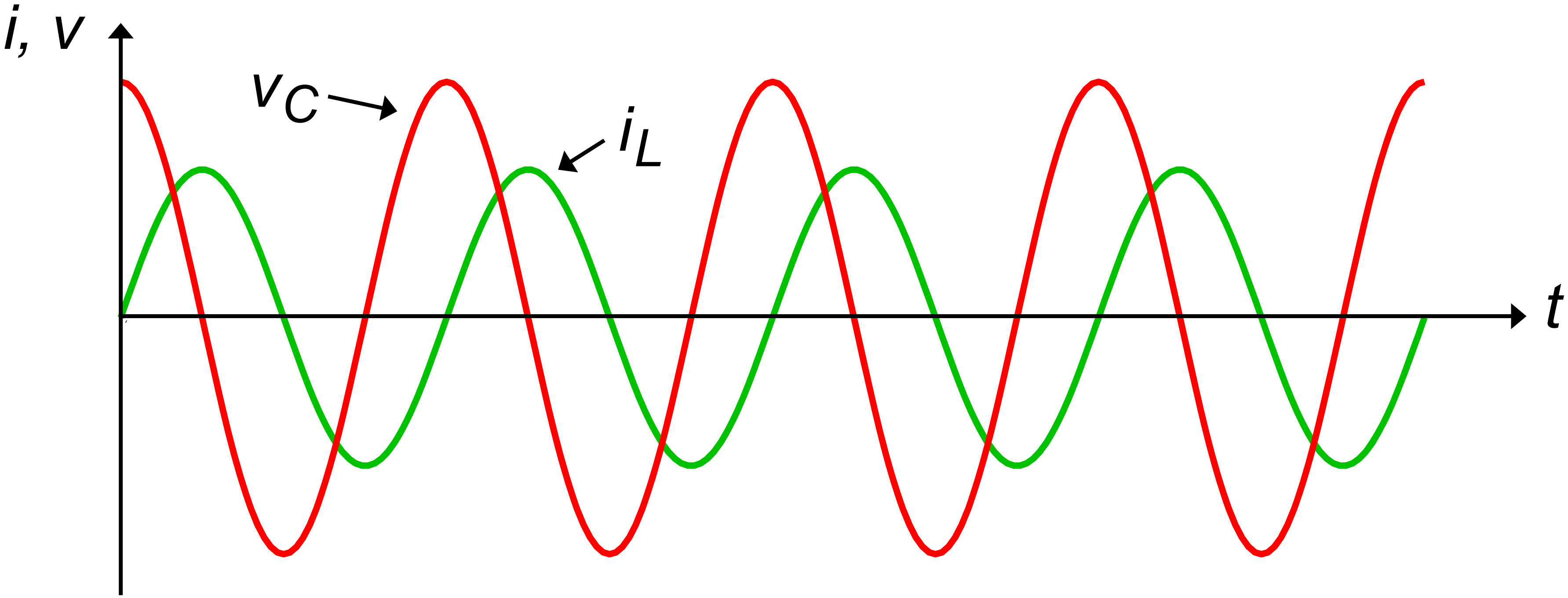

$$\frac{d\theta}{dt} = \omega \qquad \frac{d\omega}{dt} = -\frac{g\theta}{L}$$



# Resonant Circuits




$$\frac{di_L}{dt} = \frac{v_C}{L} \qquad \frac{dv_C}{dt} = -\frac{i_L}{C}$$



# Parallel Resonant Circuit

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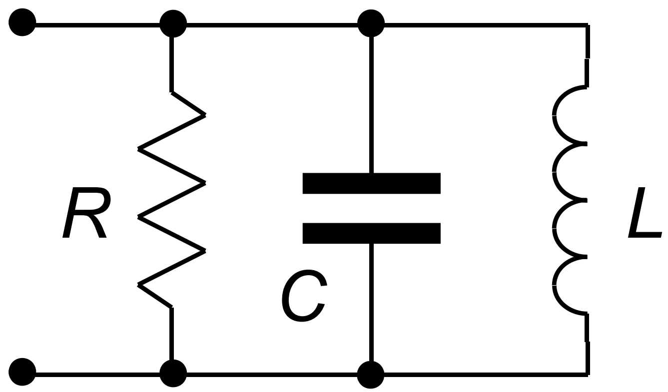



$$\frac{1}{Z} = \frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L}$$

$$= \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

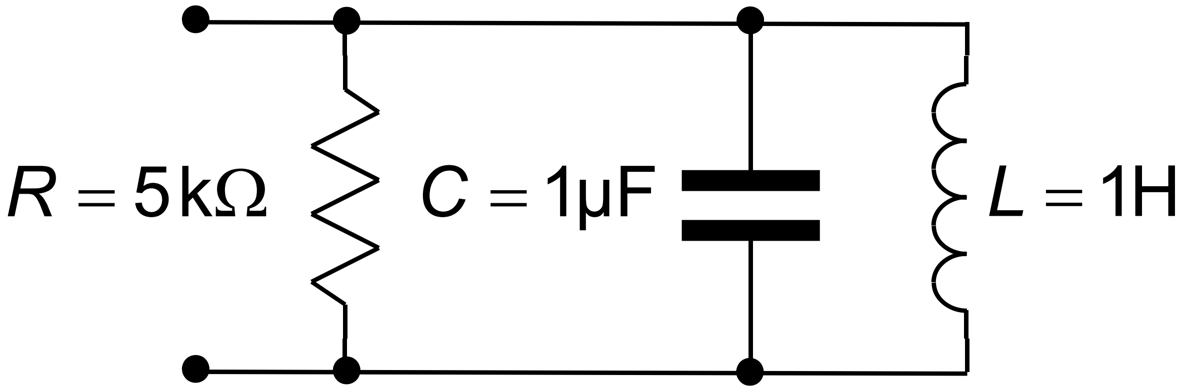
$$= \frac{j\omega L - \omega^2 LCR + R}{j\omega LR}$$

$$Z = \frac{j\omega LR}{j\omega L - \omega^2 LCR + R}$$

$$= \frac{j\omega L}{j\omega L / R - \omega^2 LC + 1}$$

$$= \frac{j\omega L}{1 + j\omega L / R - \omega^2 LC}$$

# Parallel Resonant Circuit




Impedance is a maximum (resonant frequency) when:

$$Z = \frac{j\omega L}{1 + j\omega L/R - \omega^2 LC}$$

$$= \frac{j\omega}{1 + j\omega \times 2 \times 10^{-4} - \omega^2 \times 10^{-6}}$$

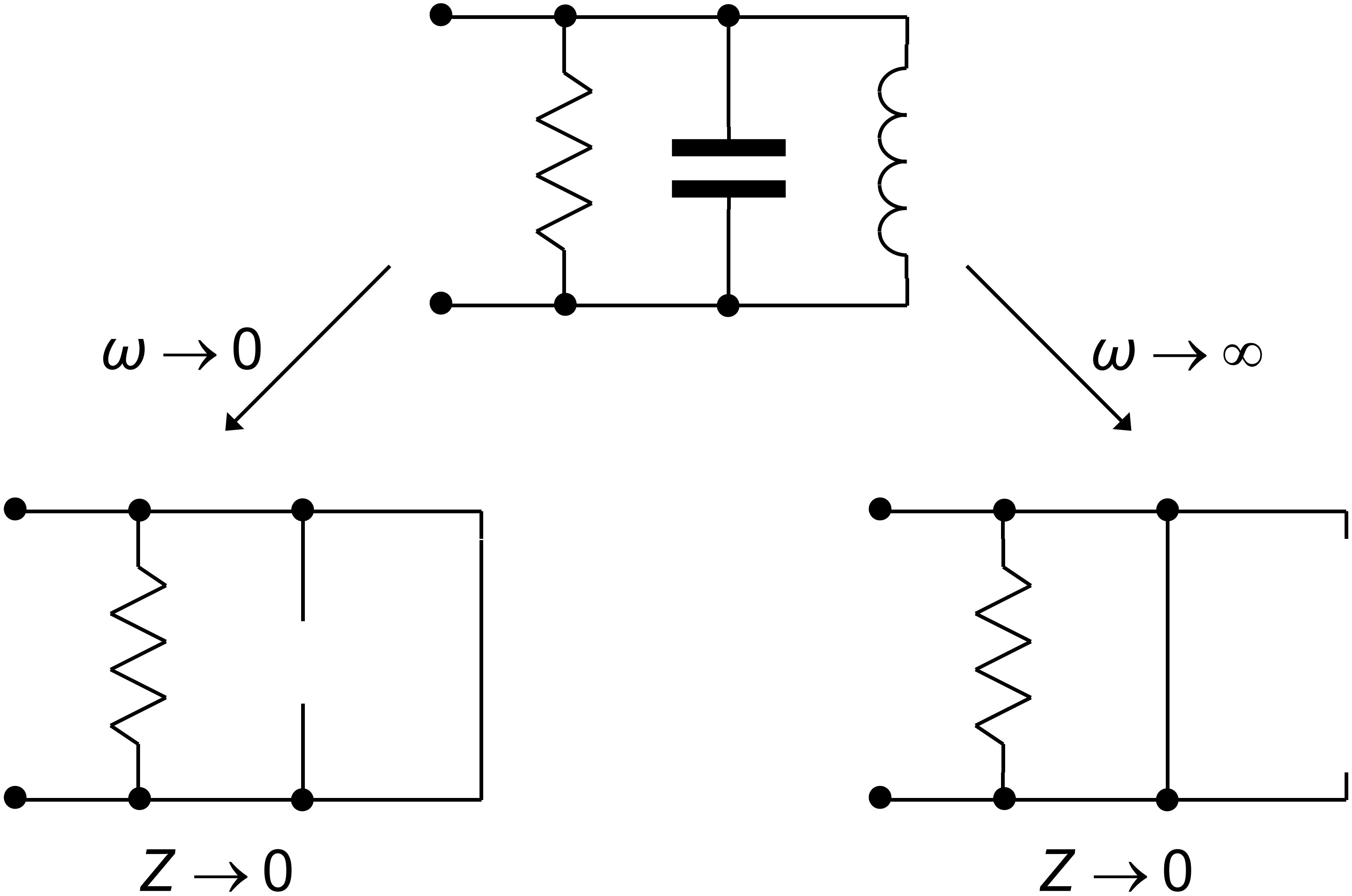
$$\omega = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{10^{-6}}}$$

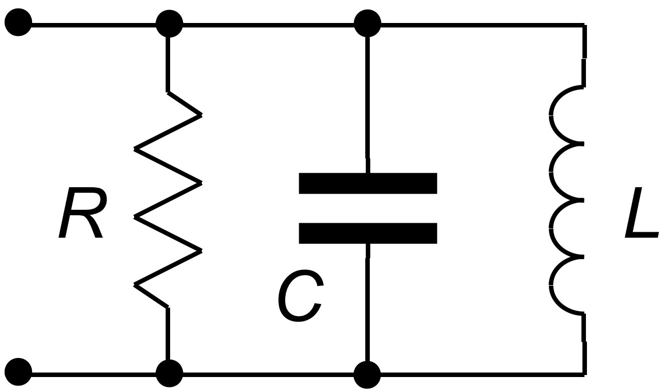
$$= 10^3$$

# Parallel Resonant Circuit





# Parallel Resonant Circuit




$$Z = \frac{j\omega L}{1 + j\omega L/R - \omega^2 LC}$$

$$\omega \to 0 \qquad Z \to \frac{j\omega L}{1} = j0$$

Resonant frequency: $$\omega = \frac{1}{\sqrt{LC}} \qquad Z = \frac{j\omega L}{j\omega L/R} = R$$

$$\omega \to \infty \qquad Z \to \frac{j\omega L}{-\omega^2 LC} = \frac{-j}{\omega C} = -j0$$

# Parallel Resonant Circuit

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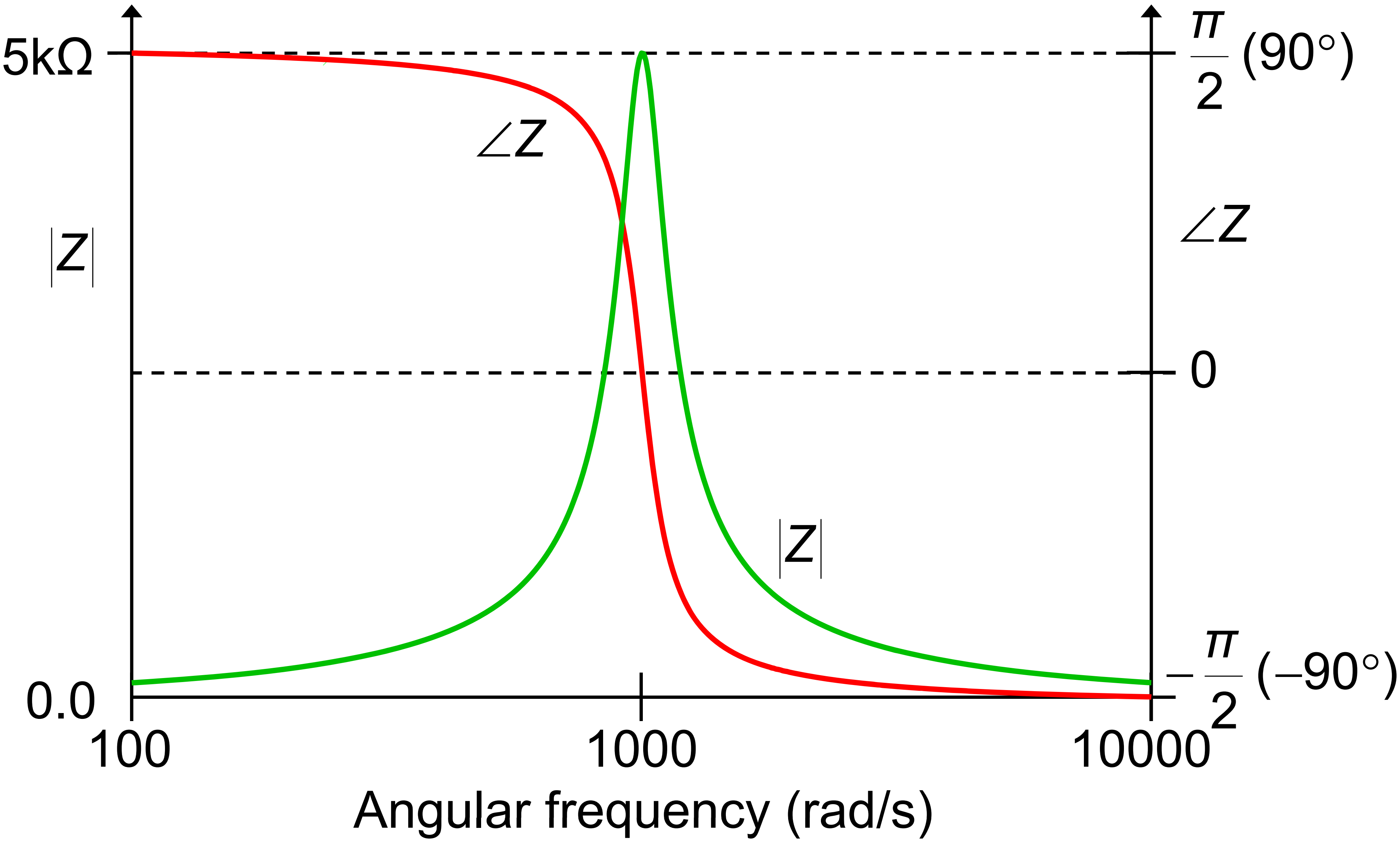


# Quality Factor



The standard form for the denominator of a second-order system is:

$$1 + j\omega / \omega_0 Q - \omega^2 / \omega_0^2$$

Compare this with the impedance $Z$:

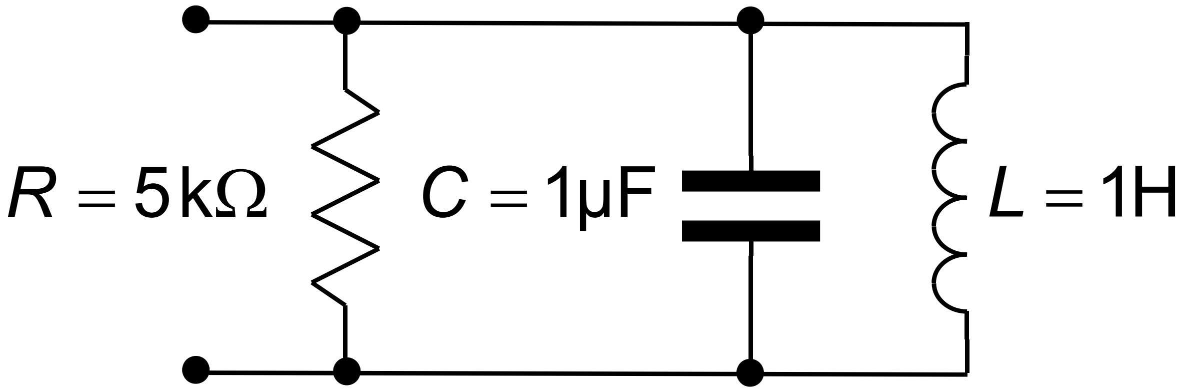
$$Z = \frac{j\omega L}{1 + j\omega L / R - \omega^2 LC}$$

So that:

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \frac{R}{\omega_0 L}$$

where $Q$ is the quality-factor and $\omega_0$ is the resonant frequency

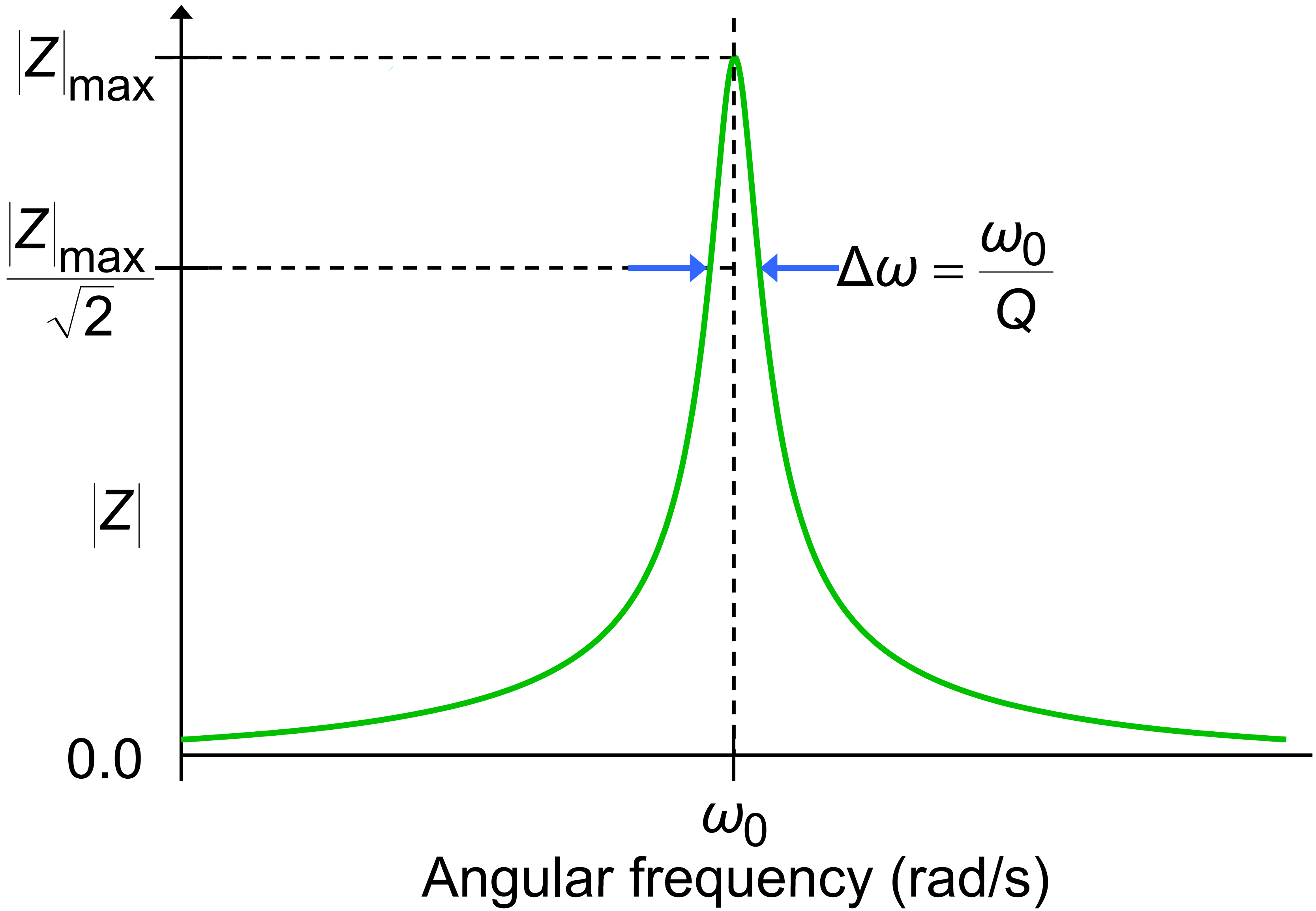
# Quality Factor



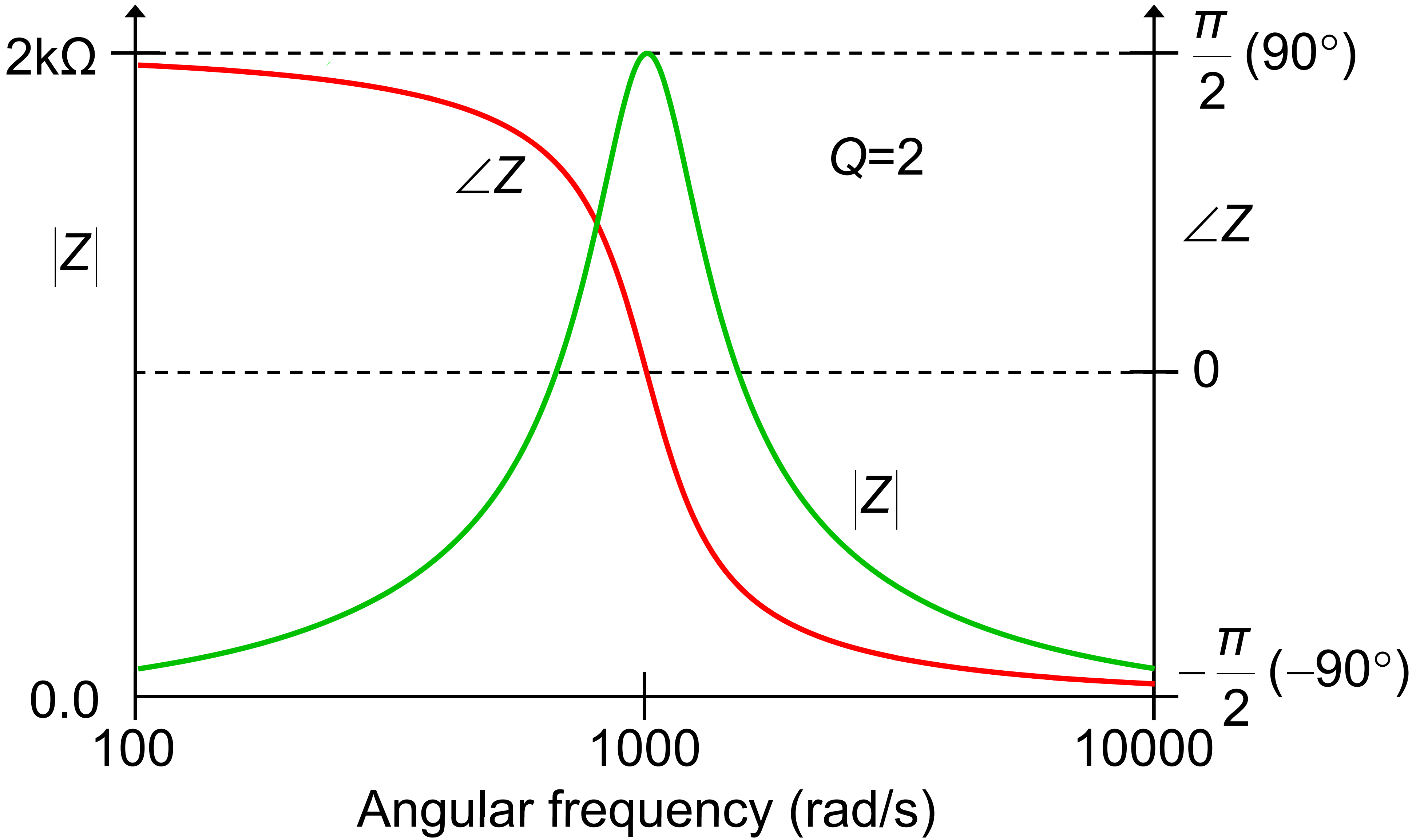

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-6}}} = 10^3$$

$$Q = \frac{R}{\omega_0 L} = \frac{5000}{1 \times 10^3} = 5$$

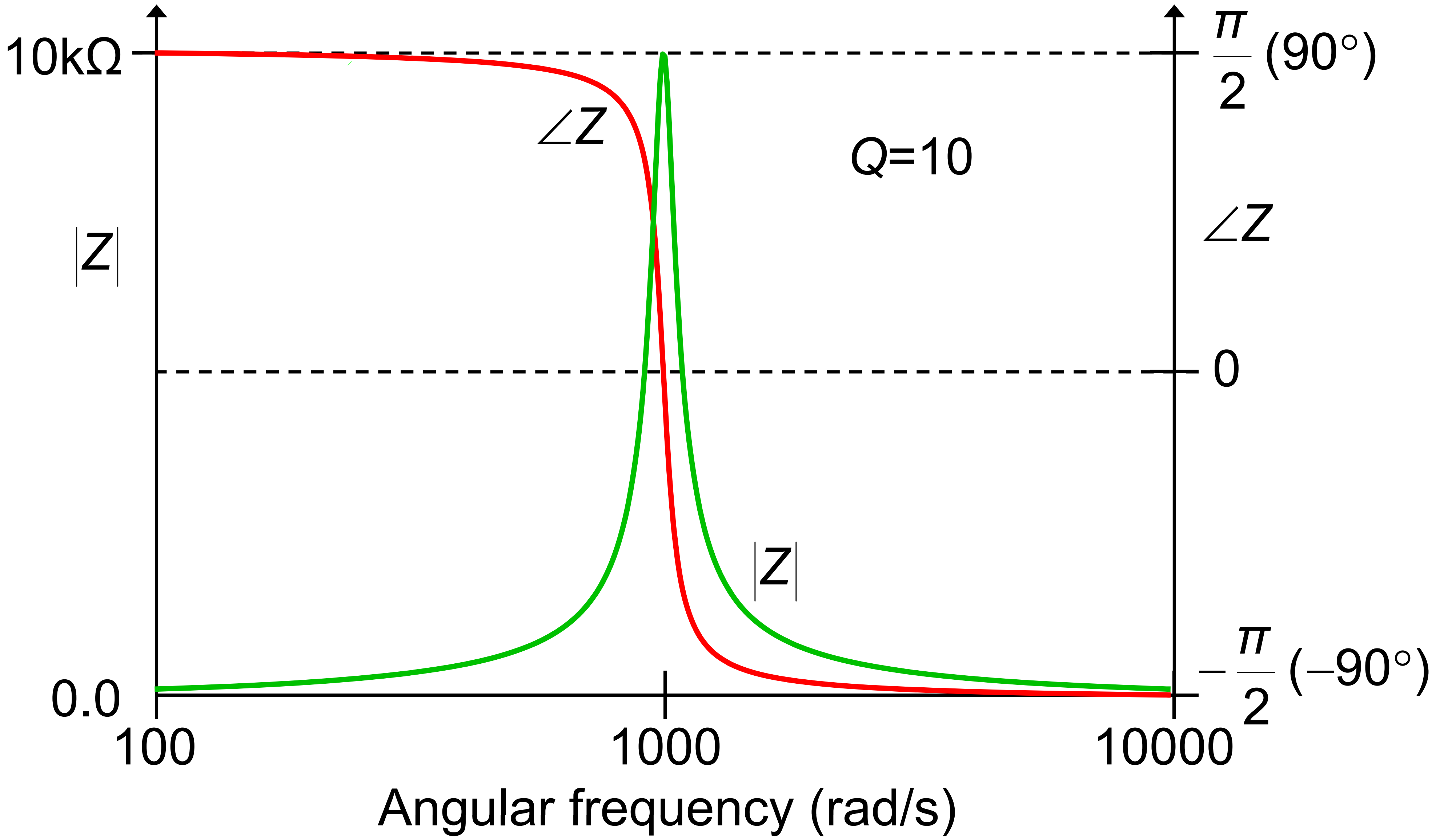
# Quality Factor



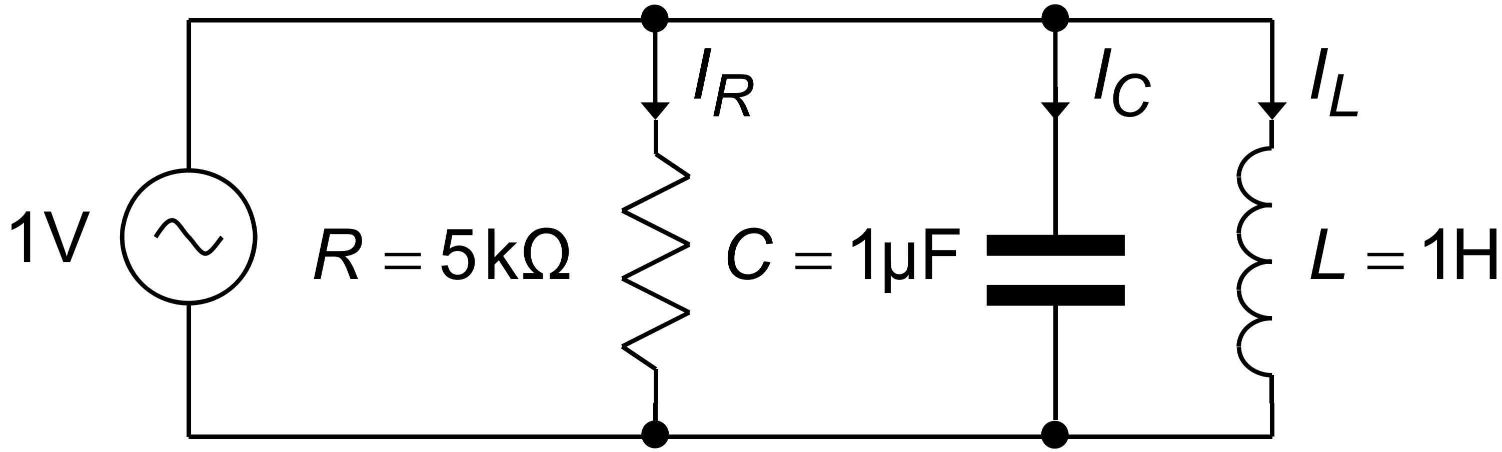
# Quality Factor



# Quality Factor



# Parallel Resonant Circuit




Resonance occurs in parallel resonant circuits because the currents in the capacitor and inductor cancel out

$$I_R = \frac{1}{R} = \frac{1}{5000} = 2 \times 10^{-4}$$

$$I_C = \frac{1}{1/j\omega C} = j\omega C = j\omega \times 10^{-6}$$

$$I_L = \frac{1}{j\omega L} = \frac{-j}{\omega L} = \frac{-j}{\omega}$$

# Parallel Resonant Circuit

At resonance:

$$\omega = 10^3 :$$

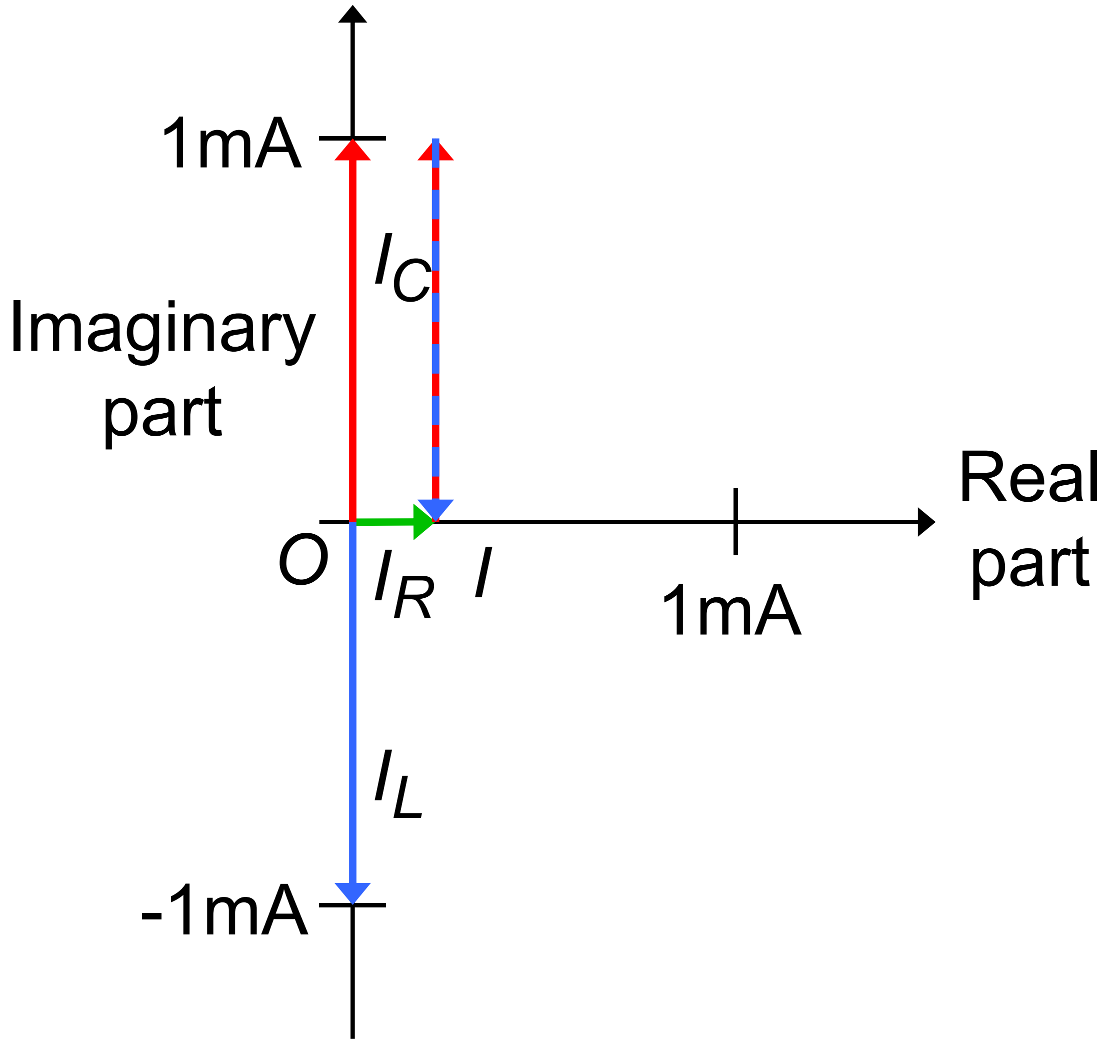
$$I_R = 2 \times 10^{-4}\,\text{A}$$

$$I_C = j\omega \times 10^{-6}$$

$$= j10^{-3}\,\text{A}$$

$$I_L = \frac{-j}{\omega}$$

$$= -j10^{-3}\,\text{A}$$



# Parallel Resonant Circuit

Below resonance:

$$\omega = 0.5 \times 10^3 :$$

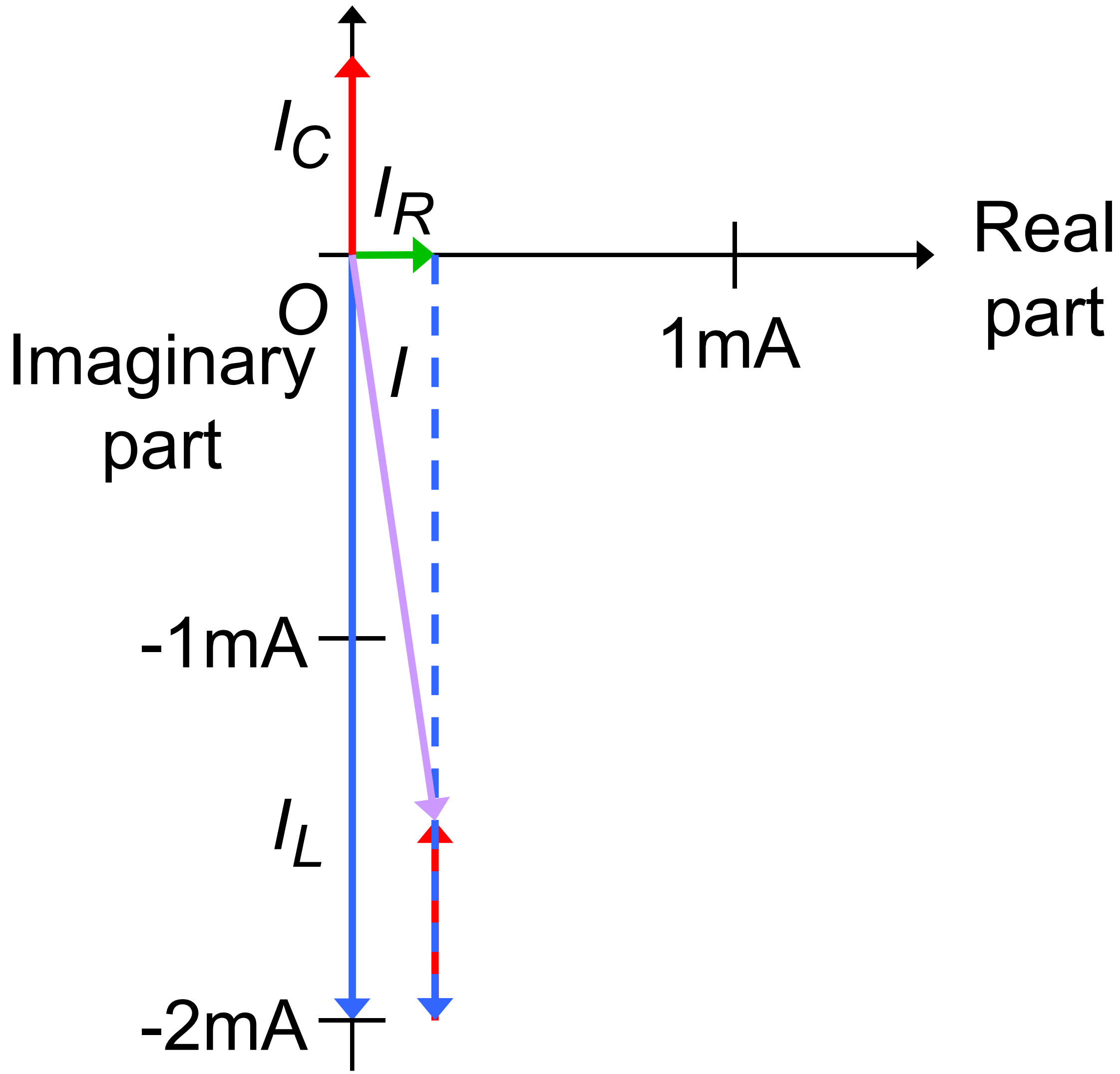
$$I_R = 2 \times 10^{-4}\,\text{A}$$

$$I_C = j\omega \times 10^{-6}$$

$$= j0.5 \times 10^{-3}\,\text{A}$$

$$I_L = \frac{-j}{\omega}$$

$$= -j2 \times 10^{-3}\,\text{A}$$



# Parallel Resonant Circuit

Above resonance:

$$\omega = 2.0 \times 10^3 :$$

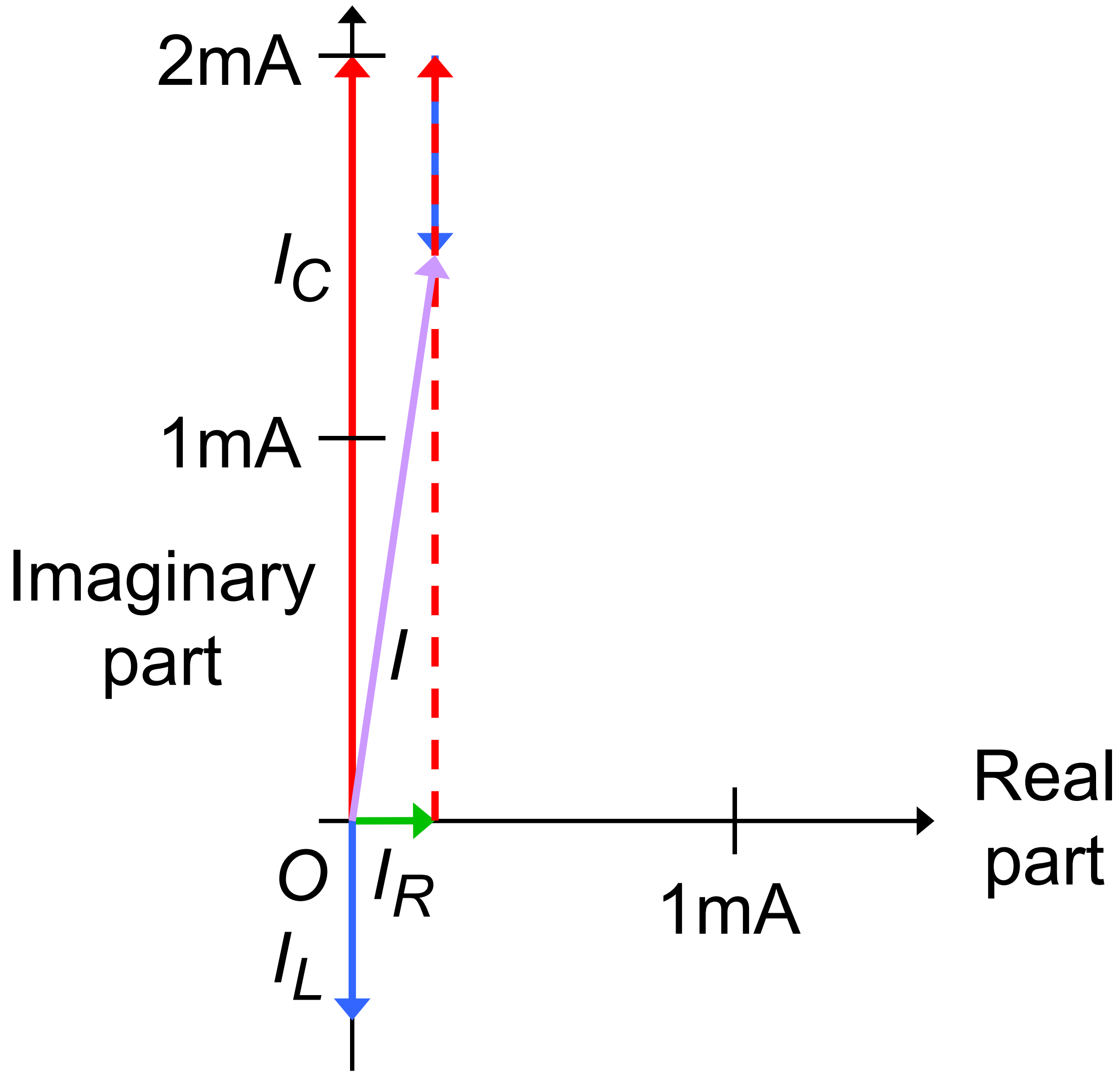
$$I_R = 2 \times 10^{-4}\,\text{A}$$

$$I_C = j\omega \times 10^{-6}$$

$$= j2.0 \times 10^{-3}\,\text{A}$$

$$I_L = \frac{-j}{\omega}$$

$$= -j0.5 \times 10^{-3}\,\text{A}$$



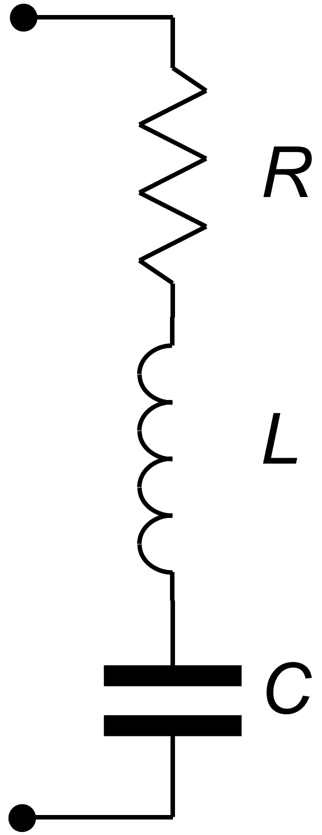
# Series Resonant Circuit

$$Z = Z_R + Z_C + Z_L$$

$$= R + \frac{1}{j\omega C} + j\omega L$$

$$= \frac{j\omega CR + 1 - \omega^2 LC}{j\omega C}$$

$$= \frac{1 + j\omega CR - \omega^2 LC}{j\omega C}$$



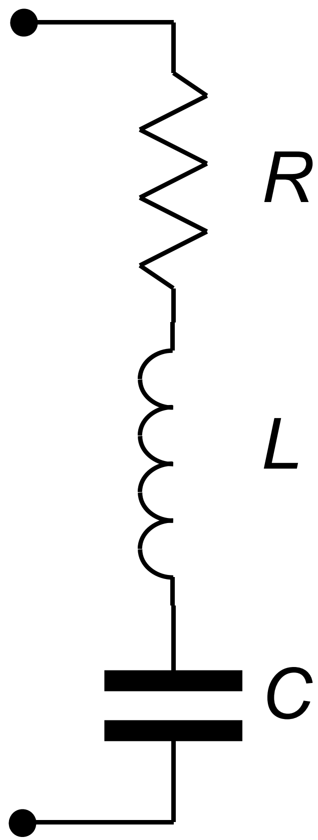
# Series Resonant Circuit

$$Z = \frac{1 + j\omega CR - \omega^2 LC}{j\omega C}$$

$$\omega \to 0 \qquad Z \to \frac{1}{j\omega C} = \frac{-j}{\omega C} = -j\infty$$

$$\omega = \frac{1}{\sqrt{LC}} \qquad Z = \frac{j\omega CR}{j\omega C} = R$$

$$\omega \to \infty \qquad Z \to \frac{-\omega^2 LC}{j\omega C} = j\omega L = j\infty$$

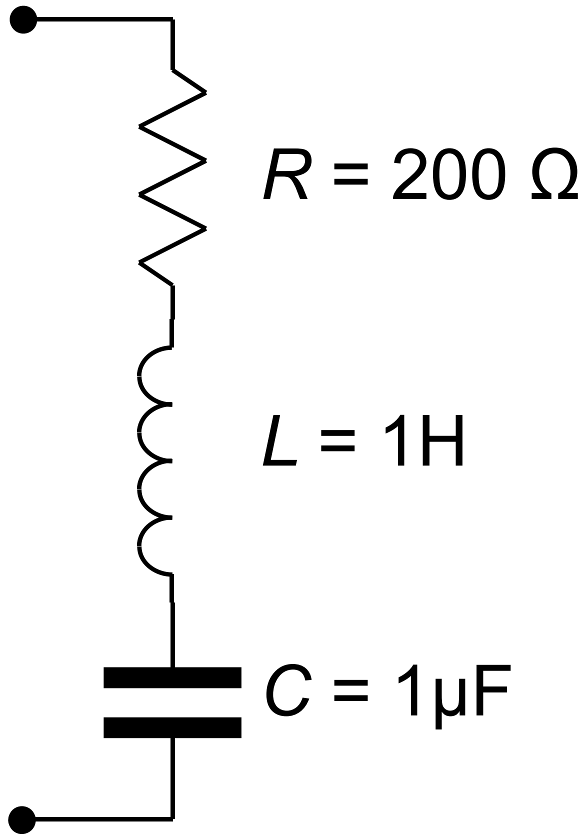


# Series Resonant Circuit

$$Z = \frac{1 + j\omega CR - \omega^2 LC}{j\omega C}$$

$$= \frac{1 + j\omega \times 200 \times 10^{-6} - \omega^2 \times 10^{-6}}{j\omega \times 10^{-6}}$$

$$= \frac{1 + j\omega \times 2 \times 10^{-4} - \omega^2 \times 10^{-6}}{j\omega \times 10^{-6}}$$



# Series Resonant Circuit

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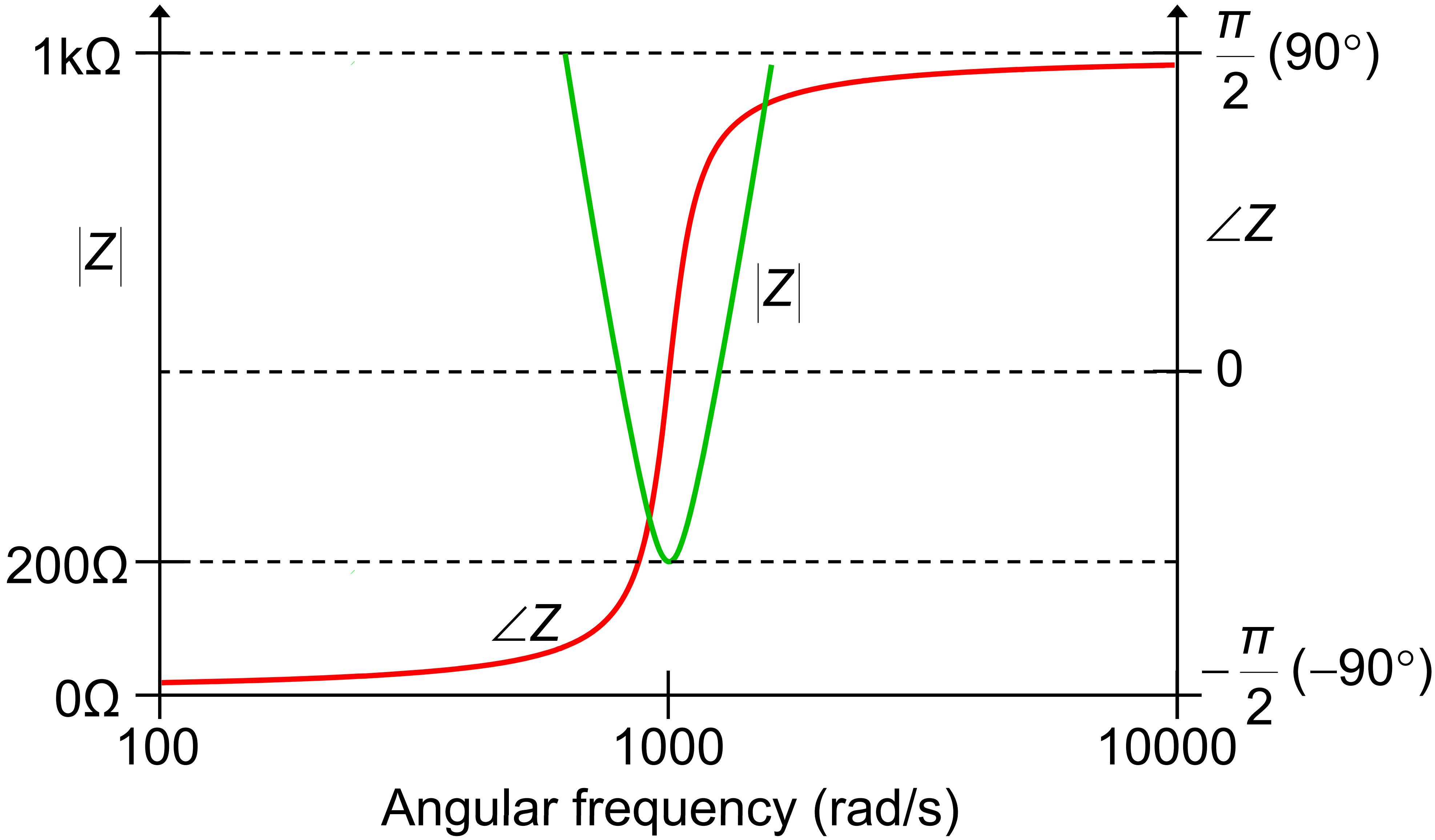


# Series Resonant Circuit

The standard form for the denominator of a second-order system is:

$$1 + j\omega / \omega_0 Q - \omega^2 / \omega_0^2$$

Compare this with the admittance $Y$ (= $1/Z$):

$$Y = \frac{j\omega C}{1 + j\omega CR - \omega^2 LC}$$

So that:

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad Q = \frac{1}{\omega_0 CR}$$

where $Q$ is the quality-factor and $\omega_0$ is the resonant frequency

# Series Resonant Circuit

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

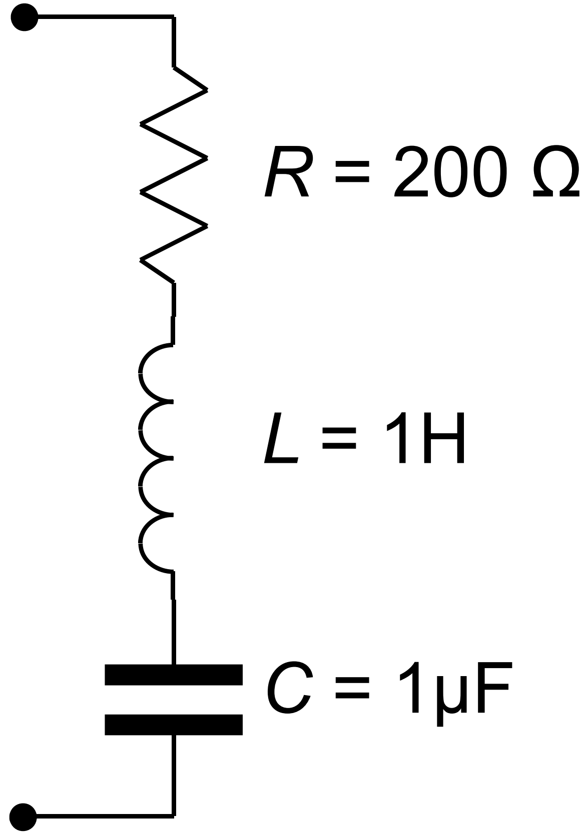
$$= \frac{1}{\sqrt{1 \times 10^{-6} \times 1}}$$

$$= 10^3 \text{ rad/s}$$

$$Q = \frac{1}{\omega_0 CR}$$

$$= \frac{1}{10^3 \times 1 \times 10^{-6} \times 200}$$

$$= 5$$



# Series Resonant Circuit

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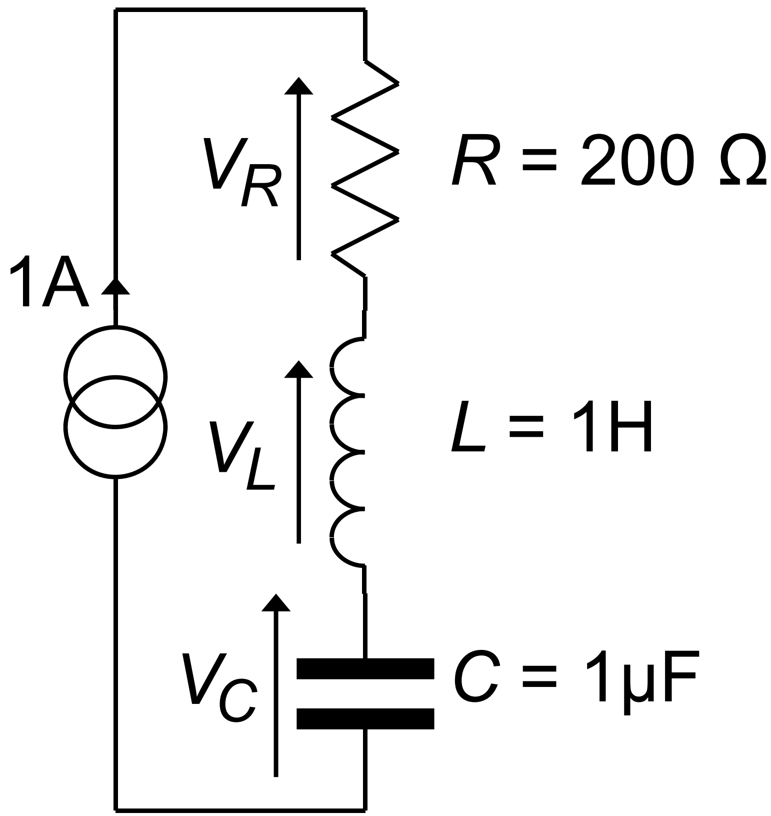
Resonance occurs in series resonant circuits because the voltages across the capacitor and inductor cancel out




$$V_R = 1 \times R = 200$$

$$V_C = \frac{1}{j\omega C} = \frac{-j}{\omega C} = \frac{-j10^6}{\omega}$$

$$V_L = 1 \times j\omega L = j\omega$$

# Series Resonant Circuit

At resonance:

$$\omega = 10^3 :$$

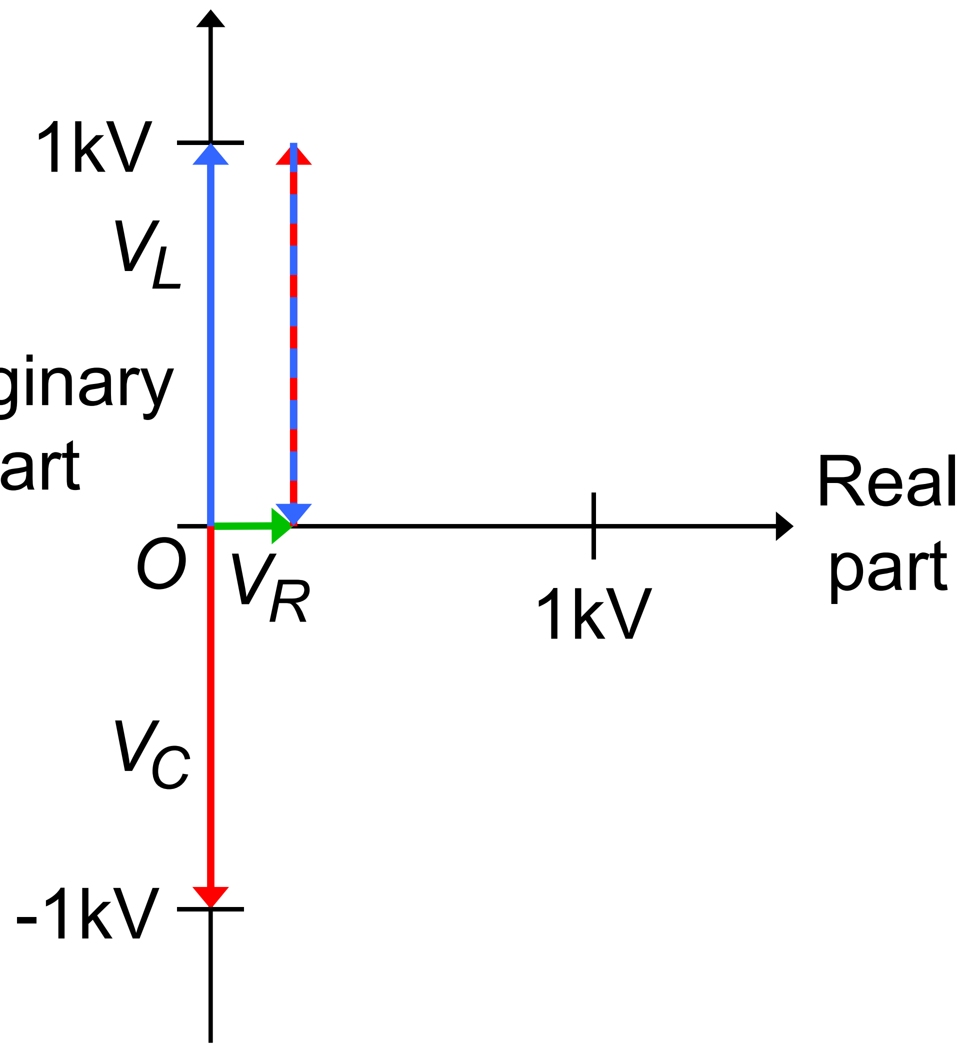
$$V_R = 200\text{V}$$

$$V_C = \frac{-j10^6}{\omega}$$

$$= -j10^3\text{V}$$

$$V_L = j\omega$$

$$= j10^3\text{V}$$



# Crystal Resonator

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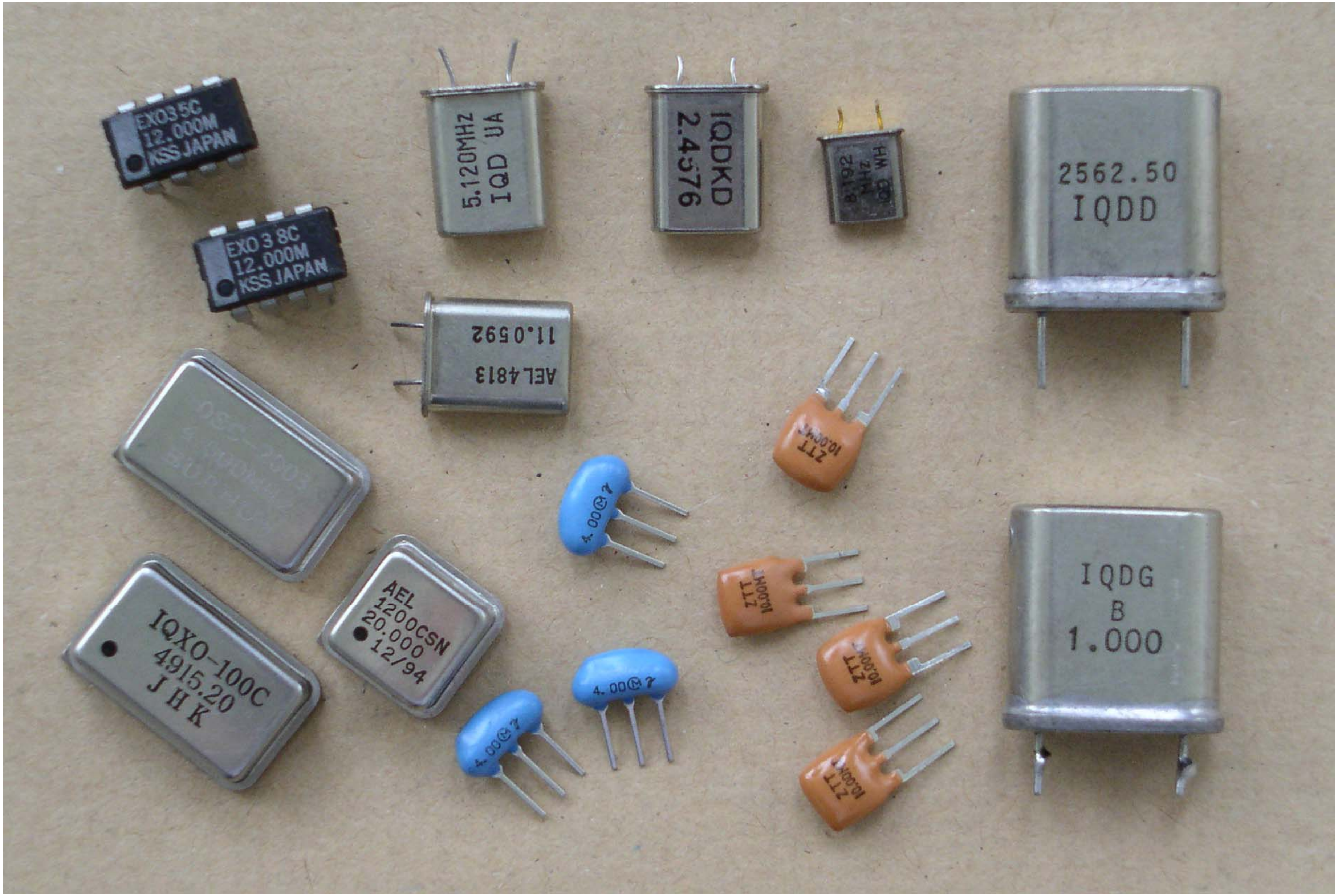
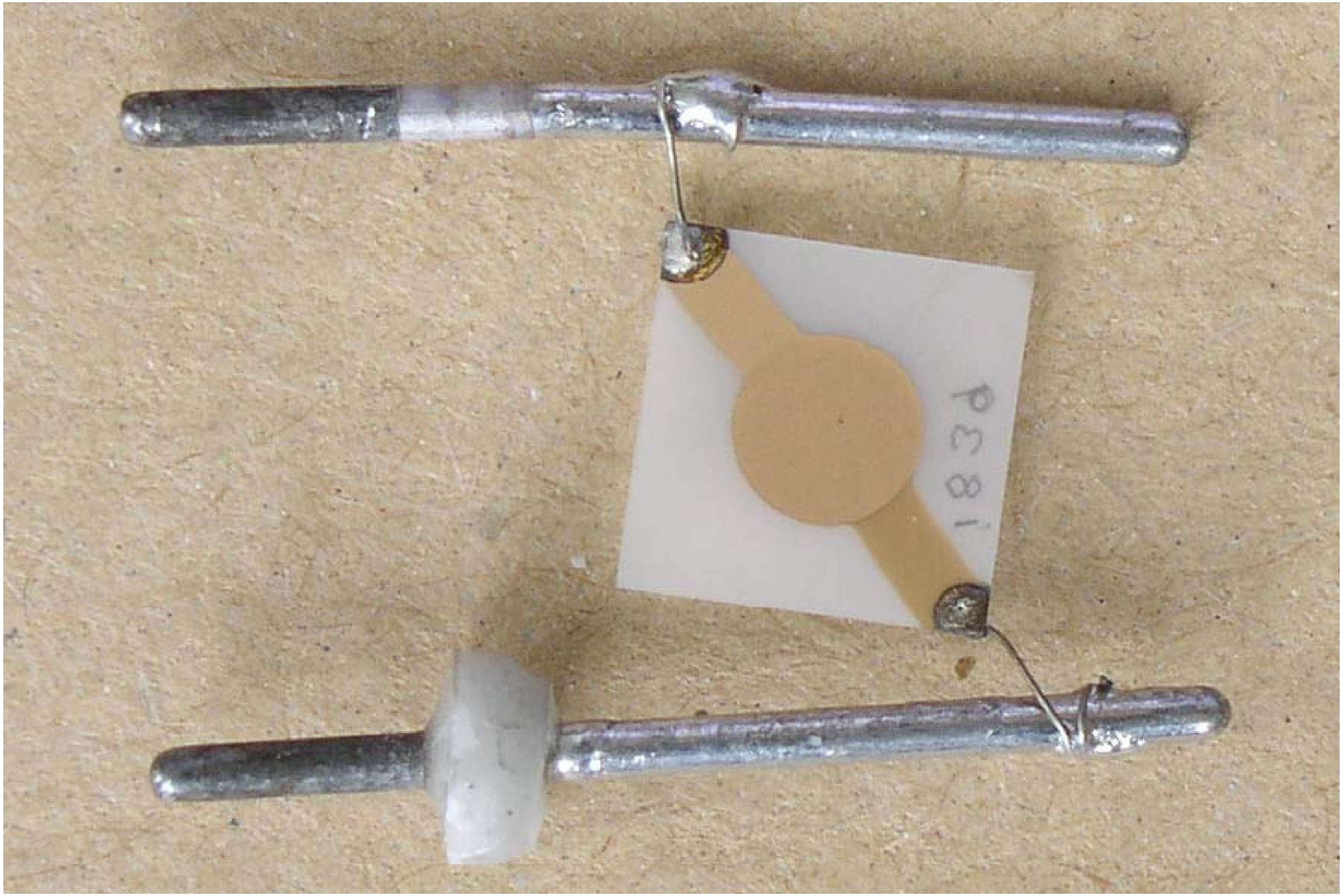


# Crystal Resonator



# Crystal Resonator

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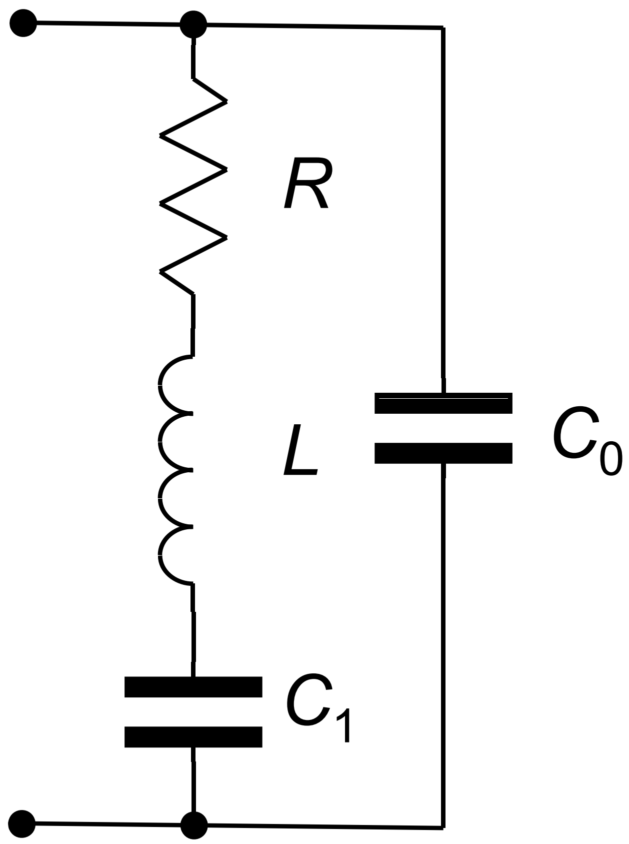
Equivalent circuit:




$f_0$ = 8.0 MHz
$R$ = 3.4 Ω
$L_1$ = 0.086 mH
$C_1$ = 4.6 pF
$C_0$ = 42 pF

$$\omega_0 = \frac{1}{\sqrt{LC}} = 5.03 \times 10^7$$

$$Q = \frac{1}{\omega_0 CR} = 1270$$

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# Lecture 6

## Frequency-Response Function
## First-Order Circuits

# Frequency-Response Function

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Frequency-response function: $$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

Voltage gain *g:* $$g = \left|\frac{Y(j\omega)}{X(j\omega)}\right| = \left|H(j\omega)\right|$$

Phase shift *φ*: $$\varphi = \angle\left(\frac{Y(j\omega)}{X(j\omega)}\right) = \angle H(j\omega)$$

# Frequency-Response Function

The order of a frequency-response function is the highest power of $j\omega$ in the denominator:

First order: $$H(j\omega) = \frac{1}{1 + j\omega/\omega_0}$$

Second order: $$H(j\omega) = \frac{1}{1 + \sqrt{2}\,j\omega/\omega_0 + (j\omega/\omega_0)^2}$$

Third order: $$H(j\omega) = \frac{1}{1 + j\omega/\omega_0 + (j\omega/\omega_0)^2 + (j\omega/\omega_0)^3}$$

The order is normally equal to (and cannot exceed) the number of reactive components

# Example 1

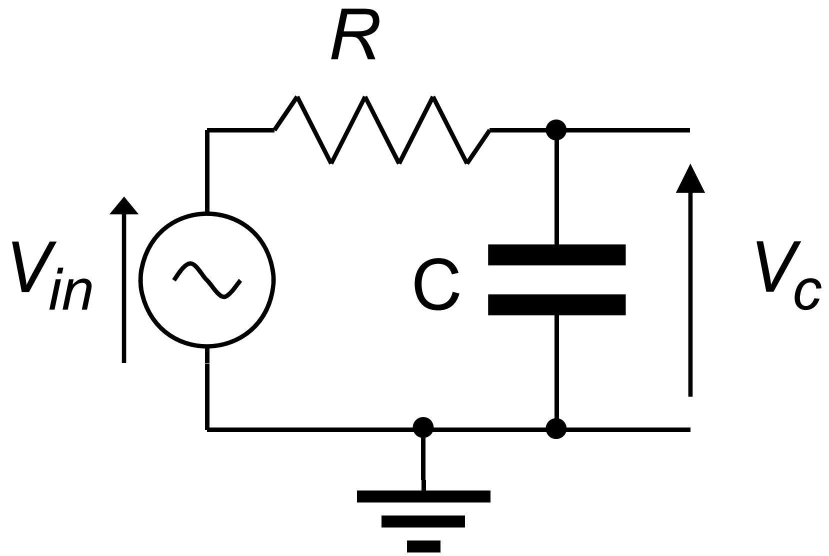
Using the potential divider formula:

$$\frac{V_c}{V_{in}} = \frac{Z_C}{Z_C + Z_R}$$

$$= \frac{1/\,j\omega C}{1/\,j\omega C + R}$$

$$H(j\omega) = \frac{1}{1 + j\omega CR}$$

$$= \frac{1}{1 + j\omega/\omega_0} \quad \text{where:} \quad \omega_0 = \frac{1}{RC}$$



# Example 1

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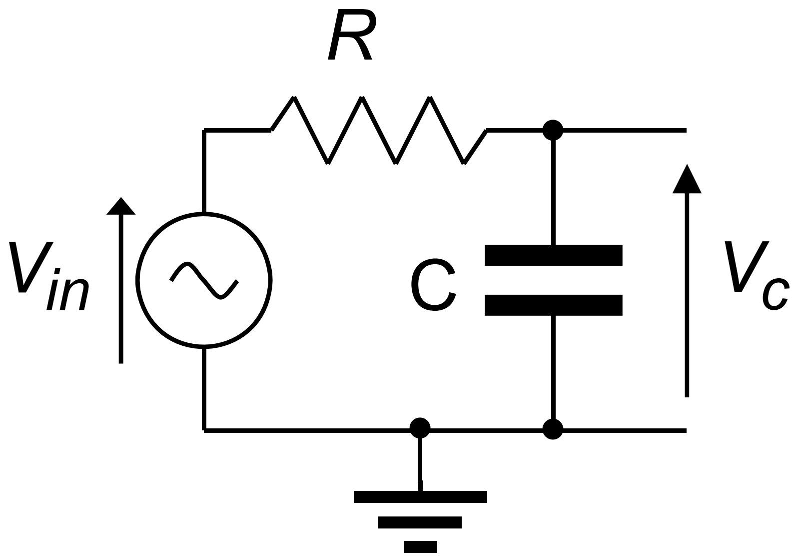
$$H(j\omega) = \frac{1}{1 + j\omega/\omega_0}$$

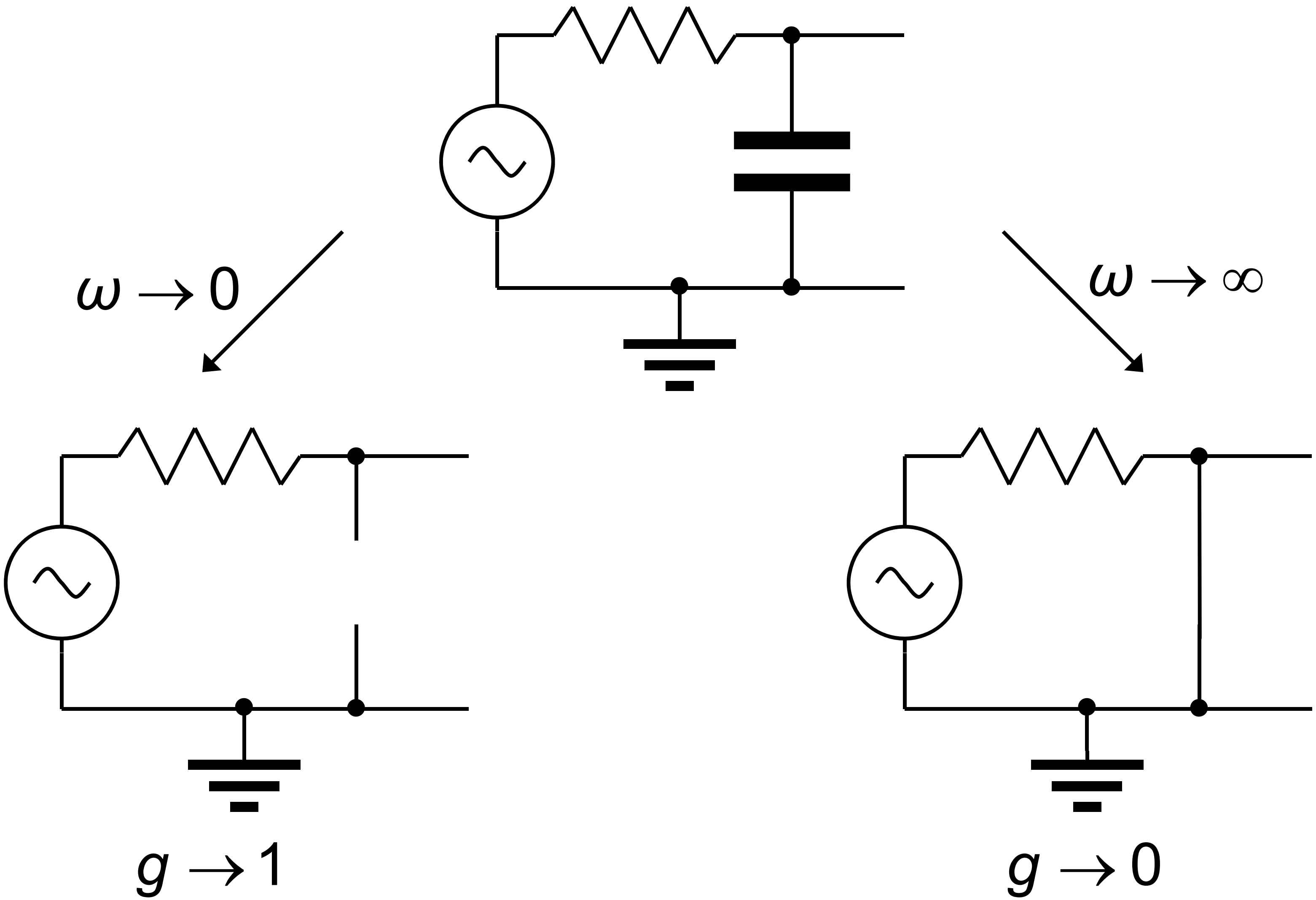
$$= \frac{1 - j\omega/\omega_0}{1 + \omega^2/\omega_0^2}$$




Gain: $$g = |H(j\omega)| = \frac{1}{\sqrt{1 + \omega^2/\omega_0^2}}$$

Phase shift: $\varphi = \angle H(j\omega)$ $\quad \tan\varphi = -\omega/\omega_0$

# Example 1



# Decibel

The decibel is a measure of the ratio of two powers $P_1$, $P_2$ :

$$\mathrm{dB} = 10\log_{10}\frac{P_1}{P_2}$$

It can also be used to measure the ratio of two voltages $V_1$, $V_2$:

$$\mathrm{dB} = 10\log_{10}\frac{V_1^2/R}{V_2^2/R} = 10\log_{10}\frac{V_1^2}{V_2^2}$$

$$\mathrm{dB} = 20\log_{10}\frac{V_1}{V_2}$$

# Decibel

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| Power ratio | Decibels |
| --- | --- |
| 1000000 | 60 dB |
| 100 | 20 dB |
| 10 | 10 dB |
| 4 | 6 dB |
| 2 | 3 dB |
| 1 | 0 dB |
| 1/2 | -3 dB |
| 1/4 | -6 dB |
| 0.01 | -20 dB |
| 0.000001 | -60 dB |

# Decibel

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| Voltage ratio | Decibels |
|---|---|
| 1000 | 60 dB |
| 10 | 20 dB |
| $\sqrt{10} = 3.162$ | 10 dB |
| 2 | 6 dB |
| $\sqrt{2} = 1.414$ | 3 dB |
| 1 | 0 dB |
| $1/\sqrt{2} = 0.7071$ | -3 dB |
| 1/2 = 0.5 | -6 dB |
| 0.1 | -20 dB |
| 0.001 | -60 dB |

# Example 1

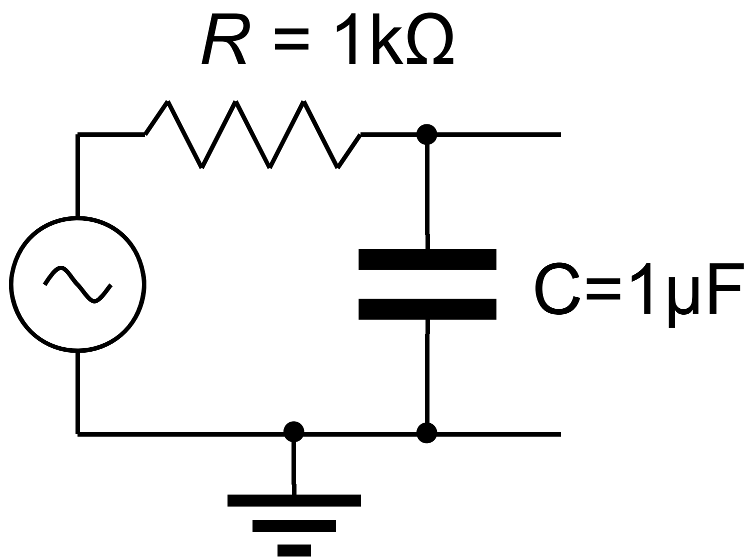
Circuit is a first-order low-pass filter:

| | | |
|---|---|---|
| | $g = \dfrac{1}{\sqrt{1+\omega^2/\omega_0^2}}$ | $\varphi = \tan^{-1} - \omega/\omega_0$ |
| $\omega << \omega_0$ | $g \approx 1\,(0\text{dB})$ | $\varphi \approx 0\ (0°)$ |
| $\omega = \omega_0$ | $g = \dfrac{1}{\sqrt{2}}\,(-3\text{dB})$ | $\varphi = -\dfrac{\pi}{4}\,(-45°)$ |
| $\omega >> \omega_0$ | $g \approx \dfrac{\omega_0}{\omega}\,(-6\text{dB}/\text{oct})$ | $\varphi \approx -\dfrac{\pi}{2}\,(-90°)$ |

# Example 1



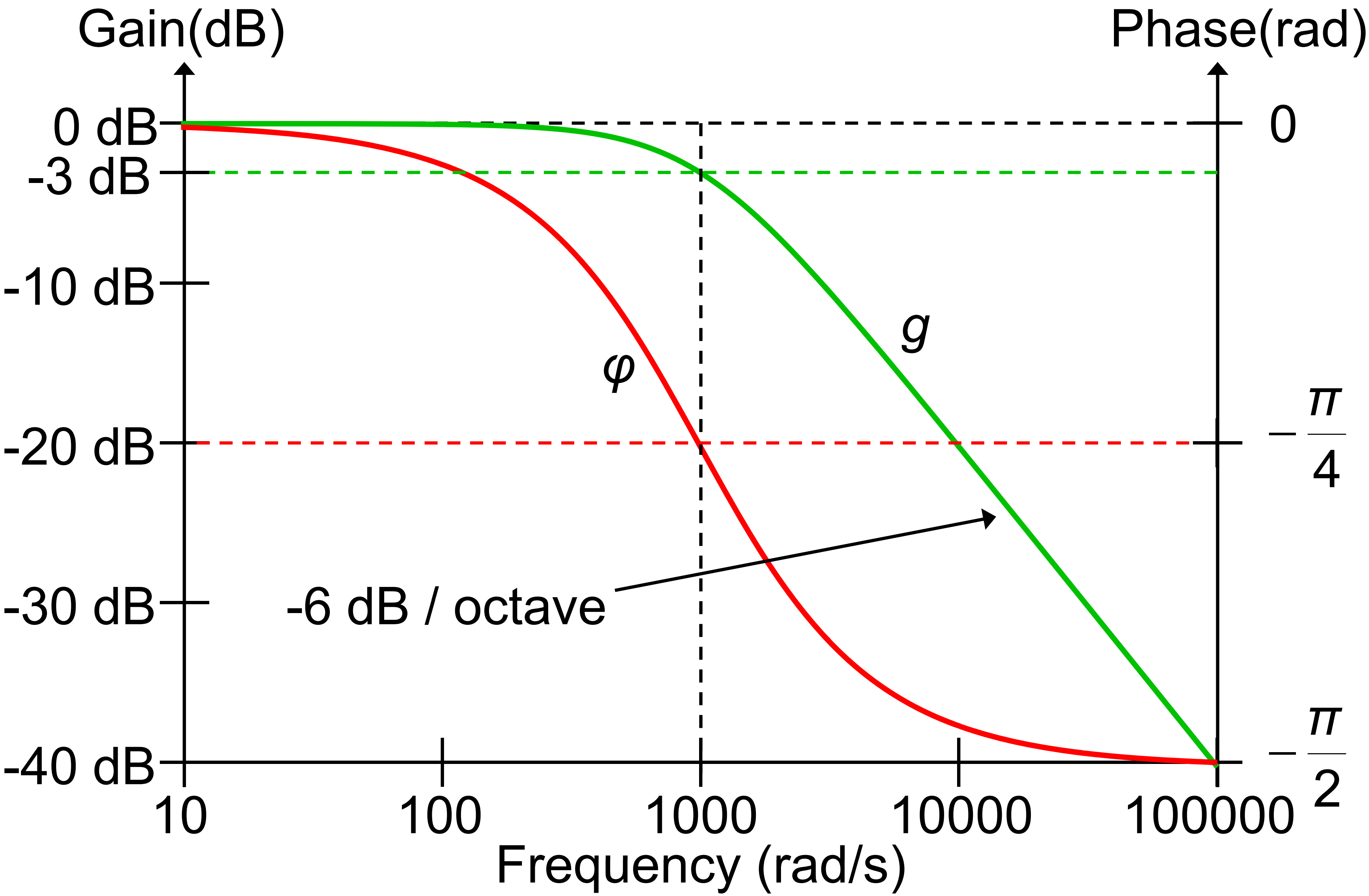
$$\omega_0 = \frac{1}{RC} = \frac{1}{10^3 \times 10^{-6}} = 10^3$$




Gain: $$g = |H(j\omega)| = \frac{1}{\sqrt{1+\omega^2/10^6}}$$

Phase shift: $$\varphi = \angle H(j\omega) \quad \tan\varphi = -\omega/10^3$$

# Bode Plot



Gain(dB)
Phase(rad)
0 dB
-3 dB
-10 dB
-20 dB
-30 dB
-40 dB
0
$-\frac{\pi}{4}$
$-\frac{\pi}{2}$
g
φ
-6 dB / octave
10
100
1000
10000
100000
Frequency (rad/s)

# Example 2

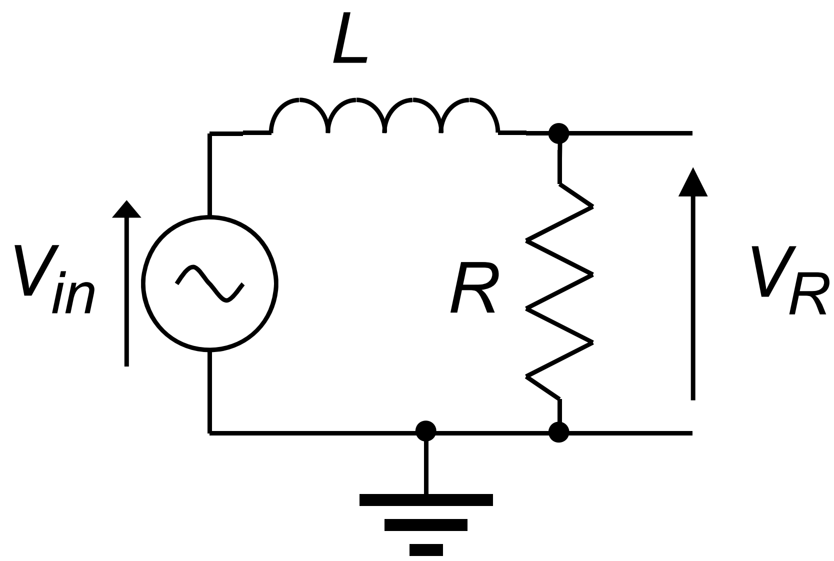
Using the potential divider formula:

$$\frac{V_R}{V_{in}} = \frac{Z_R}{Z_R + Z_L}$$

$$= \frac{R}{R + j\omega L}$$

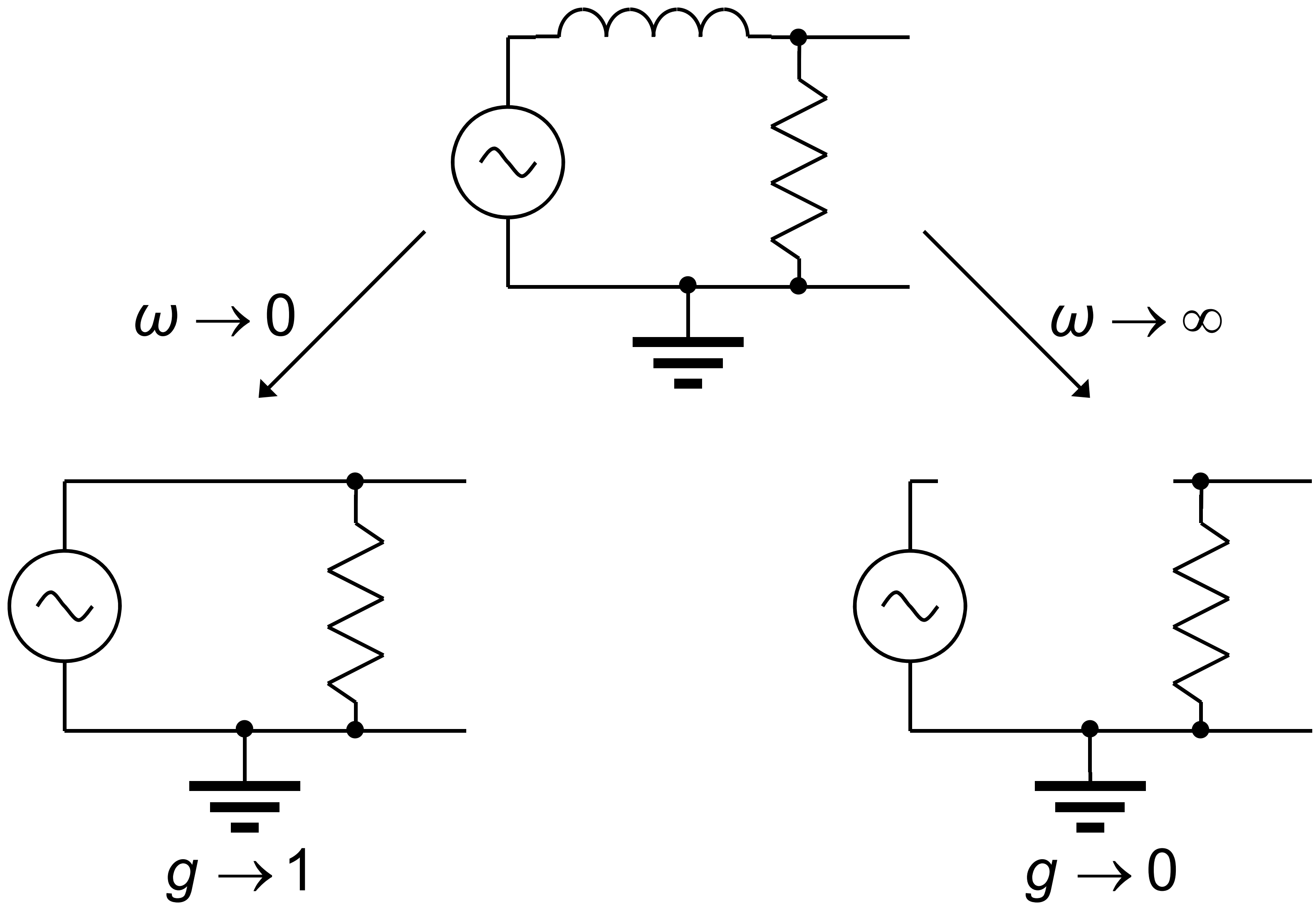
$$H(j\omega) = \frac{1}{1 + j\omega L / R}$$

$$= \frac{1}{1 + j\omega / \omega_0} \quad \text{where :} \quad \omega_0 = \frac{R}{L}$$



# Example 2

University of Reading



# Example 3

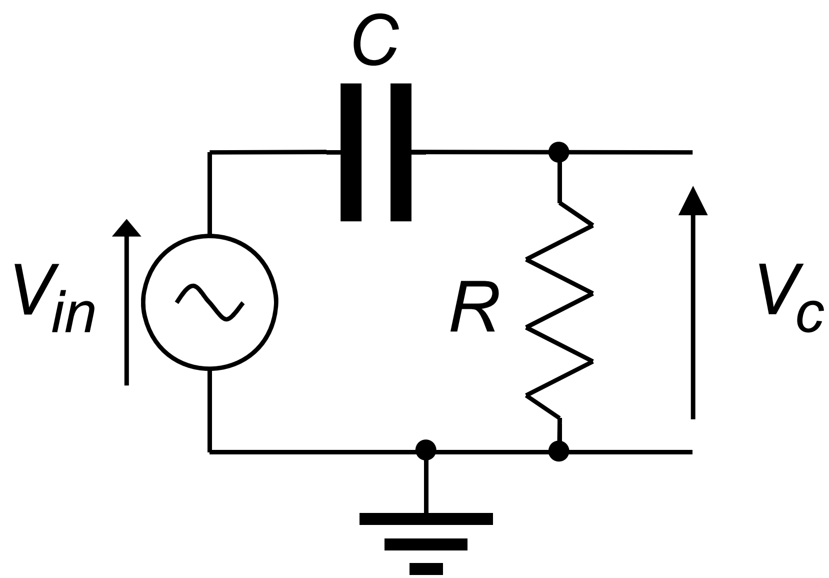
Using the potential divider formula:

$$\frac{V_c}{V_{in}} = \frac{Z_R}{Z_R + Z_C}$$

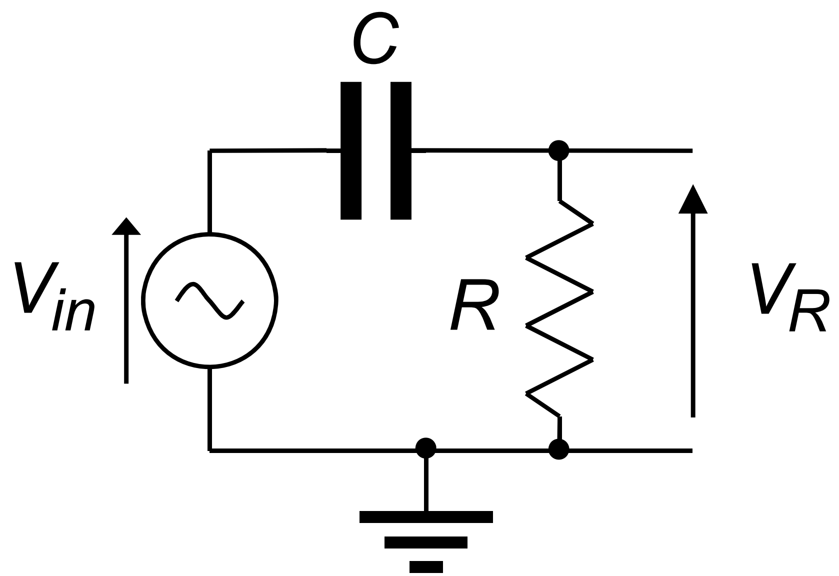
$$= \frac{R}{1/j\omega C + R}$$

$$H(j\omega) = \frac{j\omega CR}{1 + j\omega CR}$$

$$= \frac{j\omega/\omega_0}{1 + j\omega/\omega_0} \quad \text{where:} \quad \omega_0 = \frac{1}{RC}$$



# Example 3




$$H(j\omega) = \frac{j\omega/\omega_0}{1 + j\omega/\omega_0}$$

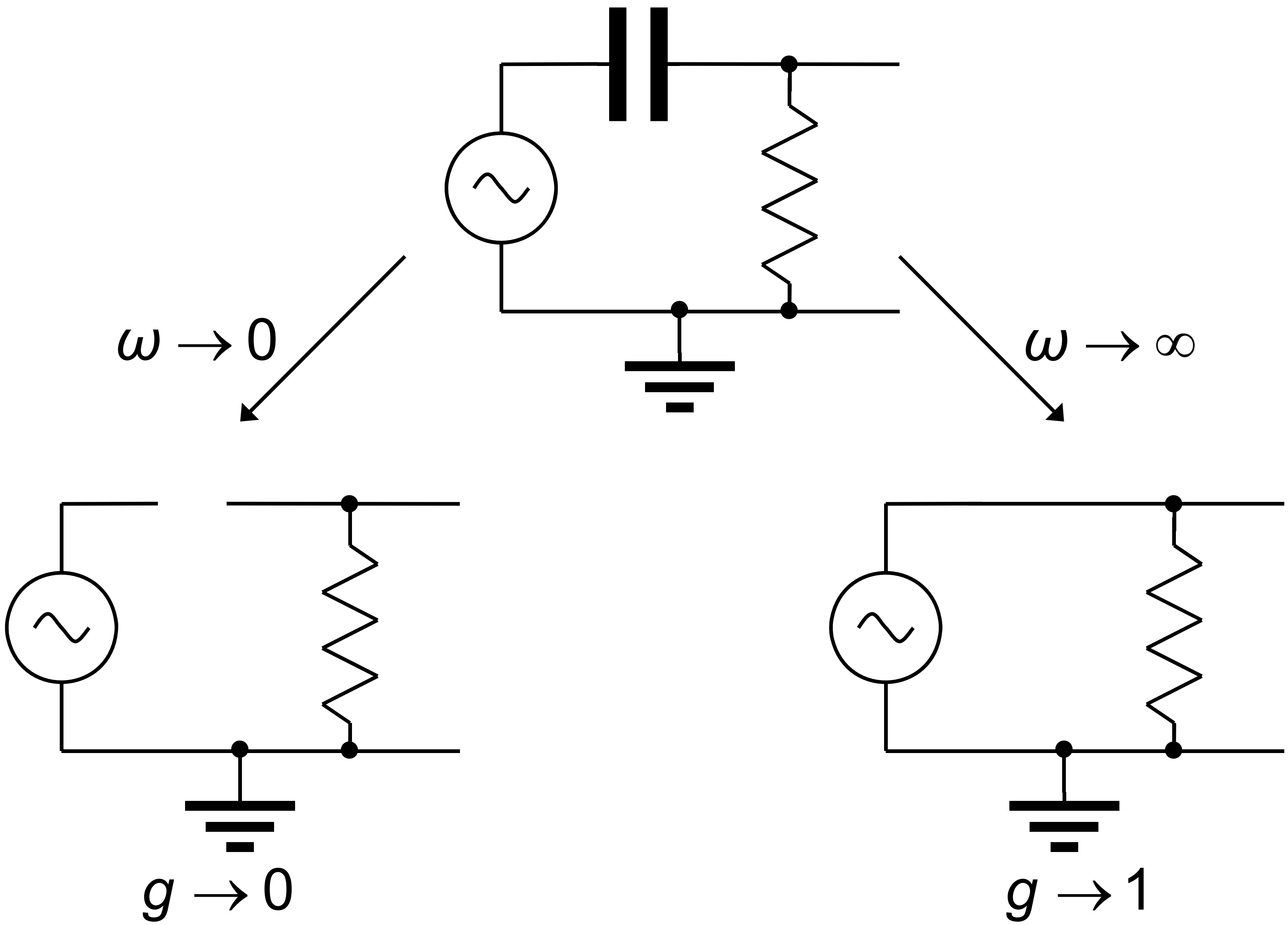
$$= \frac{1}{1 - j\omega_0/\omega}$$

$$= \frac{1 + j\omega_0/\omega}{1 + \omega_0^2/\omega^2}$$

Gain: $$g = |H(j\omega)| = \frac{1}{\sqrt{1 + \omega_0^2/\omega^2}}$$

Phase shift: $\varphi = \angle H(j\omega) \quad \tan\varphi = \omega_0/\omega$

# Example 3



# Example 4

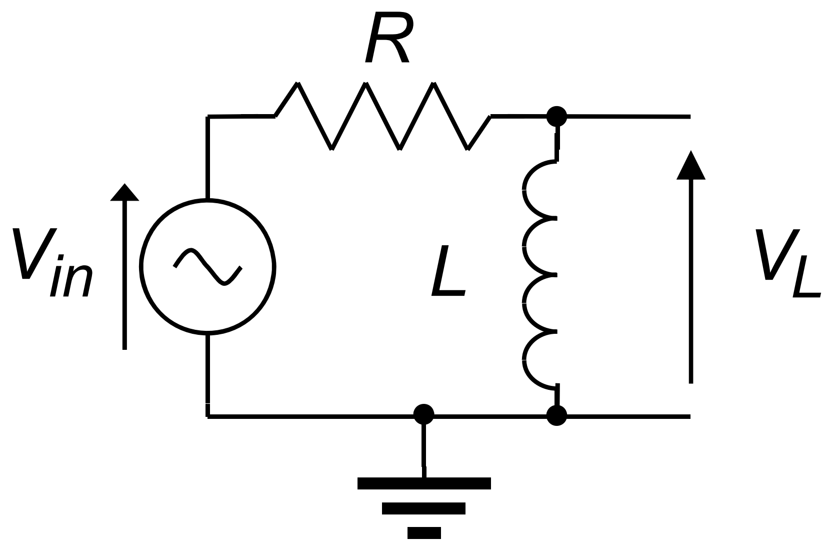
Using the potential divider formula:

$$\frac{V_L}{V_{in}} = \frac{Z_L}{Z_L + Z_R}$$

$$= \frac{j\omega L}{j\omega L + R}$$

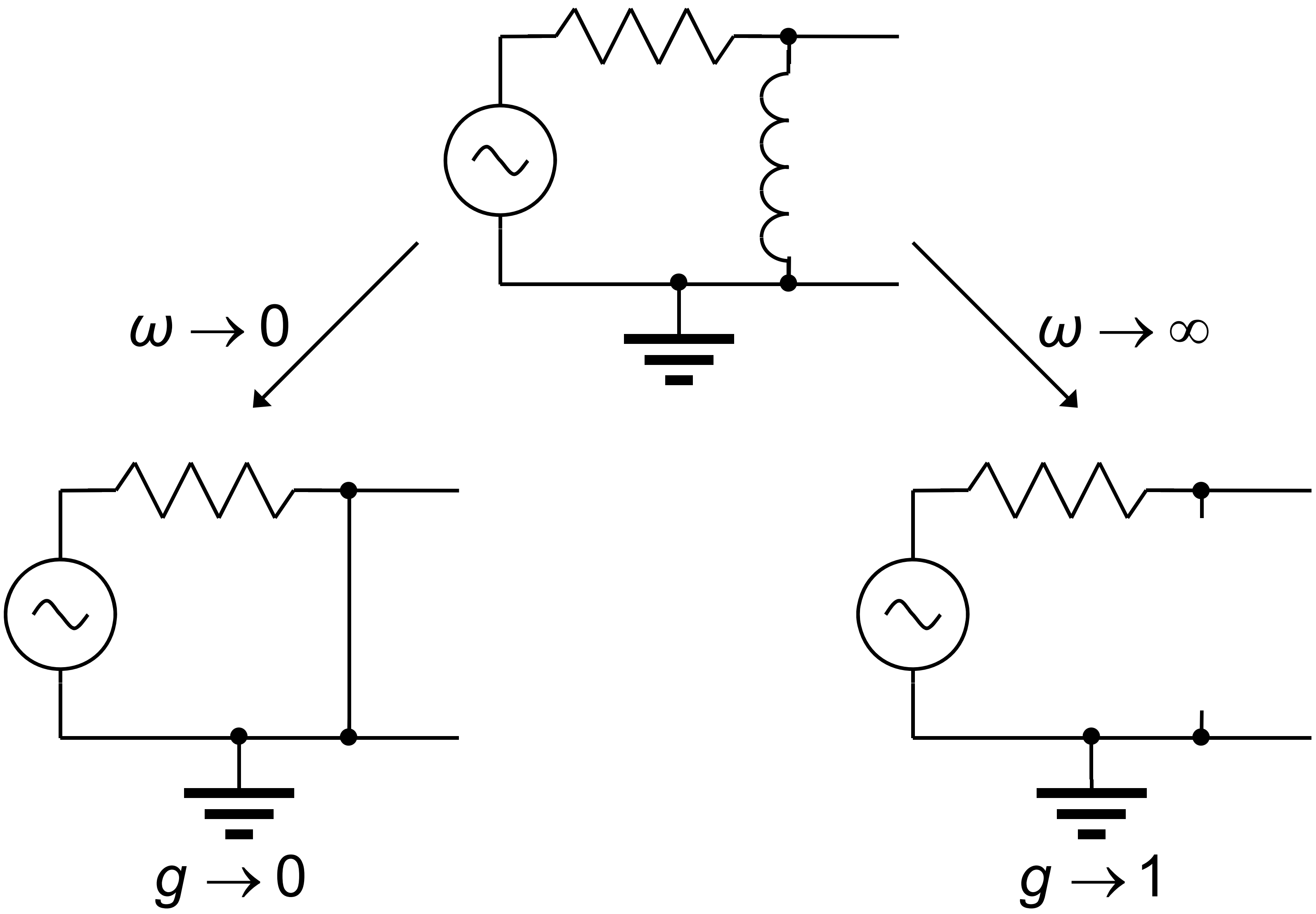
$$H(j\omega) = \frac{j\omega L / R}{1 + j\omega L / R}$$

$$= \frac{j\omega / \omega_0}{1 + j\omega / \omega_0} \quad \text{where :} \quad \omega_0 = \frac{R}{L}$$



# Example 4

University of Reading



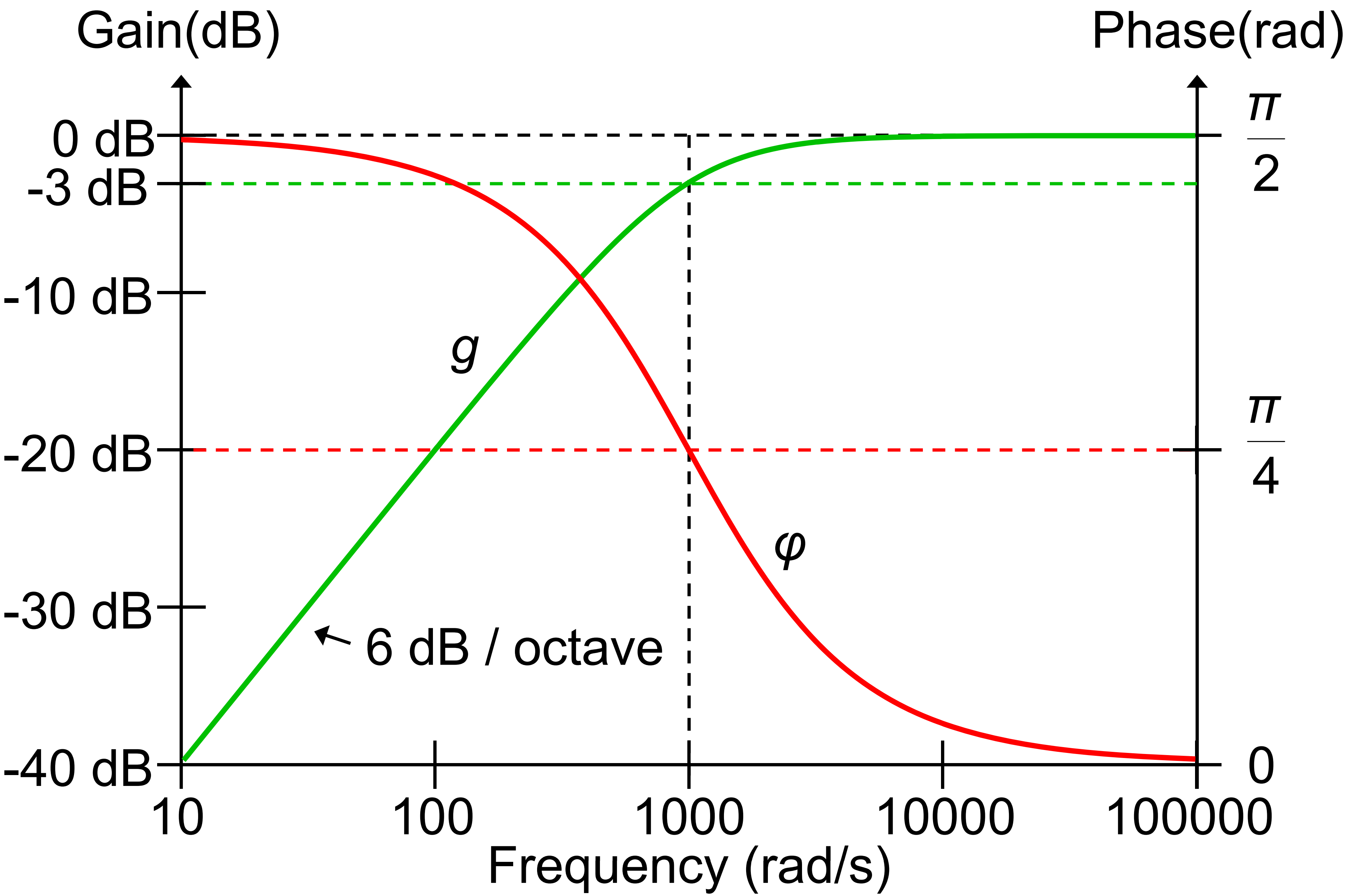
# Example 4

Circuit is a first-order high-pass filter:

| | | |
|---|---|---|
| | $g = \dfrac{1}{\sqrt{1+\omega_0^2/\omega^2}}$ | $\varphi = \tan^{-1}\omega_0/\omega$ |
| $\omega << \omega_0$ | $g \approx \dfrac{\omega}{\omega_0}$ (6dB/oct) | $\varphi \approx \dfrac{\pi}{2}$ (90°) |
| $\omega = \omega_0$ | $g = \dfrac{1}{\sqrt{2}}$ (–3dB) | $\varphi = \dfrac{\pi}{4}$ (45°) |
| $\omega >> \omega_0$ | $g \approx 1$ (0dB) | $\varphi \approx 0$ (0°) |

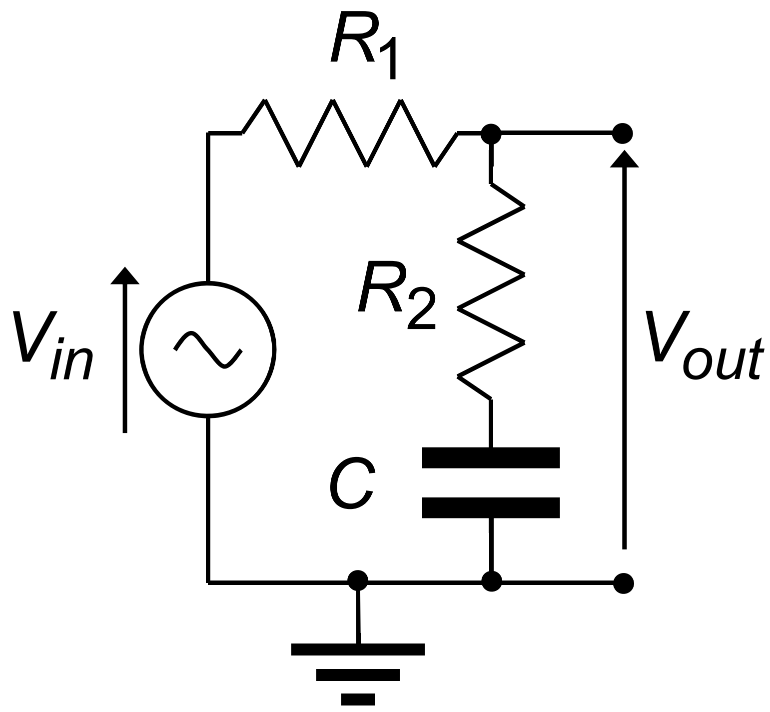
# Bode Plot

University of Reading



# Example 5

Using the potential divider formula:




$$\frac{V_c}{V_{in}} = \frac{R_2 + 1/j\omega C}{R_2 + 1/j\omega C + R_1}$$

$$= \frac{j\omega CR_2 + 1}{j\omega CR_2 + 1 + j\omega CR_1}$$

$$H(j\omega) = \frac{1 + j\omega CR_2}{1 + j\omega C(R_1 + R_2)}$$

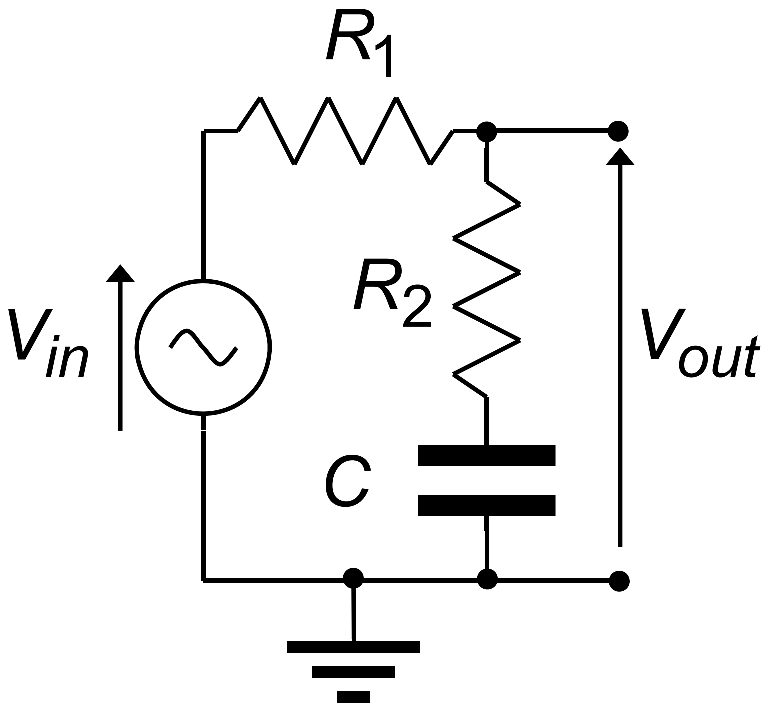
$$= \frac{1 + j\omega/\omega_2}{1 + j\omega/\omega_1} \quad \text{where:} \quad \omega_1 = \frac{1}{C(R_1 + R_2)} \quad \omega_2 = \frac{1}{CR_2}$$

# Example 5

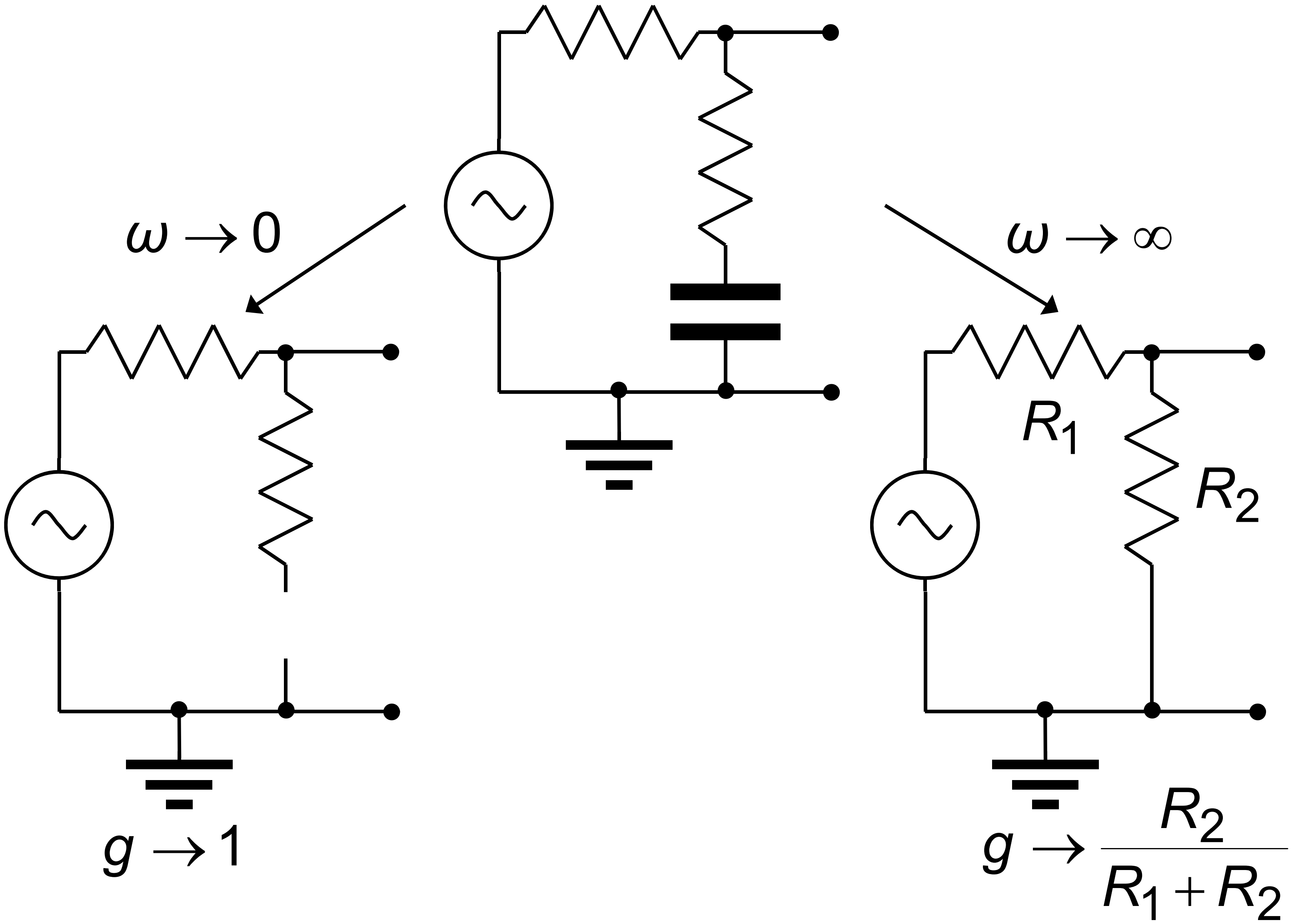


$$H(j\omega) = \frac{1 + j\omega / \omega_2}{1 + j\omega / \omega_1}$$

$$g = \left| H(j\omega) \right| = \frac{\sqrt{1 + \omega^2 / \omega_2^2}}{\sqrt{1 + \omega^2 / \omega_1^2}}$$



# Example 5



# Example 5

Assuming that $\omega_1 << \omega_2$:

| | |
|---|---|
| | $g = \dfrac{\sqrt{1+\omega^2/\omega_2^2}}{\sqrt{1+\omega^2/\omega_1^2}}$ |
| $\omega << \omega_1$ | $g \approx 1\,(0\text{dB})$ |
| $\omega_1 << \omega << \omega_2$ | $g \approx \dfrac{\omega_1}{\omega}\,(-6\text{dB/oct})$ |
| $\omega >> \omega 2$ | $g \approx \dfrac{\omega_1}{\omega_2} = \dfrac{R_2}{R_2+R_1}$ |

# Example 5



$$H(j\omega) = \frac{1 + j\omega/\omega_2}{1 + j\omega/\omega_1}$$

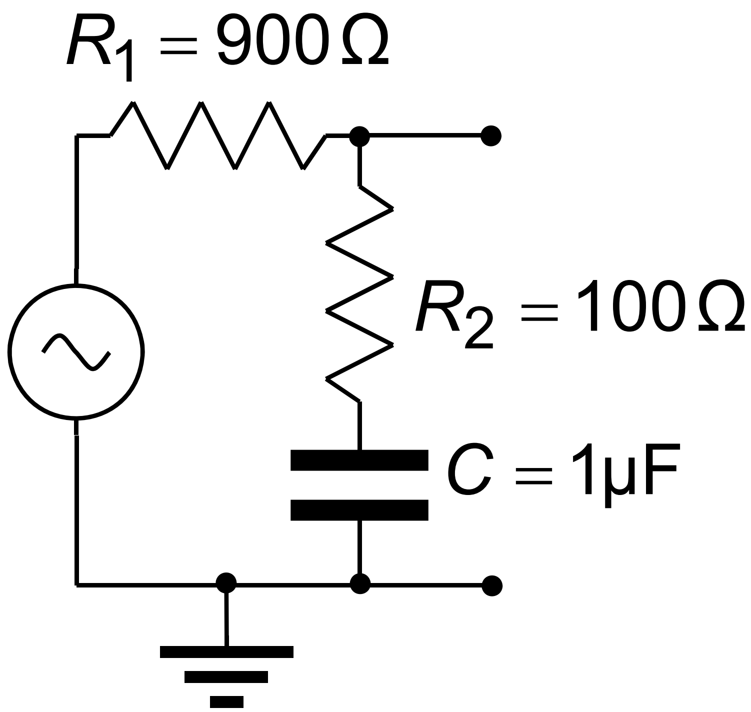
$$\omega_1 = \frac{1}{C(R_1 + R_2)}$$

$$= \frac{1}{10^{-6}(900 + 100)}$$

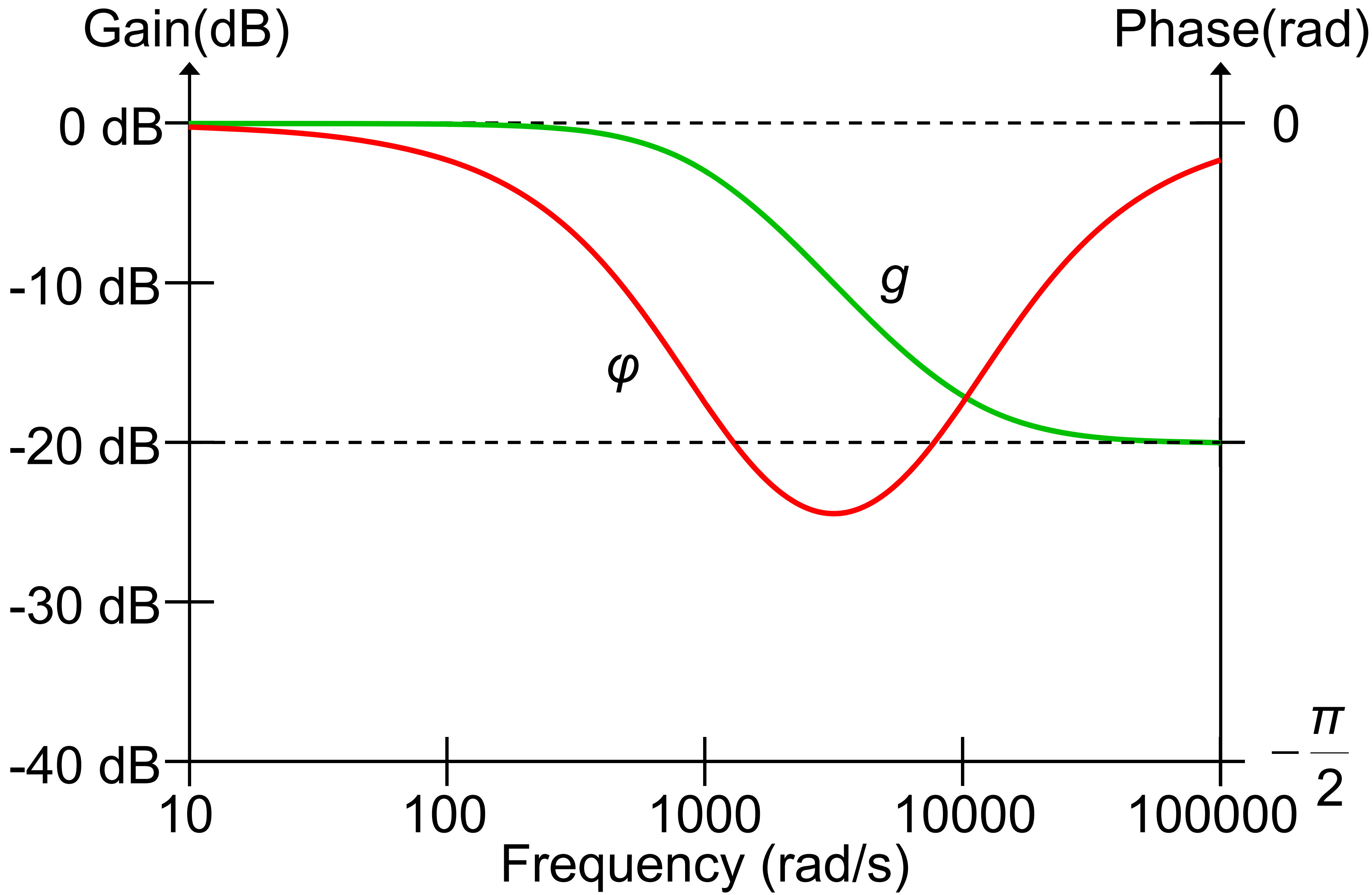
$$= 10^3 \text{ rad/s}$$

$$\omega_2 = \frac{1}{CR_2} = \frac{1}{10^{-6} \times 100}$$

$$= 10^4 \text{ rad/s}$$



# Bode Plot

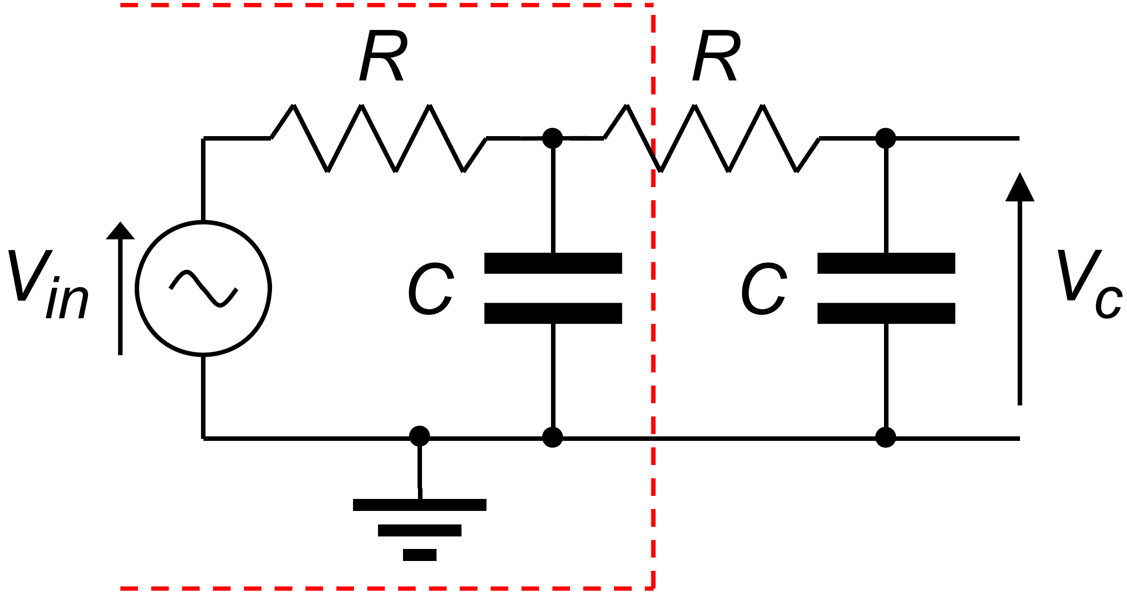

Gain(dB)
Phase(rad)
0 dB
-10 dB
-20 dB
-30 dB
-40 dB
0
$-\frac{\pi}{2}$
g
φ
10
100
1000
10000
100000
Frequency (rad/s)

# Lecture 7

## Second-Order Circuits

# Example 1

University of Reading

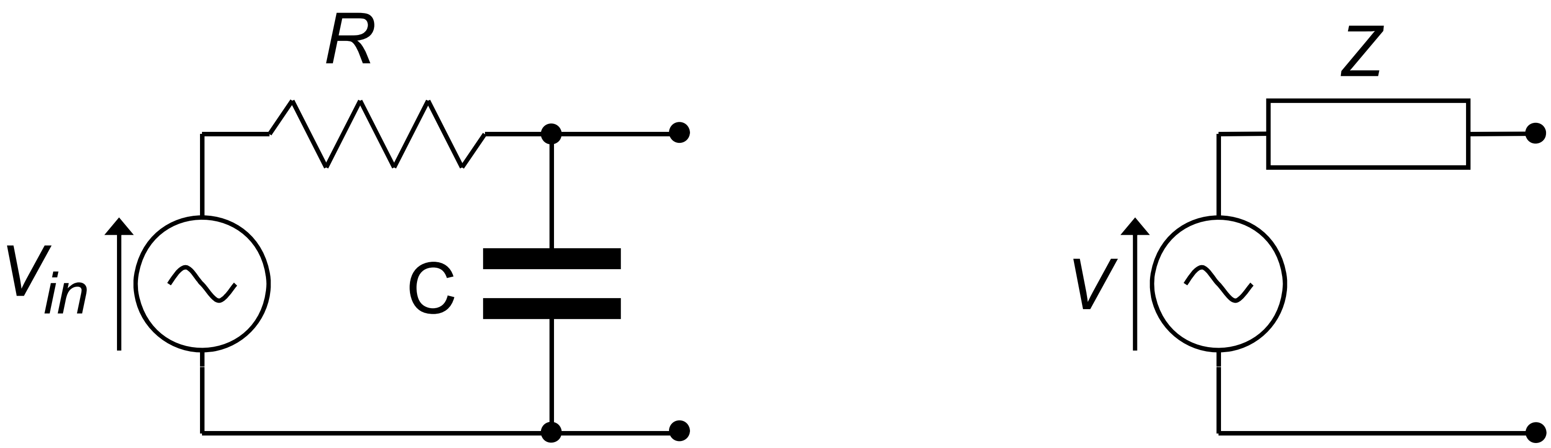



This circuit must be simplified before the frequency response function can be determined

A Thévenin equivalent circuit is created of the components to the left of the red line

# Example 1

Thévenin equivalent circuit:



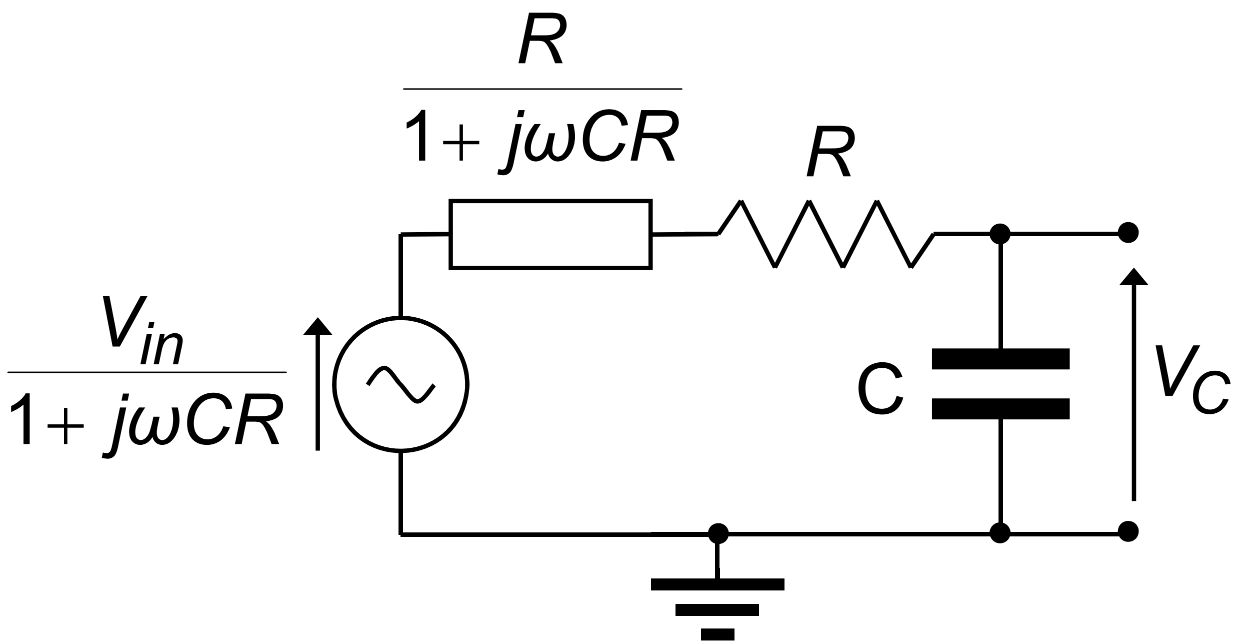

$$V = V_{in} \frac{1/\,j\omega C}{1/\,j\omega C + R} = \frac{V_{in}}{1 + j\omega CR}$$

$$\frac{1}{Z} = \frac{1}{R} + j\omega C = \frac{1 + j\omega CR}{R}$$

$$Z = \frac{R}{1 + j\omega CR}$$

# Example 1

University of Reading




$$V_C = \frac{V_{in}}{1+j\omega CR} \times \frac{1/j\omega C}{1/j\omega C + R + \dfrac{R}{1+j\omega CR}}$$

$$= \frac{V_{in}}{1+j\omega CR} \times \frac{1}{1+j\omega CR + \dfrac{j\omega CR}{1+j\omega CR}}$$

# Example 1



Frequency-response function:

$$V_C = \frac{V_{in}}{1+j\omega CR} \times \frac{1}{1+j\omega CR + \dfrac{j\omega CR}{1+j\omega CR}}$$

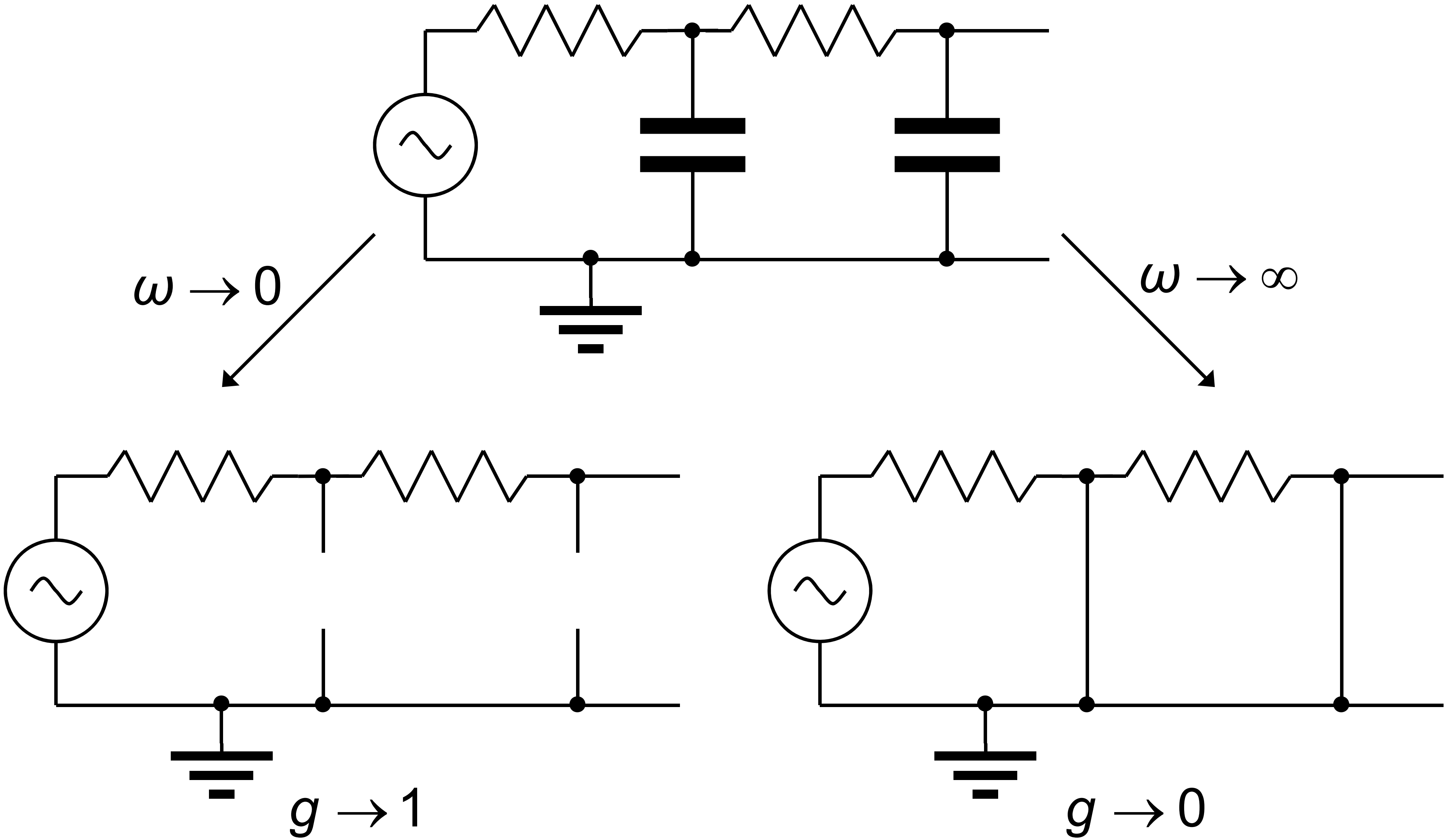
$$= \frac{V_{in}}{(1+j\omega CR)\times(1+j\omega CR)+j\omega CR}$$

$$H(j\omega) = \frac{1}{1+3j\omega CR - \omega^2 C^2 R^2}$$

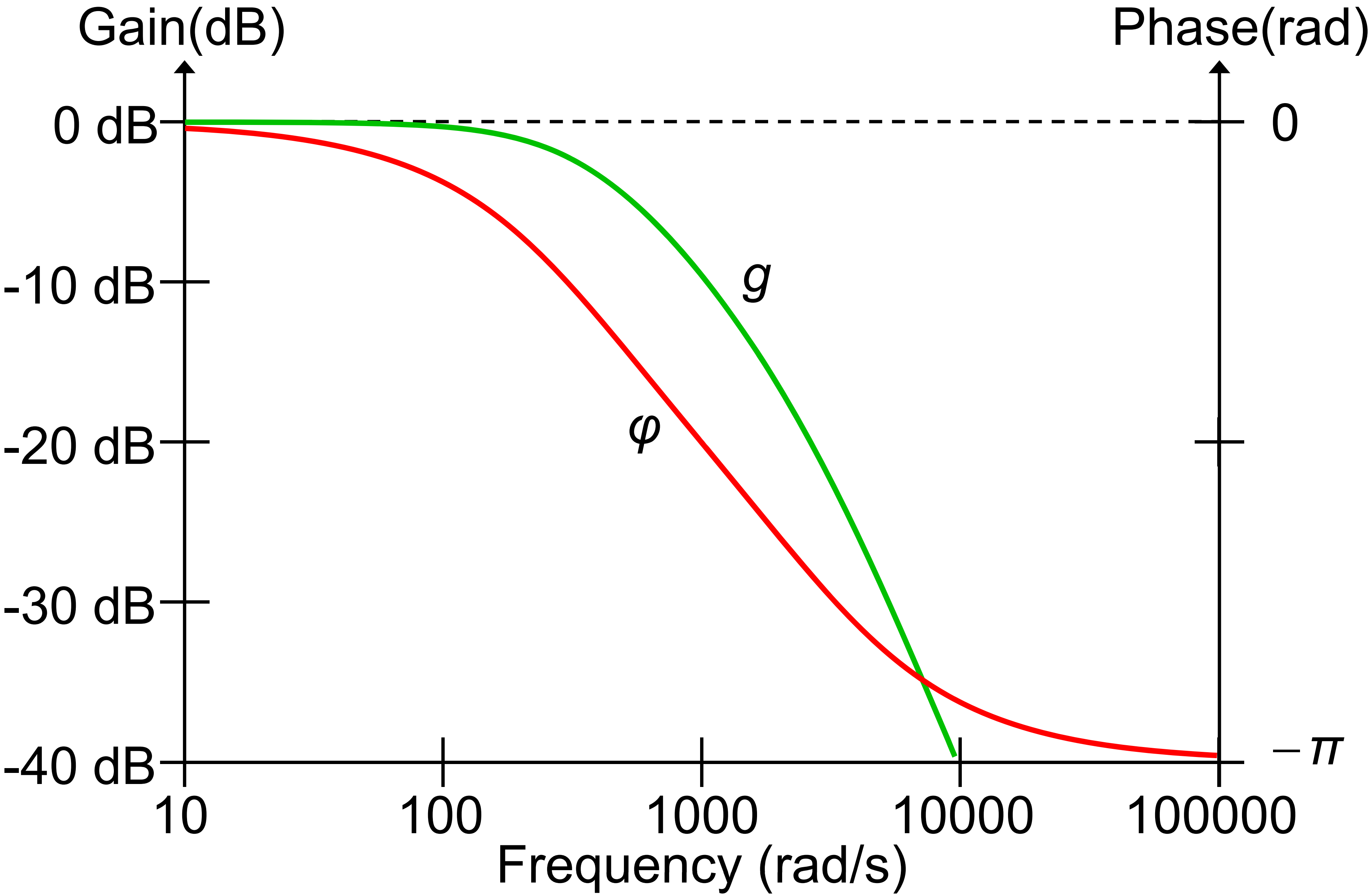
$R$ = 1 kΩ, $C$ = 1μF:

$$H(j\omega) = \frac{1}{1+j\omega\times 3\times 10^{-3} - \omega^2 \times 10^{-6}}$$

# Example 1

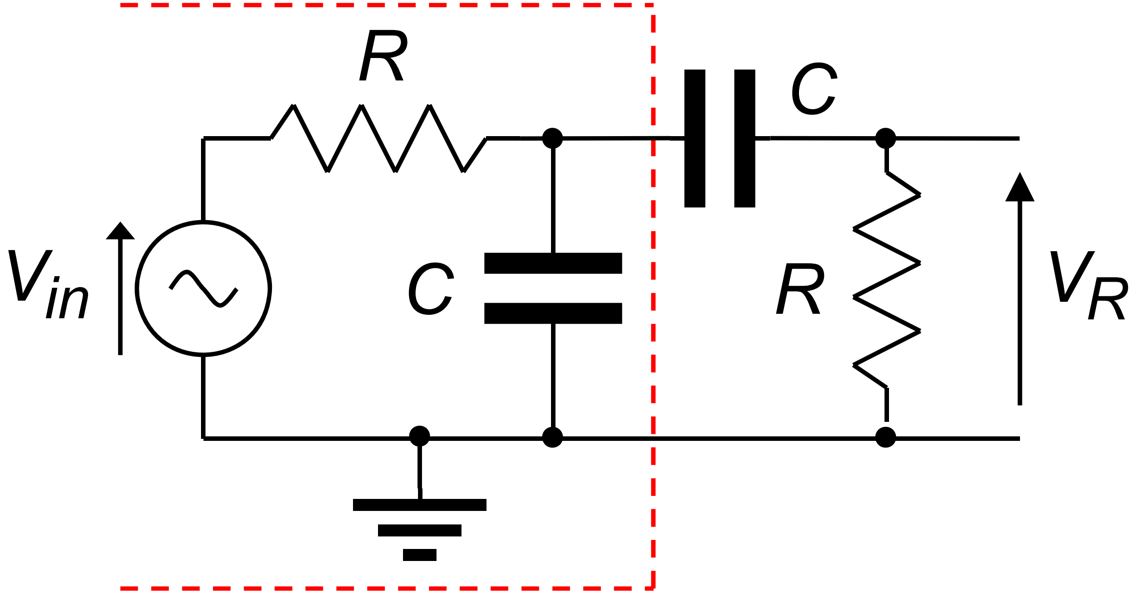


# Bode Plot



Gain(dB)
Phase(rad)
0 dB
-10 dB
-20 dB
-30 dB
-40 dB
0
$-\pi$
g
φ
10
100
1000
10000
100000
Frequency (rad/s)

# Example 2

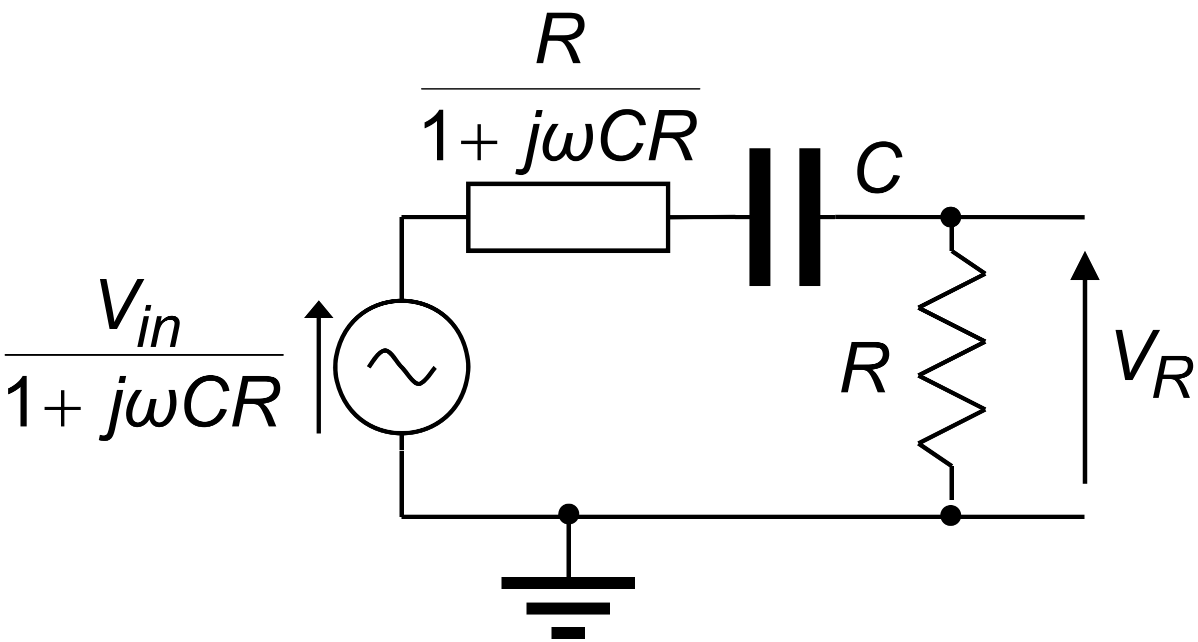



This circuit must be simplified before the frequency response function can be determined

A Thévenin equivalent circuit is created of the components to the left of the red line

# Example 2






$$V_R = \frac{V_{in}}{1+j\omega CR} \times \frac{R}{R + 1/j\omega C + \dfrac{R}{1+j\omega CR}}$$

$$= \frac{V_{in}}{1+j\omega CR} \times \frac{j\omega CR}{j\omega CR + 1 + \dfrac{j\omega CR}{1+j\omega CR}}$$

# Example 2

Frequency-response function:

$$V_R = \frac{V_{in}}{1 + j\omega CR} \times \frac{j\omega CR}{1 + j\omega CR + \dfrac{j\omega CR}{1 + j\omega CR}}$$

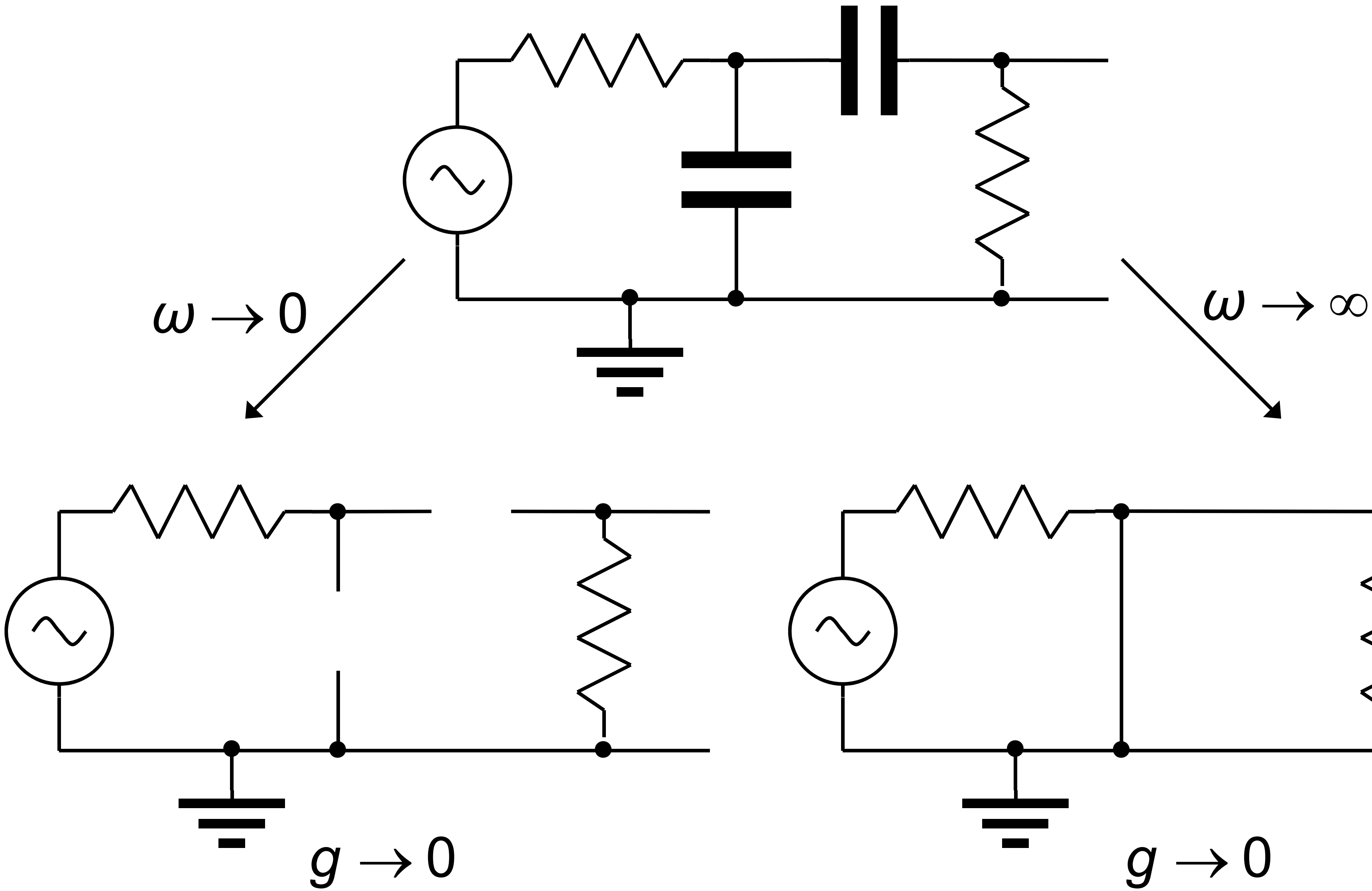
$$= \frac{V_{in}\, j\omega CR}{(1 + j\omega CR) \times (1 + j\omega CR) + j\omega CR}$$

$$H(j\omega) = \frac{j\omega CR}{1 + 3\, j\omega CR - \omega^2 C^2 R^2}$$

$R$ = 1 kΩ, $C$ = 1μF:

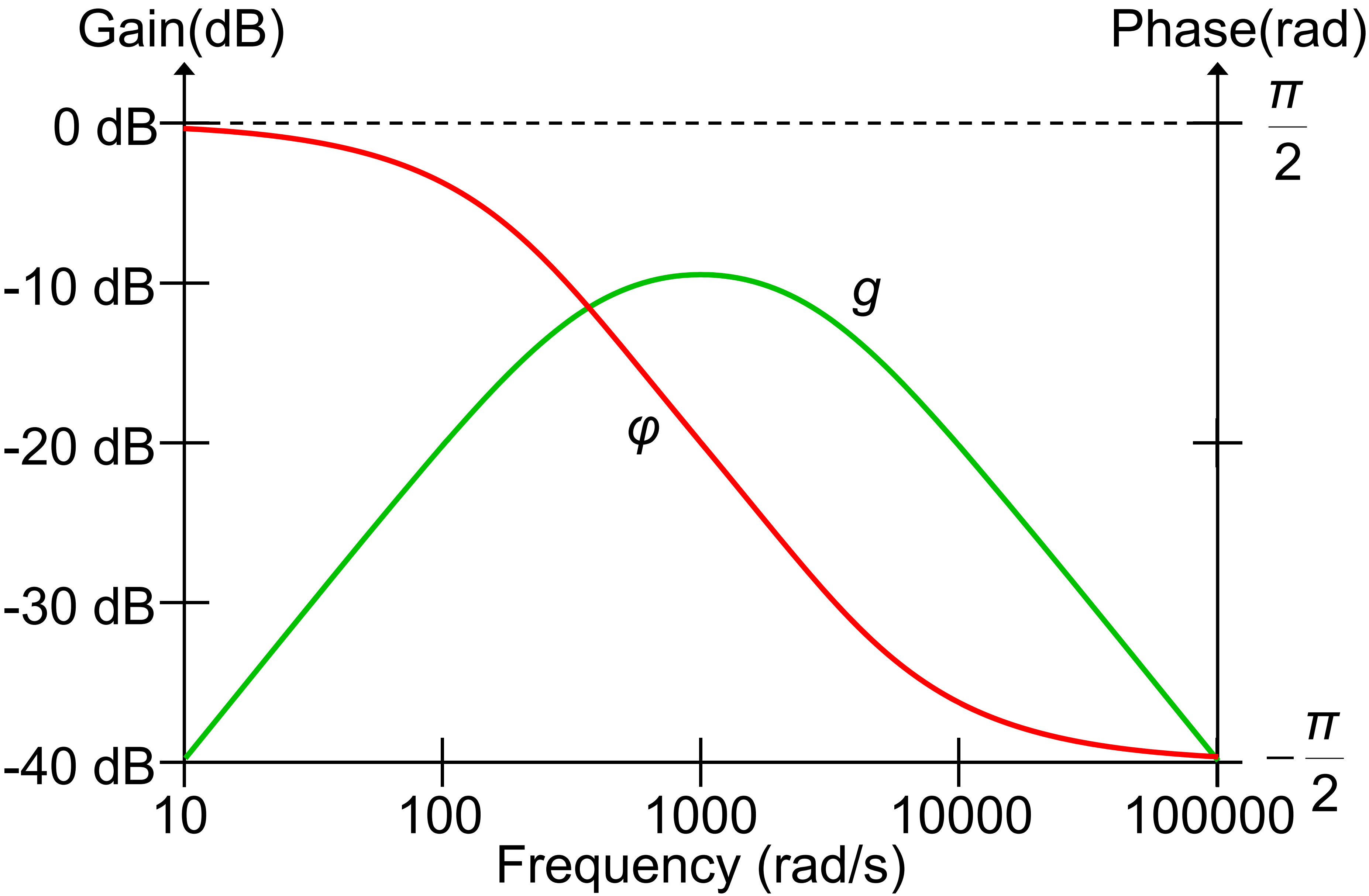
$$H(j\omega) = \frac{j\omega \times 10^{-3}}{1 + j\omega \times 3 \times 10^{-3} - \omega^2 \times 10^{-6}}$$

# Example 2



# Bode Plot

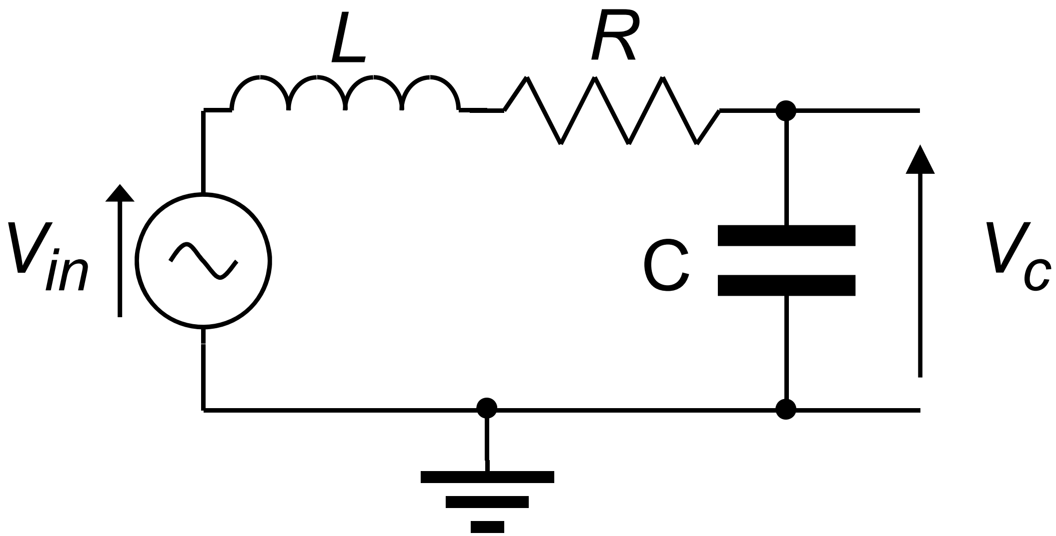
University of Reading



# Example 3



Using the potential divider formula:




$$\frac{V_C}{V_{in}} = \frac{1/j\omega C}{1/j\omega C + j\omega L + R}$$

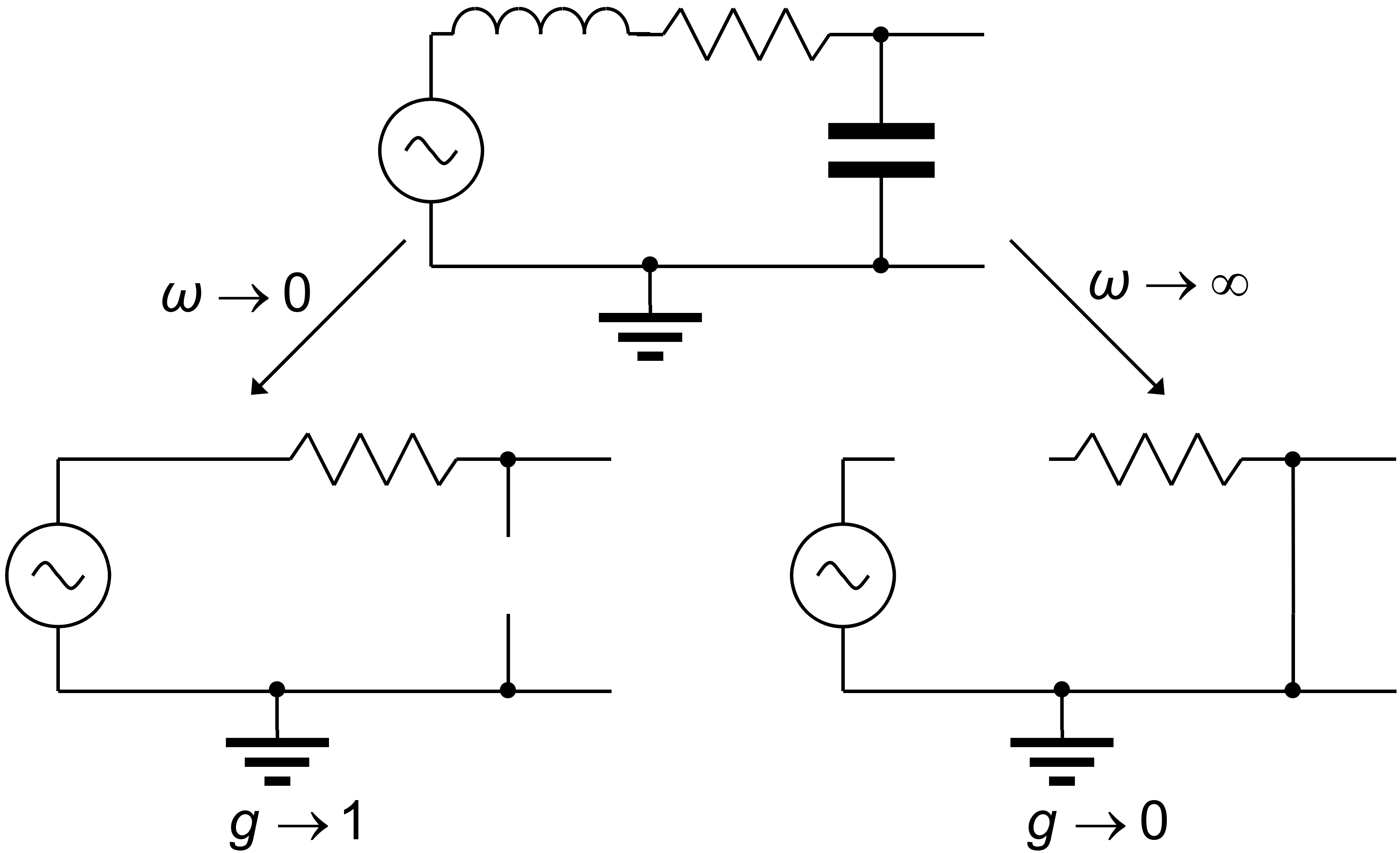
$$H(j\omega) = \frac{1}{1 + j\omega CR - \omega^2 LC}$$

$$= \frac{1}{1 + j\omega/(\omega_0 Q) - \omega^2/\omega_0^2}$$

where: $\omega_0 = \frac{1}{\sqrt{LC}}$ and: $Q = \frac{1}{\omega_0 CR} = \frac{1}{R}\sqrt{\frac{L}{C}}$

# Example 3

University of Reading



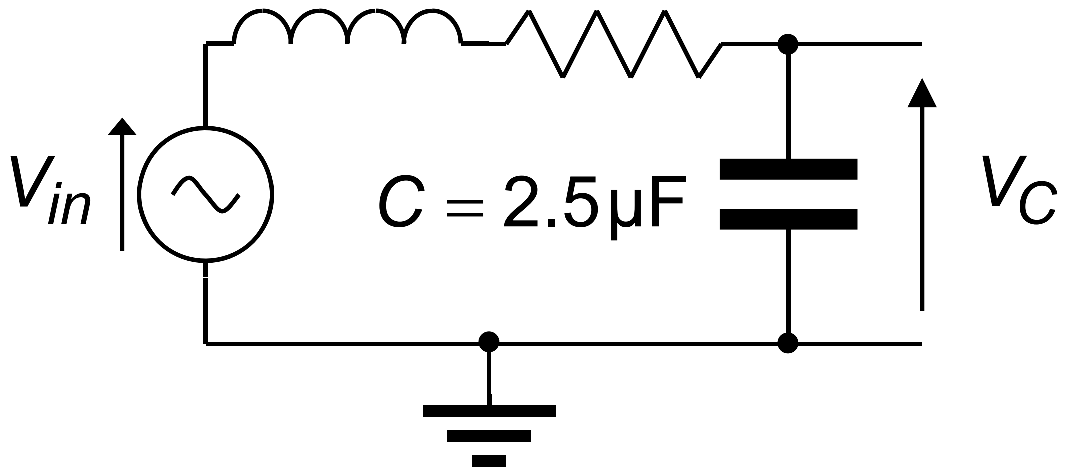
# Example 3

Circuit is a second-order low-pass filter:

| | | |
|---|---|---|
| | $H(j\omega) = \dfrac{1}{1 + j\omega/(\omega_0 Q) - \omega^2/\omega_0^2}$ | $g = \lvert H(j\omega) \rvert$ |
| $\omega << \omega_0$ | $H(j\omega) \approx 1$ | $g = 1\,(0\text{dB})$ |
| $\omega = \omega_0$ | $H(j\omega) = -jQ$ | $g = Q$ |
| $\omega >> \omega_0$ | $H(j\omega) \approx \dfrac{-\omega_0^2}{\omega^2}$ | $g \approx \dfrac{\omega_0^2}{\omega^2}\,(-12\text{dB/oct})$ |

# Example 3

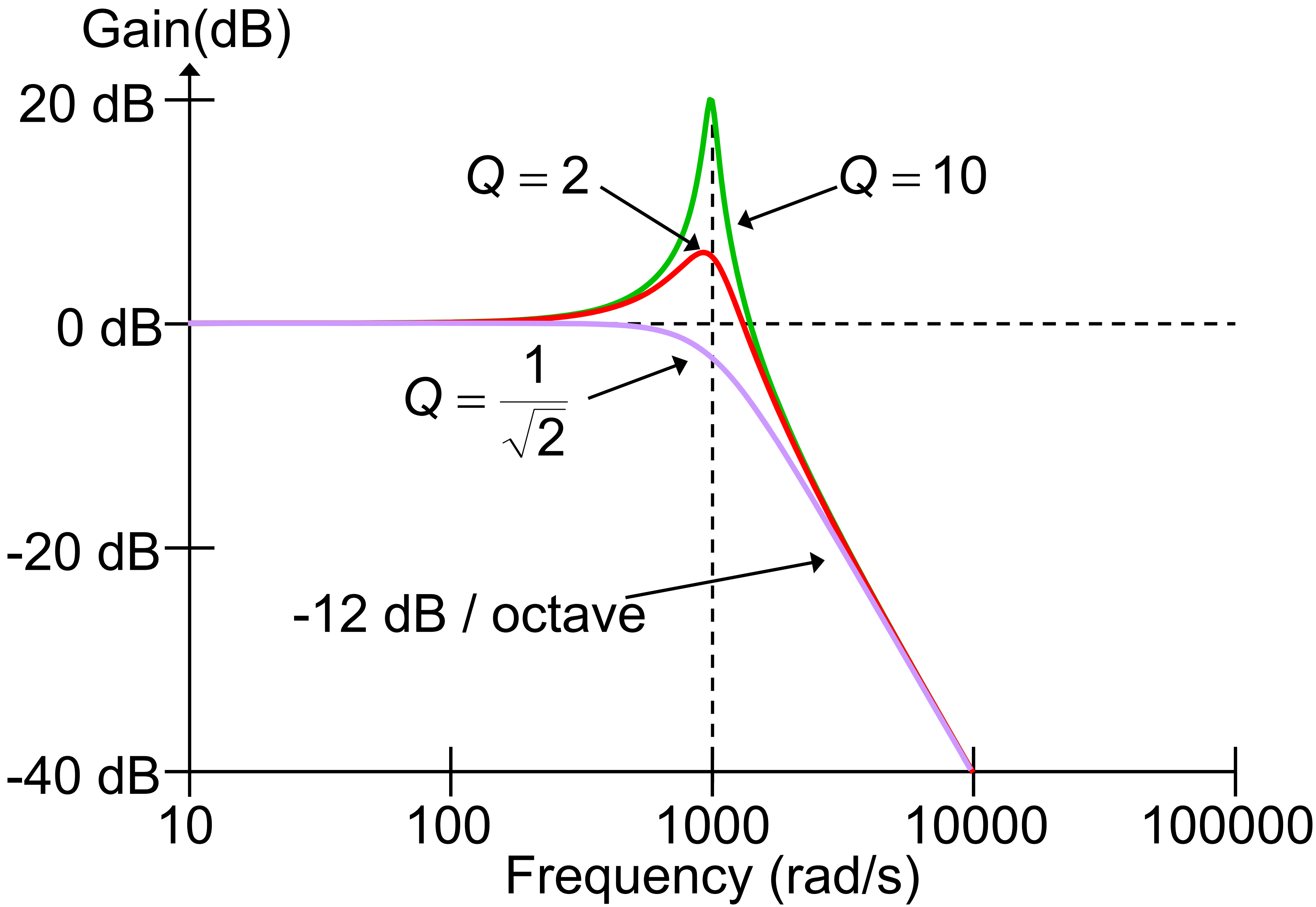





$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{400\times 10^{-3}\times 2.5\times 10^{-6}}} = \frac{1}{\sqrt{10^{-6}}} = 10^3$$

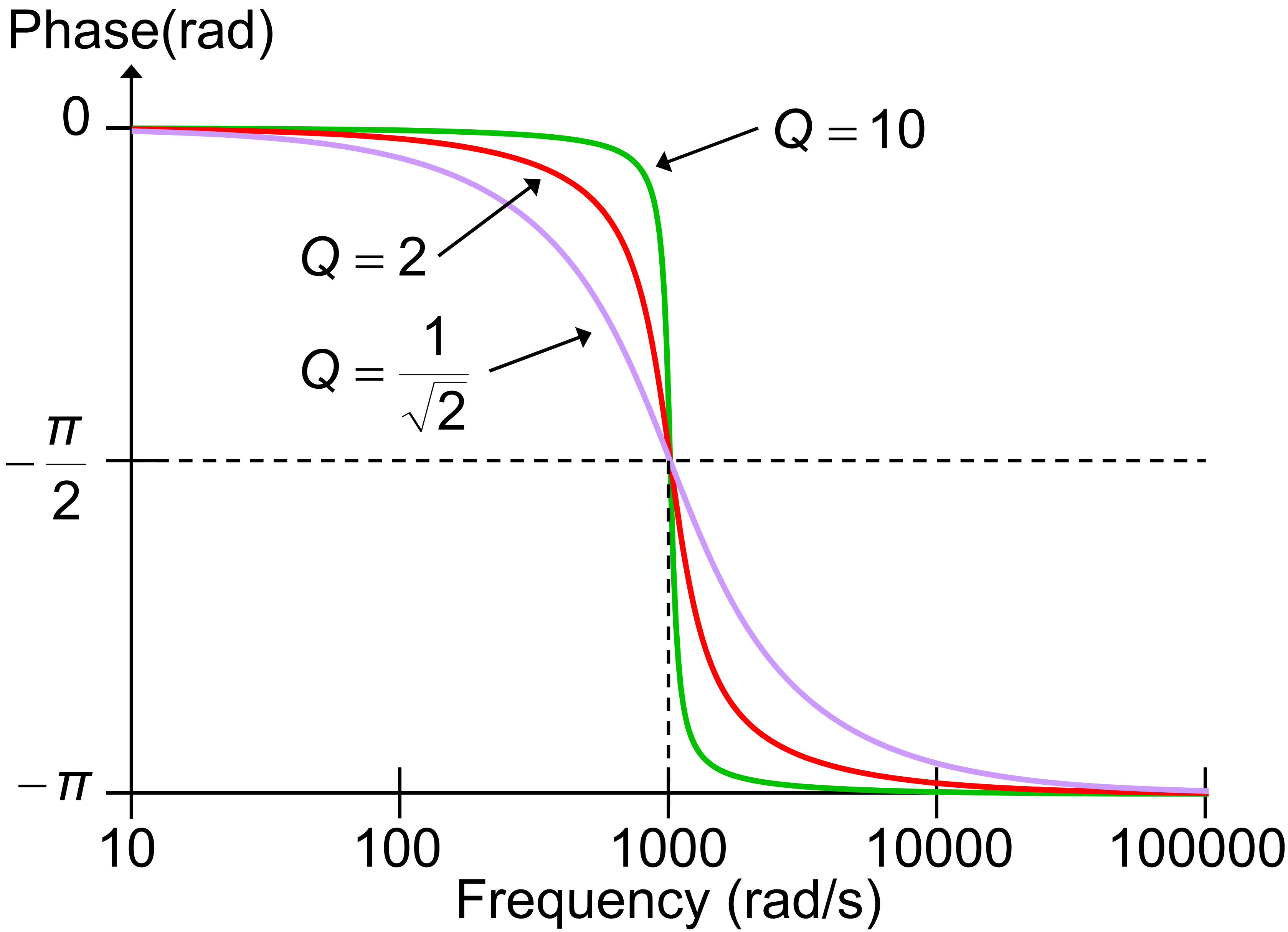
$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{200}\sqrt{\frac{400\times 10^{-3}}{2.5\times 10^{-6}}} = \frac{1}{200}\sqrt{1.6\times 10^5} = 2$$

# Bode Plot



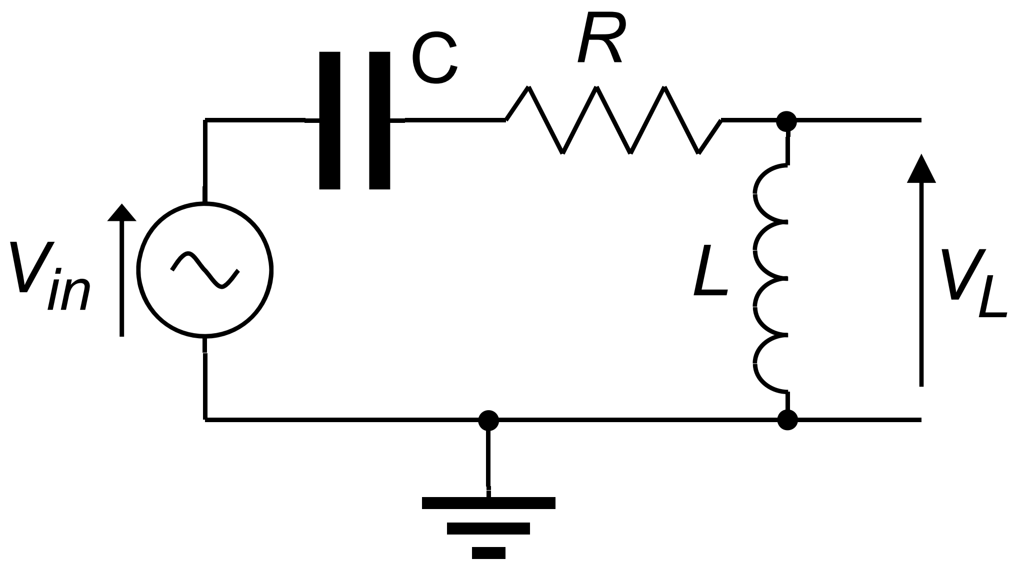
Gain(dB)
20 dB
0 dB
-20 dB
-40 dB
Q = 2
Q = 10
$Q = \frac{1}{\sqrt{2}}$
-12 dB / octave
10
100
1000
10000
100000
Frequency (rad/s)

# Bode Plot


Phase(rad)
0
$-\frac{\pi}{2}$
$-\pi$
$Q = 10$
$Q = 2$
$Q = \frac{1}{\sqrt{2}}$
10
100
1000
10000
100000
Frequency (rad/s)

# Example 4

Using the potential divider formula:



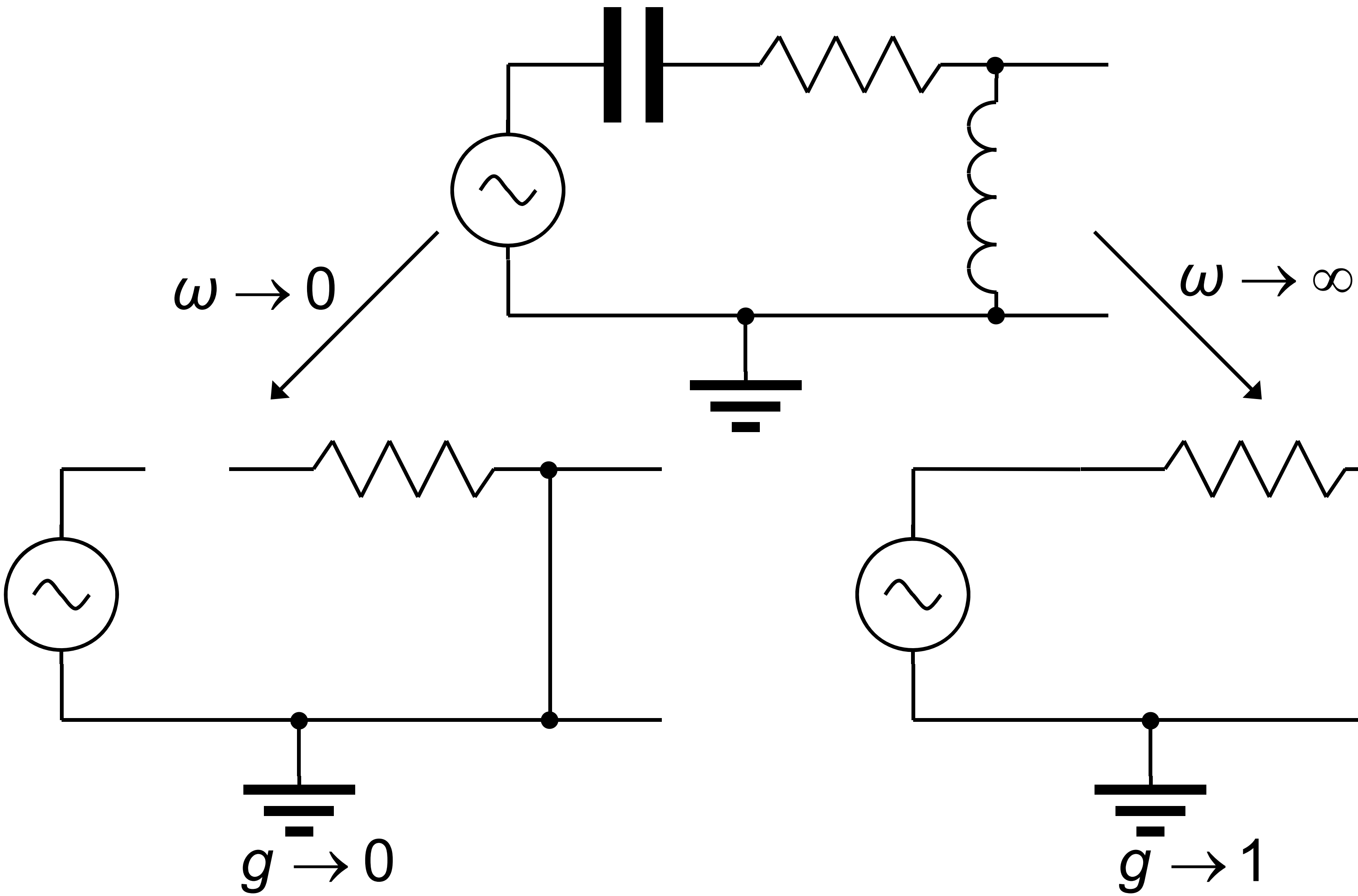

$$\frac{V_L}{V_{in}} = \frac{j\omega L}{1/j\omega C + j\omega L + R}$$

$$H(j\omega) = \frac{-\omega^2 LC}{1 + j\omega CR - \omega^2 LC}$$

$$= \frac{-\omega^2/\omega_0^2}{1 + j\omega/(\omega_0 Q) - \omega^2/\omega_0^2}$$

$$\text{where:} \quad \omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and:} \quad Q = \frac{1}{\omega_0 CR} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

# Example 4

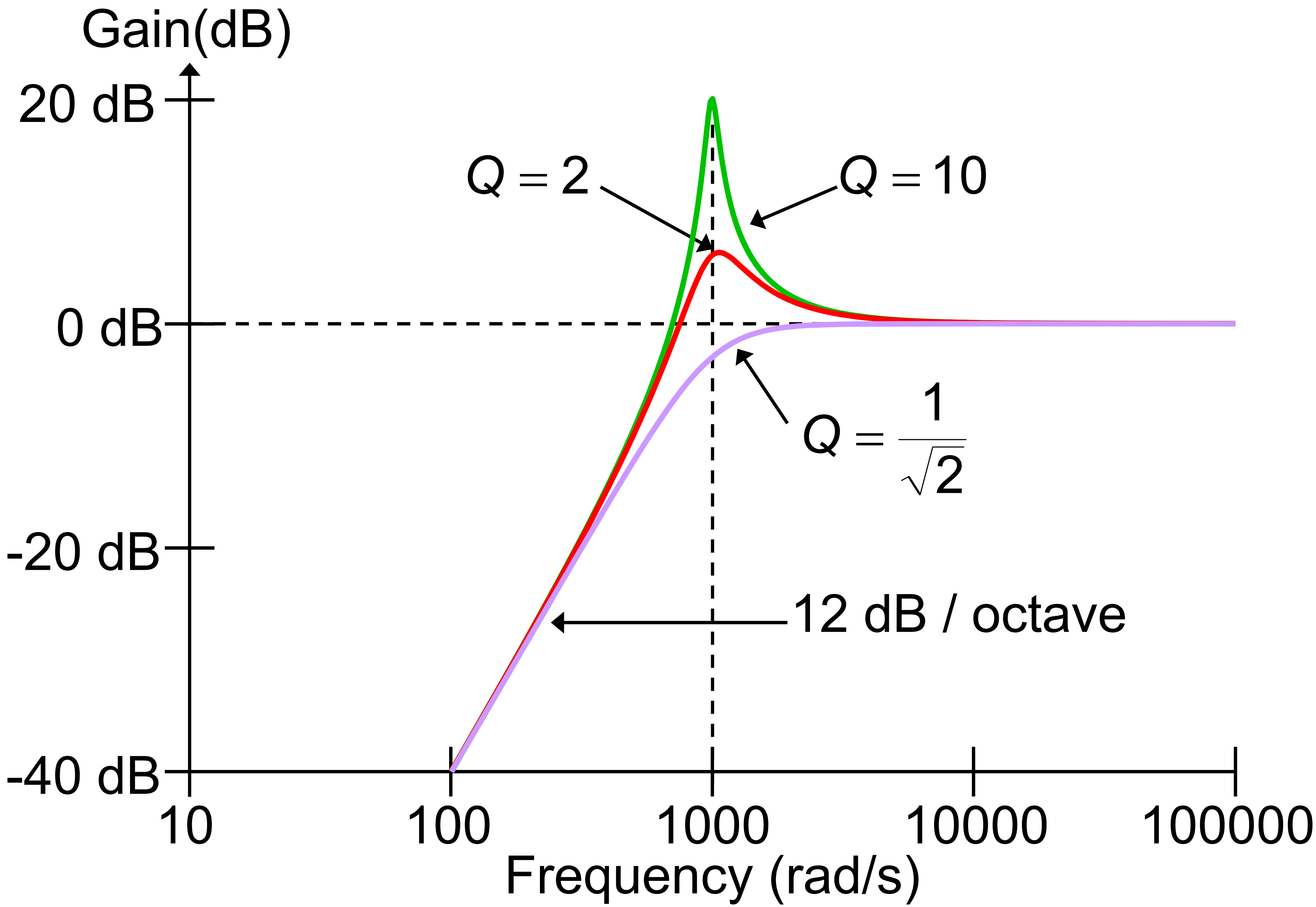


# Example 4

Circuit is a second-order high-pass filter:

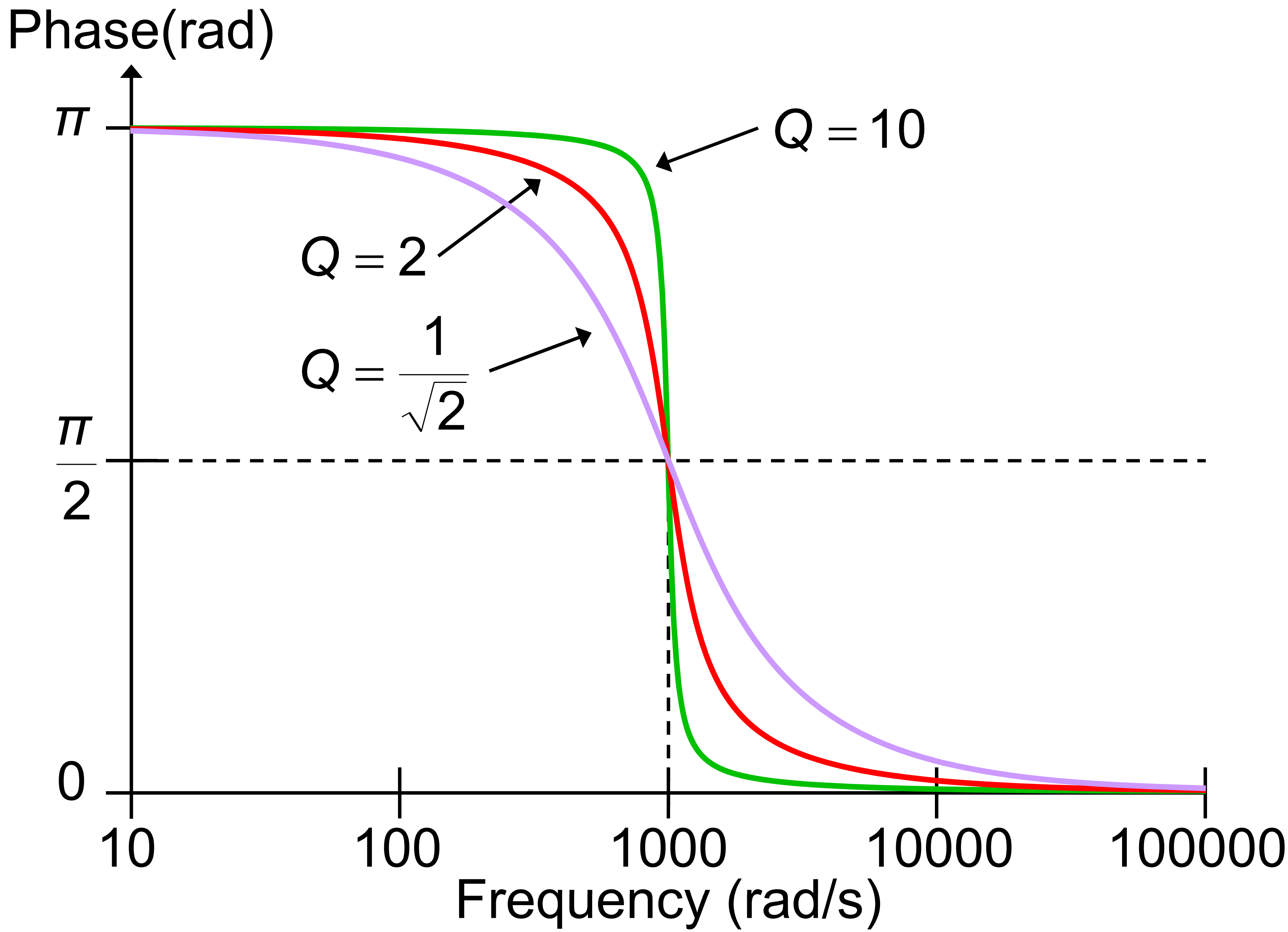
| | | |
|---|---|---|
| | $H(j\omega) = \dfrac{-\omega^2/\omega_0^2}{1 + j\omega/(\omega_0 Q) - \omega^2/\omega_0^2}$ | $g = \lvert H(j\omega) \rvert$ |
| $\omega << \omega_0$ | $H(j\omega) \approx \dfrac{-\omega^2}{\omega_0^2}$ | $g \approx \dfrac{\omega^2}{\omega_0^2}$ (12dB / oct) |
| $\omega = \omega_0$ | $H(j\omega) = jQ$ | $g = Q$ |
| $\omega >> \omega_0$ | $H(j\omega) \approx 1$ | $g = 1$ (0dB) |

# Bode Plot



Gain(dB)
20 dB
0 dB
-20 dB
-40 dB
Q = 2
Q = 10
Q = 1/√2
12 dB / octave
10
100
1000
10000
100000
Frequency (rad/s)

# Bode Plot



Phase(rad)
π
π/2
0
Q = 10
Q = 2
Q = 1/√2
10
100
1000
10000
100000
Frequency (rad/s)

# Example 5

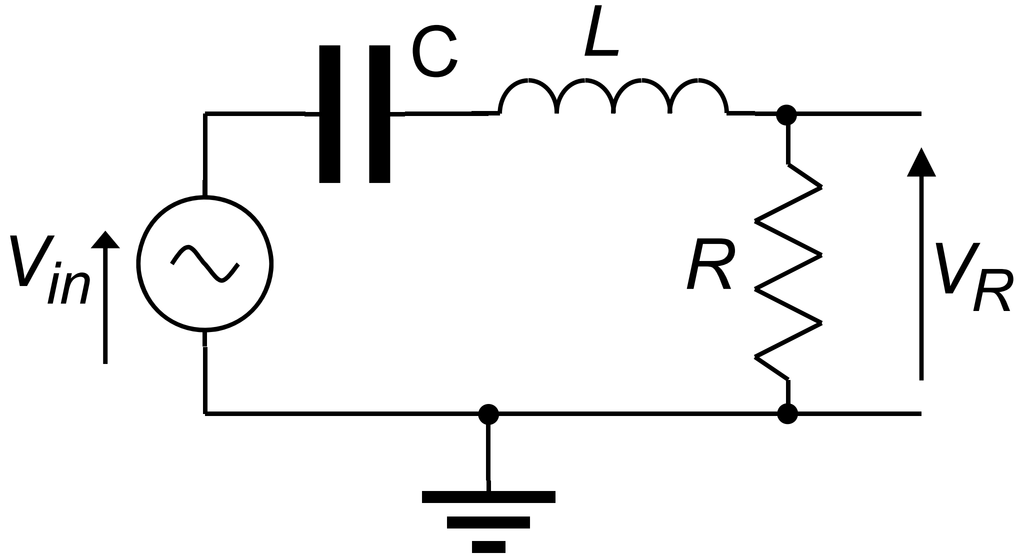


Using the potential divider formula:

$$\frac{V_R}{V_{in}} = \frac{R}{1/\,j\omega C + j\omega L + R}$$

$$H(j\omega) = \frac{j\omega CR}{1 + j\omega CR - \omega^2 LC}$$

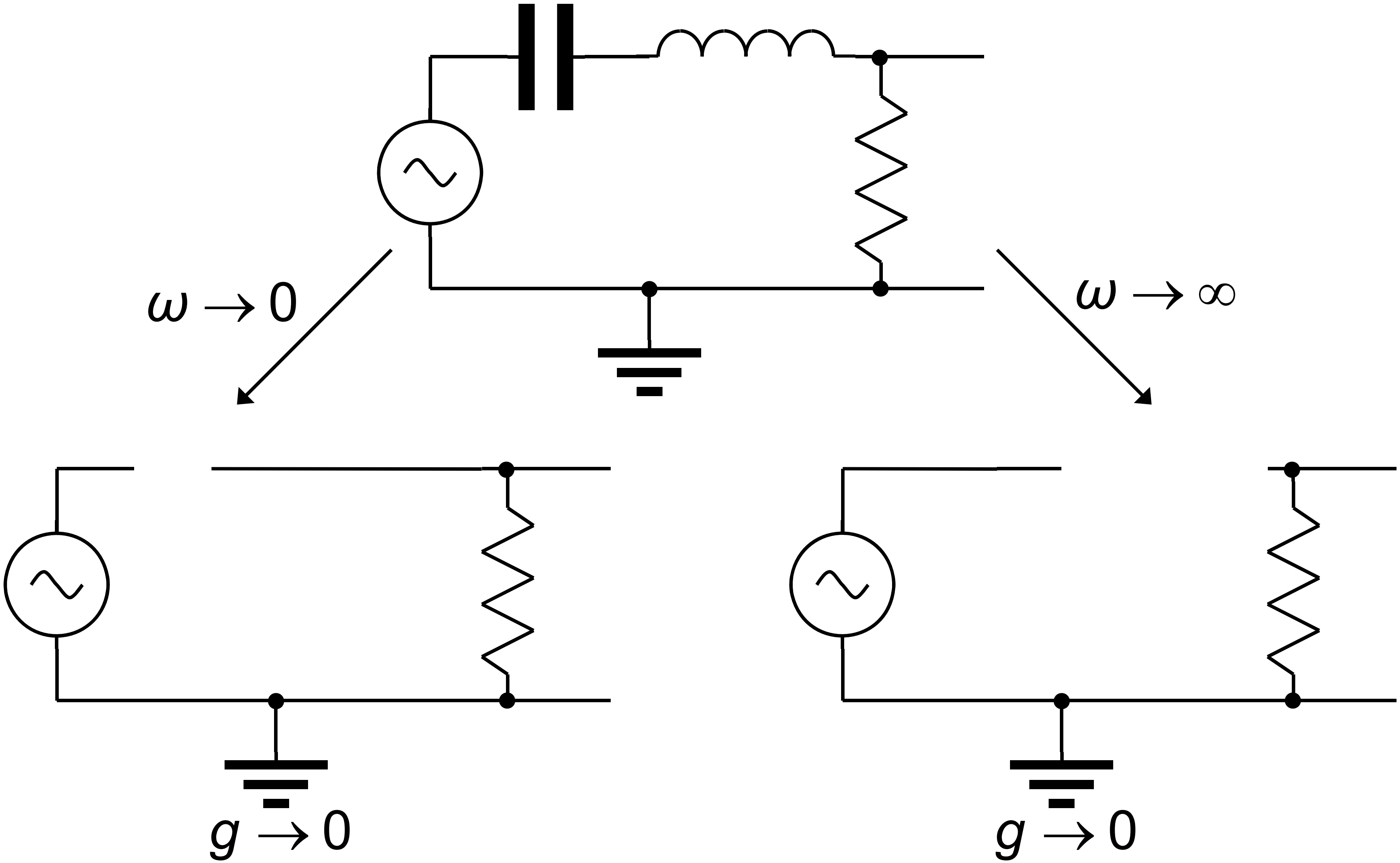
$$= \frac{j\omega/(\omega_0 Q)}{1 + j\omega/(\omega_0 Q) - \omega^2/\omega_0^2}$$




where: $\omega_0 = \dfrac{1}{\sqrt{LC}}$ and: $Q = \dfrac{1}{\omega_0 CR} = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$

# Example 5



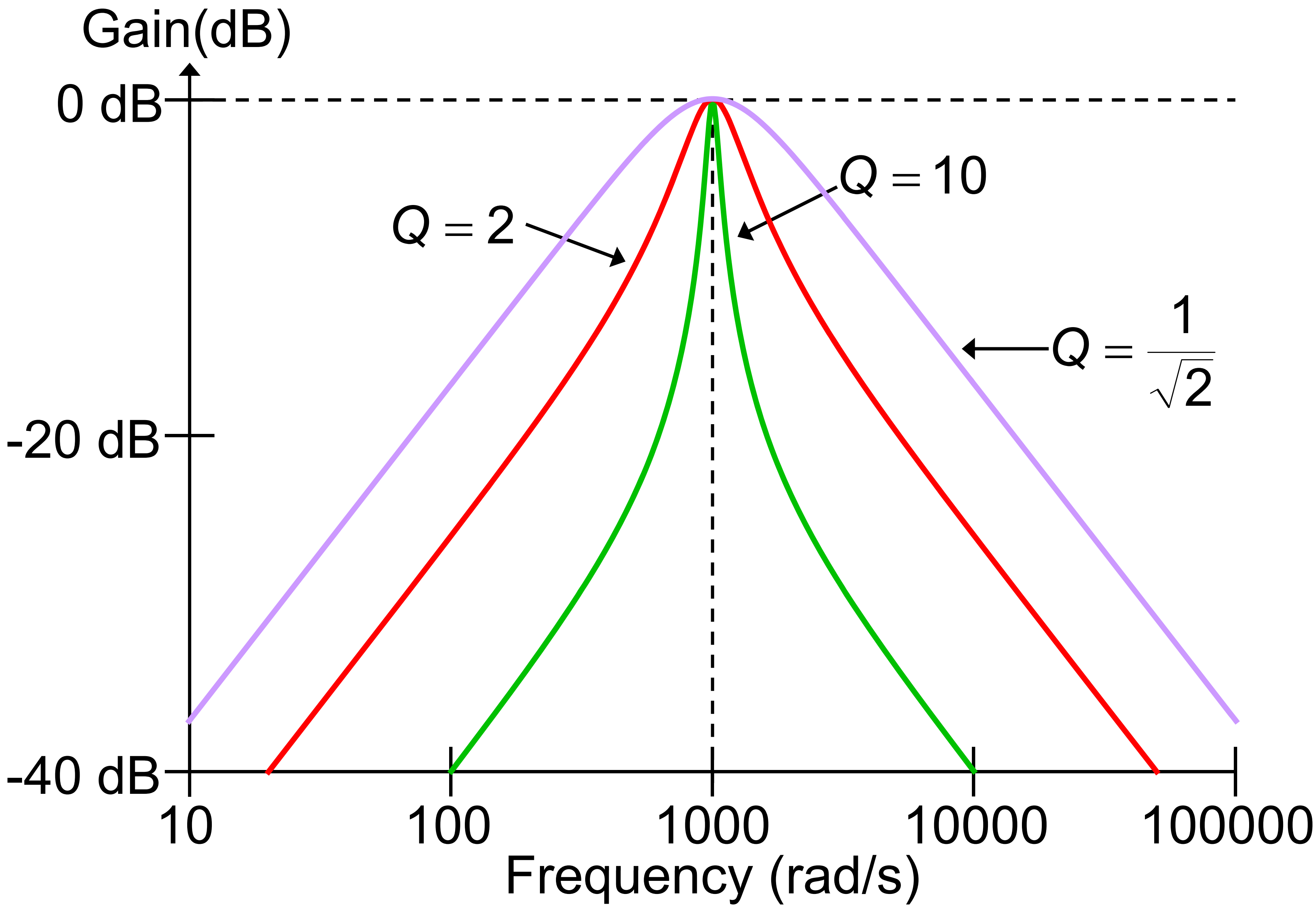


# Example 5

Circuit is a second-order band-pass filter:

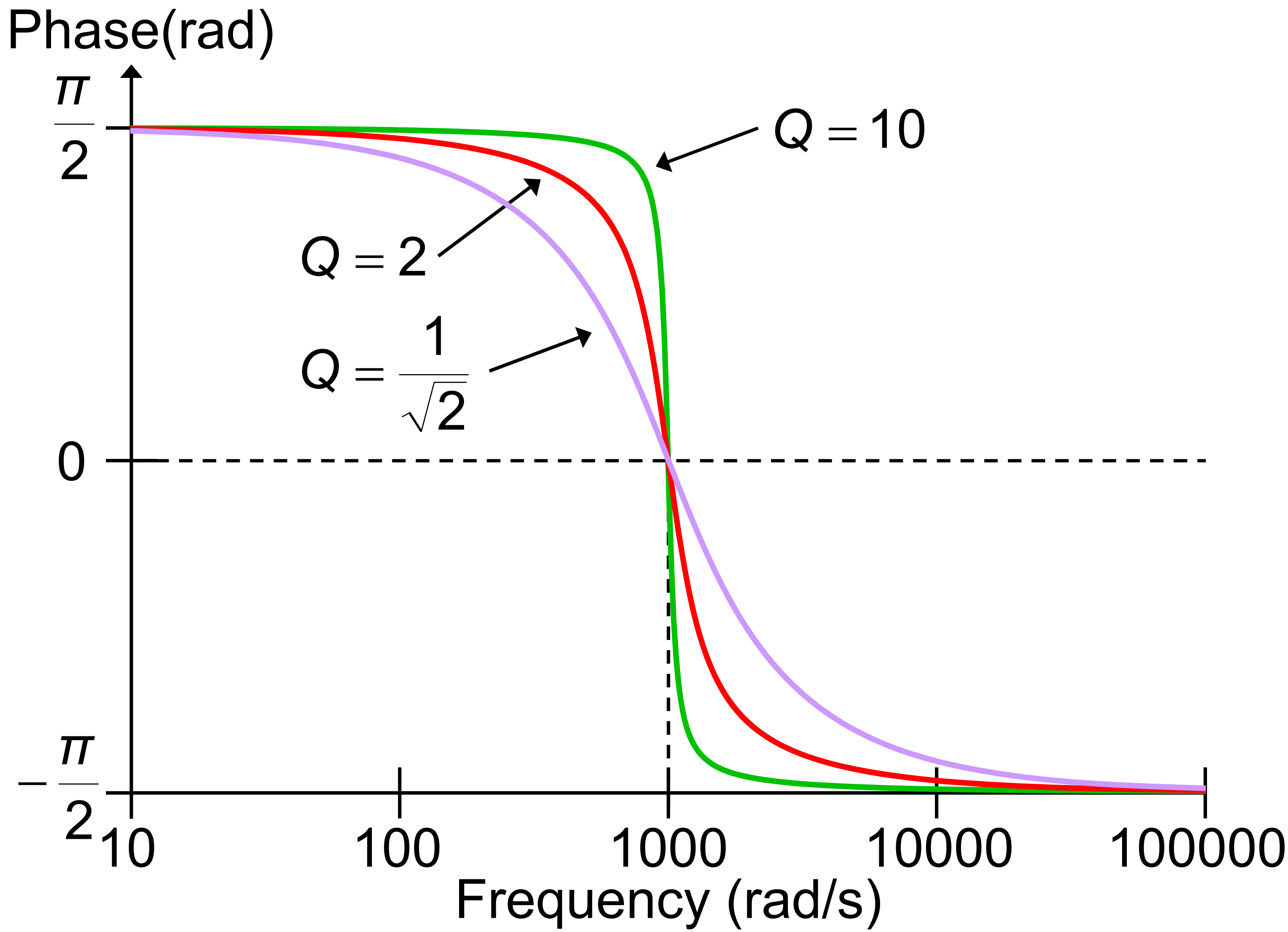
| | | |
|---|---|---|
| | $H(j\omega) = \dfrac{j\omega/(\omega_0 Q)}{1 + j\omega/(\omega_0 Q) - \omega^2/\omega_0^2}$ | $g = \lvert H(j\omega) \rvert$ |
| $\omega << \omega_0$ | $H(j\omega) \approx \dfrac{j\omega}{\omega_0 Q}$ | $g \approx \dfrac{\omega}{\omega_0 Q}$ (6dB/oct) |
| $\omega = \omega_0$ | $H(j\omega) = 1$ | $g = 1$ (0dB) |
| $\omega >> \omega_0$ | $H(j\omega) \approx \dfrac{-j\omega_0}{\omega Q}$ | $g \approx \dfrac{\omega_0}{\omega Q}$ (–6dB/oct) |

# Bode Plot



Gain(dB)
0 dB
-20 dB
-40 dB
$Q = 2$
$Q = 10$
$Q = \frac{1}{\sqrt{2}}$
10
100
1000
10000
100000
Frequency (rad/s)

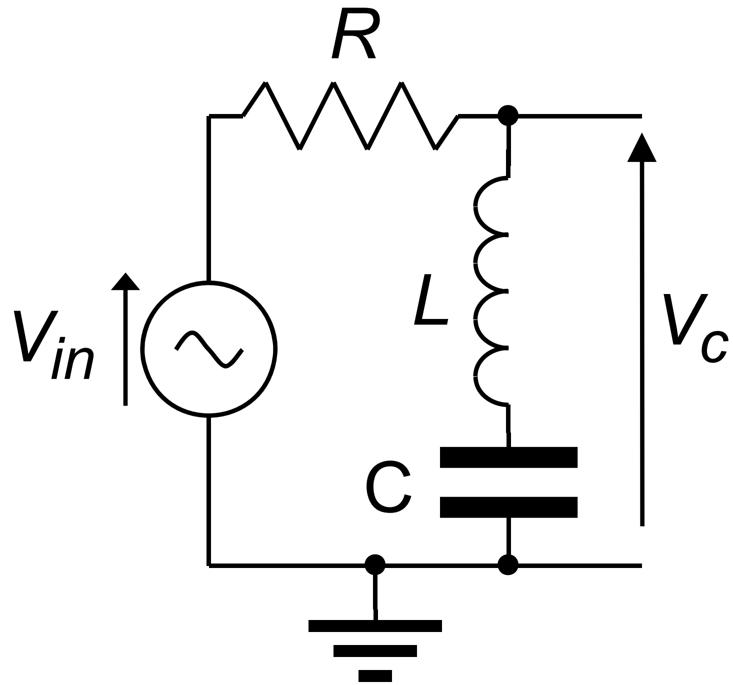
# Bode Plot



Phase(rad)
$\frac{\pi}{2}$
0
$-\frac{\pi}{2}$
$Q = 10$
$Q = 2$
$Q = \frac{1}{\sqrt{2}}$
10
100
1000
10000
100000
Frequency (rad/s)

# Example 6

Using the potential divider formula:



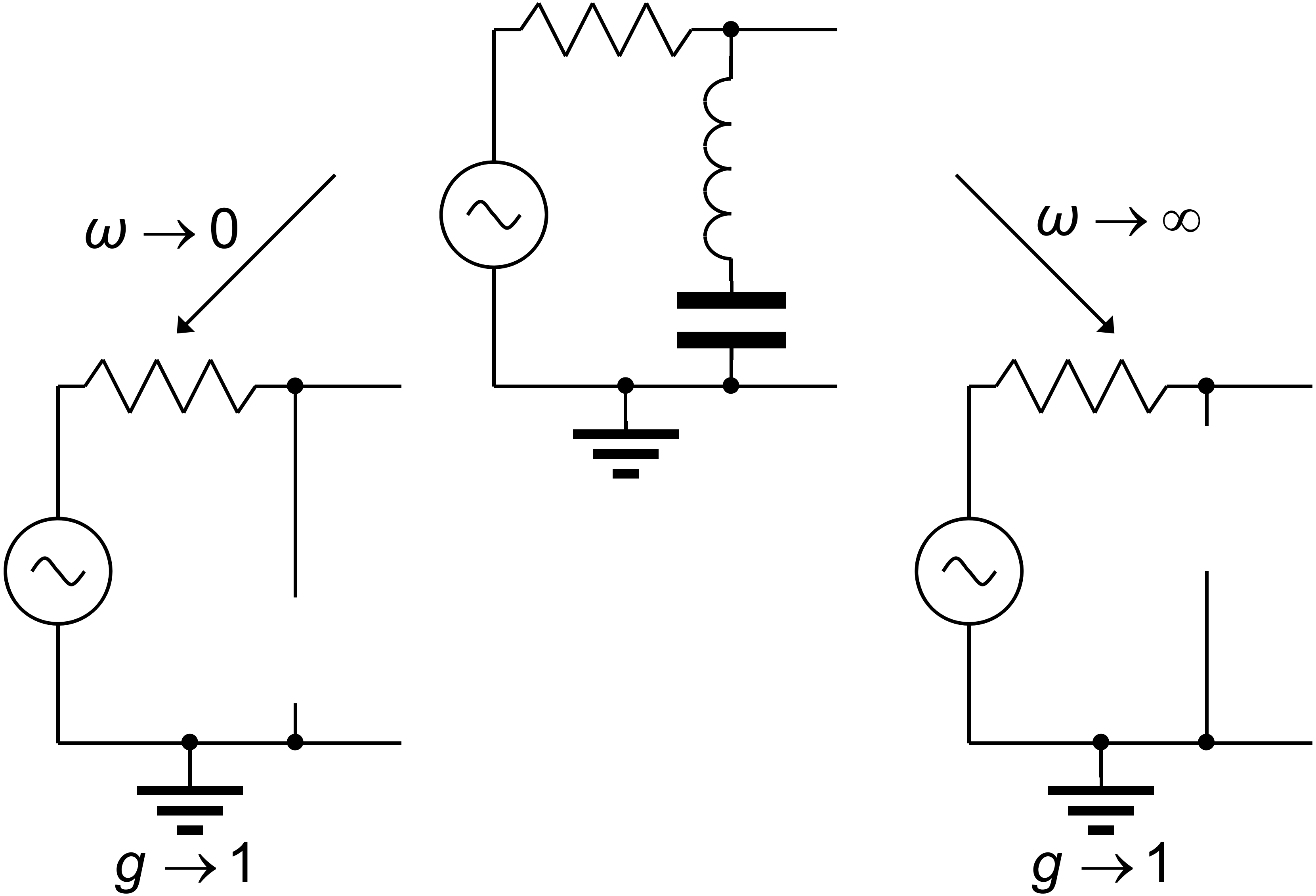

$$\frac{V_C}{V_{in}} = \frac{1/\,j\omega C + j\omega L}{1/\,j\omega C + j\omega L + R}$$

$$H(j\omega) = \frac{1 - \omega^2 LC}{1 + j\omega CR - \omega^2 LC}$$

$$= \frac{1 - \omega^2 LC}{1 + j\omega/(\omega_0 Q) - \omega^2/\omega_0^2}$$

where : $\omega_0 = \frac{1}{\sqrt{LC}}$ and : $Q = \frac{1}{\omega_0 CR} = \frac{1}{R}\sqrt{\frac{L}{C}}$

# Example 6

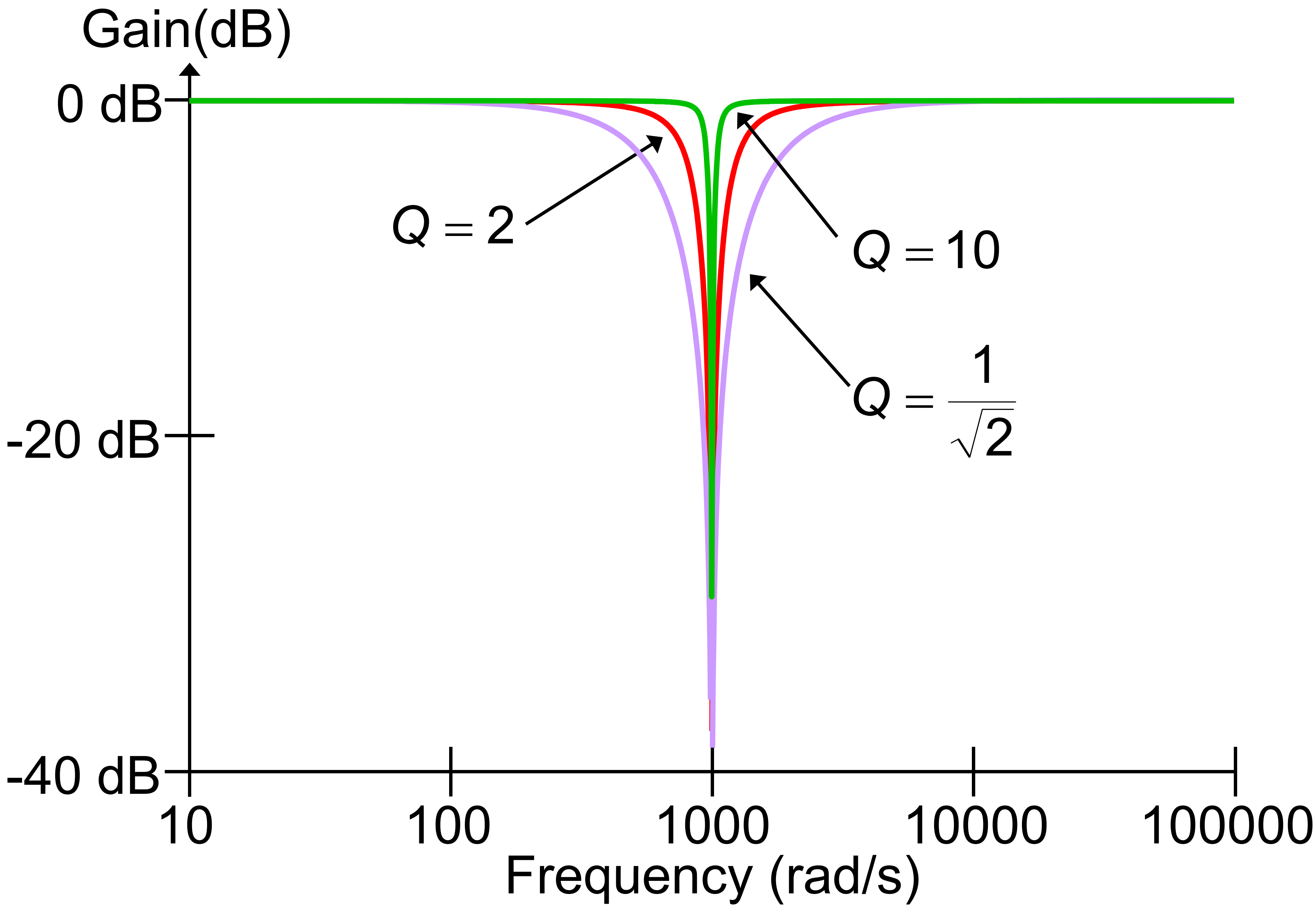


# Example 6

Circuit is a second-order band-stop filter:

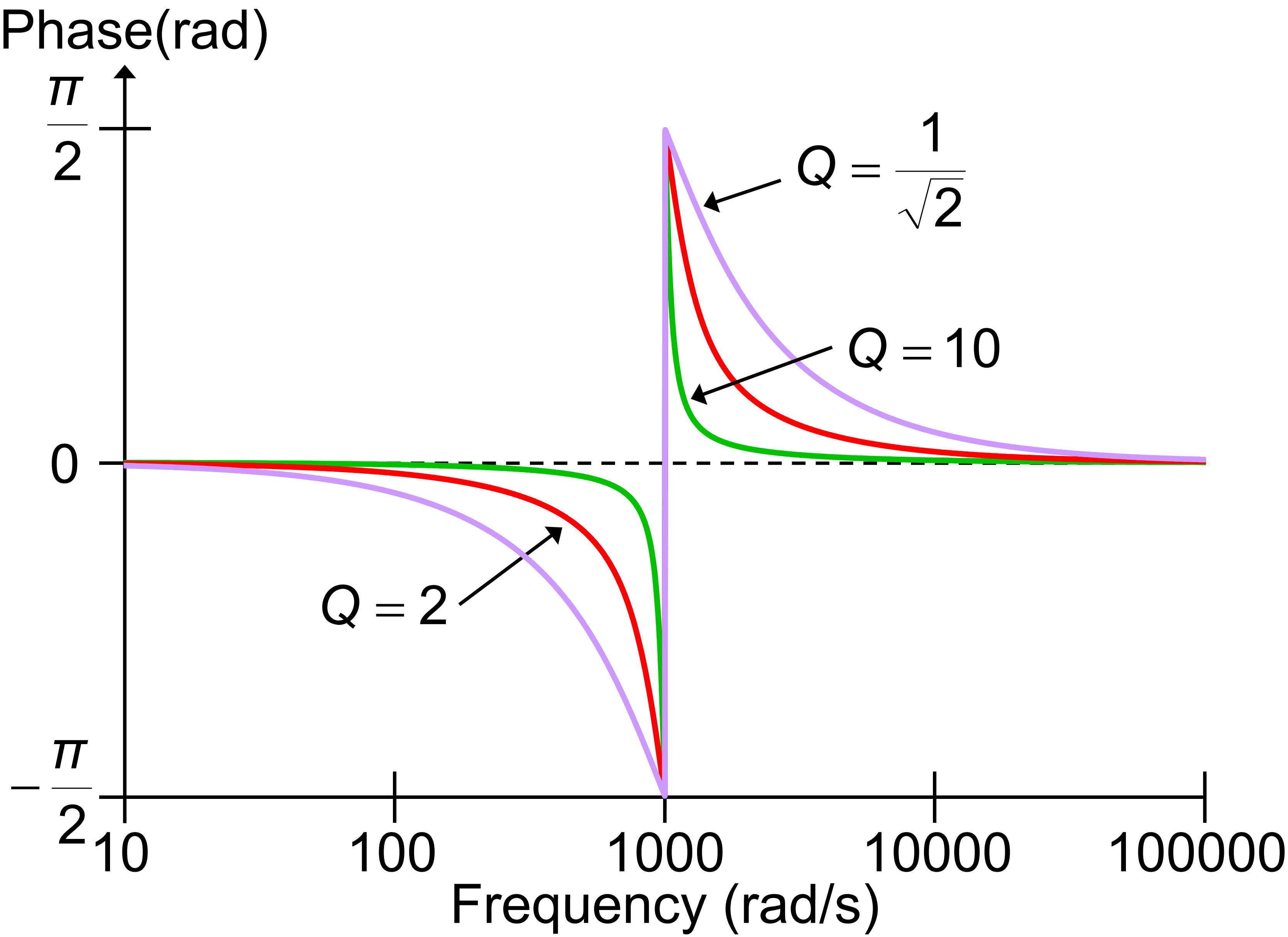
| | | |
|---|---|---|
| | $H(j\omega) = \dfrac{1-\omega^2/\omega_0^2}{1+j\omega/(\omega_0 Q)-\omega^2/\omega_0^2}$ | $g = \lvert H(j\omega)\rvert$ |
| $\omega << \omega_0$ | $H(j\omega) \approx 1$ | $g \approx 1\,(0\,\text{dB})$ |
| $\omega = \omega_0$ | $H(j\omega) = 0$ | $g = 0\;(-\infty\,\text{dB})$ |
| $\omega >> \omega_0$ | $H(j\omega) \approx 1$ | $g \approx 1\,(0\text{dB})$ |

# Bode Plot



Gain(dB)
0 dB
-20 dB
-40 dB
$Q = 2$
$Q = 10$
$Q = \frac{1}{\sqrt{2}}$
10
100
1000
10000
100000
Frequency (rad/s)

# Bode Plot



Phase(rad)
$\frac{\pi}{2}$
0
$-\frac{\pi}{2}$
$Q = \frac{1}{\sqrt{2}}$
$Q = 10$
$Q = 2$
10
100
1000
10000
100000
Frequency (rad/s)

# Lecture 8

## Power in AC Circuits

# Power in AC Circuits

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To calculate the power in a circuit we shall need to make use of some trigonometric identities:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

Adding:

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

$$\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$$

so that:

$$\cos^2 A = \frac{1}{2}\{\cos 2A + \cos 0\} = \frac{1}{2} + \frac{1}{2}\cos 2A$$

# rms Voltages and Currents

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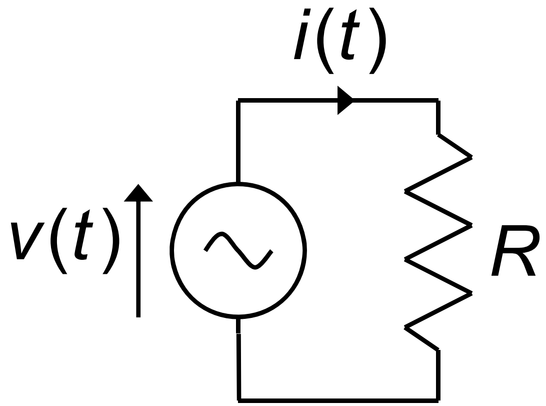
The average power in a resistor is given by:

$$P = \frac{1}{T}\int_0^T v(t)i(t)dt$$

$$= \frac{1}{T}\int_0^T \frac{v^2(t)}{R}dt$$

$$= \frac{1}{R}\frac{1}{T}\int_0^T v^2(t)dt$$

$$= \frac{V_{rms}^2}{R} \quad \text{where :} \quad V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt}$$



# rms Voltages and Currents

The root-mean-square voltage $V_{rms}$ determines the power dissipated in a circuit:

$$P = \frac{V_{rms}^2}{R}$$

There is a similar expression for the power dissipated when a current $I_{rms}$ flows through a circuit:

$$P = RI_{rms}^2$$

These expressions apply to any waveform

# rms Voltages and Currents

The rms value of a sinusoid of amplitude (peak) value $v_0$:

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$

$$= \sqrt{\frac{1}{T}\int_0^T v_0^2 \cos^2(\omega t)\,dt}$$

$$= \sqrt{v_0^2 \frac{1}{T}\int_0^T \frac{1}{2} + \frac{1}{2}\cos(2\omega t)\,dt}$$

$$= \sqrt{\frac{v_0^2}{2}} = \frac{v_0}{\sqrt{2}}$$

Averages to zero over a complete cycle:
$T = 2\pi/\omega$

# rms Voltages and Currents

The UK mains power was until recently supplied at 240 V rms and that in Europe 220 V rms

On 1 January 1995 the nominal voltage across Europe was harmonised at 230 V rms.

This corresponds to an amplitude of:

$$
\begin{aligned}
v_0 &= \sqrt{2} \times V_{rms} \\
&= \sqrt{2} \times 230 \\
&= 325 \text{ V}
\end{aligned}
$$

# rms Voltages and Currents

A mains power (230 V rms) electric fire has a resistance of 52 Ω:

$$P = \frac{V_{rms}^2}{R} = \frac{230^2}{52} = 1.017\ \text{kW}$$

An audio amplifier which drives a 4 Ω loudspeaker at up to 150 W must supply a sinusoidal output voltage:

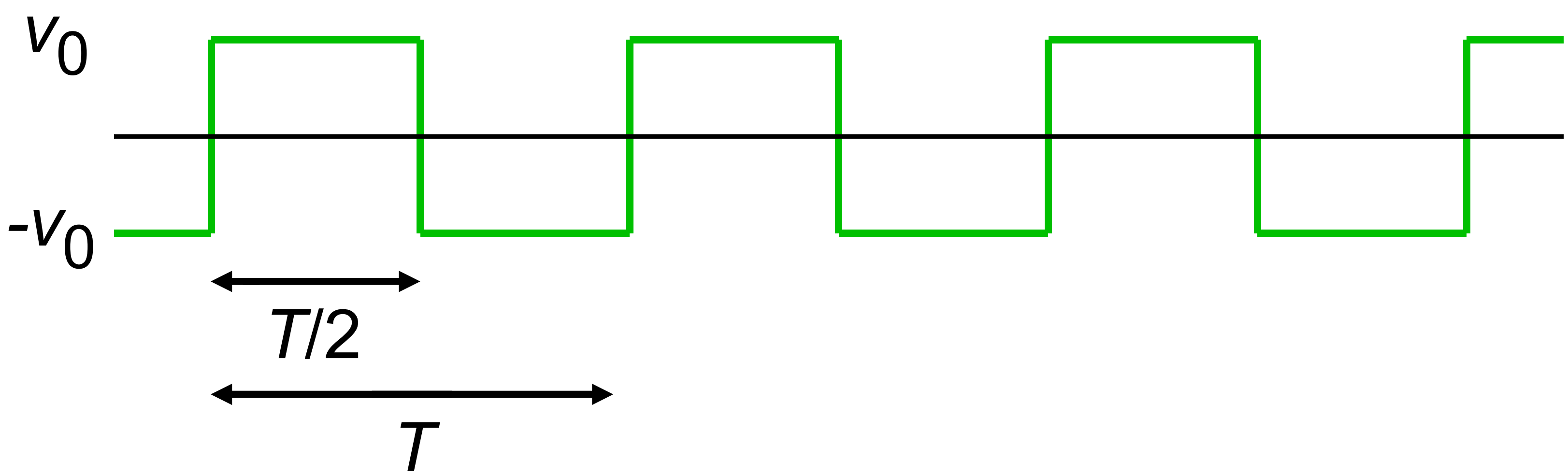
$$V_{rms}^2 = P.R = 150 \times 4 = 600$$

$$V_{rms} = 24.5\ \text{V}$$

This corresponds to a sinusoid of peak value 34.6 V

# rms Voltages and Currents

Square wave of amplitude $\pm v_0$:




$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt} = \sqrt{\frac{1}{T}\int_0^{T/2} v_0^2\,dt} + \sqrt{\frac{1}{T}\int_{T/2}^{T} (-v_0)^2\,dt}$$

$$= \sqrt{v_0^2 \frac{1}{T}\int_0^T dt} = \sqrt{v_0^2} = v_0$$

# Crest Factor

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The ratio between the peak voltage and the rms voltage is known as the crest factor:

$$cf = \frac{V_{peak}}{V_{rms}}$$

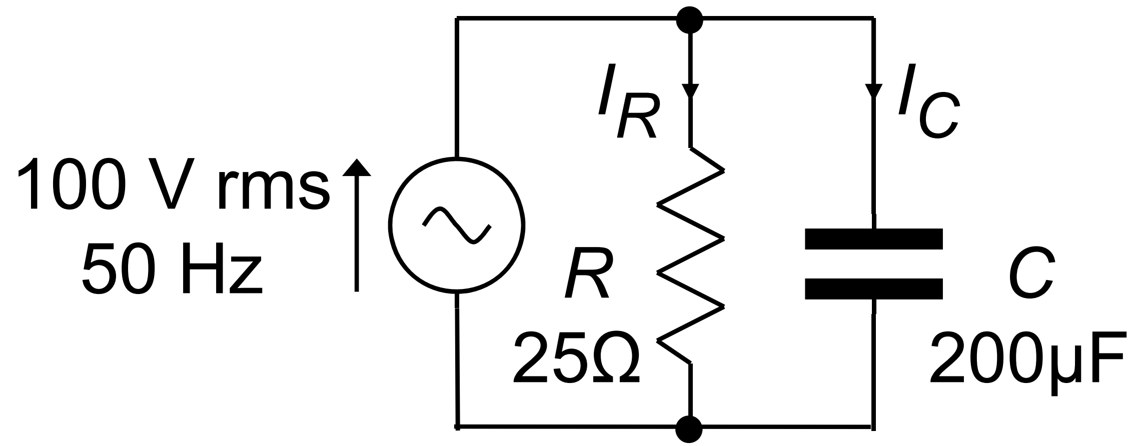
For a sinusoid the crest factor is √2; for a square wave the crest factor is 1

For audio signals the crest factor depends on the source but is commonly 2 or higher

150 W of audio into 4 Ω loudspeakers would therefore require peak voltages of 50 V or greater

# Power in a Reactive Load

Capacitors and inductors store energy, but do not dissipate power




$$I_R = \frac{100}{25} = 4 \text{ A}$$

$$I_C = \frac{100}{|Z_C|} = 100 \times 2\pi \times 50 \times 200 \times 10^{-6} \text{ A} = 6.28 \text{ A}$$

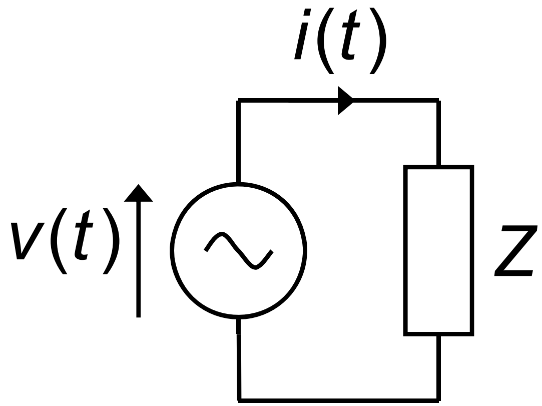
$$P = \frac{100^2}{25} = 400 \text{ W}$$

# Instantaneous Power

For sinusoidal voltages and currents:

$$v(t) = v_0 \cos(\omega t)$$

$$i(t) = i_0 \cos(\omega t + \varphi)$$




Instantaneous power:

$$\begin{aligned} p(t) &= v(t) \times i(t) \\ &= v_0 \cos(\omega t) i_0 \cos(\omega t + \varphi) \\ &= v_0 i_0 \cos(\omega t) \cos(\omega t + \varphi) \\ &= \frac{1}{2} v_0 i_0 \left\{ \cos(2\omega t + \varphi) + \cos\varphi \right\} \end{aligned}$$

# Average Power

Average power:

$$P = \frac{1}{T}\int_0^T p(t)\,dt$$

$$= \frac{1}{T}\int_0^T v_0 \cos(\omega t) i_0 \cos(\omega t + \varphi)\,dt$$

$$= \frac{1}{2} v_0 i_0 \frac{1}{T}\int_0^T \cos(2\omega t + \varphi)\,dt + \frac{1}{2} v_0 i_0 \frac{1}{T}\int_0^T \cos\varphi\,dt$$

If $T >> 1/\omega$:

$$P = \frac{1}{2} v_0 i_0 \frac{1}{T}\int_0^T \cos\varphi\,dt$$

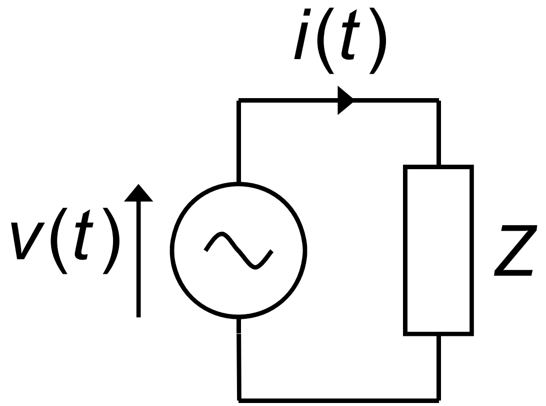
$$= \frac{1}{2} v_0 i_0 \cos\varphi$$

# Average Power

$$P = \frac{1}{2} v_0 i_0 \cos\varphi$$

$$= \frac{1}{2} \frac{v_0^2}{|Z|} \cos\varphi$$

$$= \frac{1}{2} i_0^2 |Z| \cos\varphi$$



# Average Power

Average power:

$$P = \frac{1}{2} v_0 i_0 \cos\varphi$$

For a resistor:

$$\varphi = 0 \quad \rightarrow \quad P = \frac{1}{2} v_0 i_0 = \frac{1}{2} \frac{v_0^2}{R} = \frac{1}{2} R i_0^2$$

For a capacitor:

$$\varphi = \frac{\pi}{2} \quad \rightarrow \quad P = 0$$

For an inductor:

$$\varphi = -\frac{\pi}{2} \quad \rightarrow \quad P = 0$$

# rms Voltages and Currents

Power expressed in terms of rms voltages and currents:

$$P = \frac{1}{2} v_0 i_0 \cos\varphi$$

$$= \frac{1}{2} V_{rms}\sqrt{2}\; I_{rms}\sqrt{2} \cos\varphi$$

$$= V_{rms} I_{rms} \cos\varphi \quad \text{(W)}$$

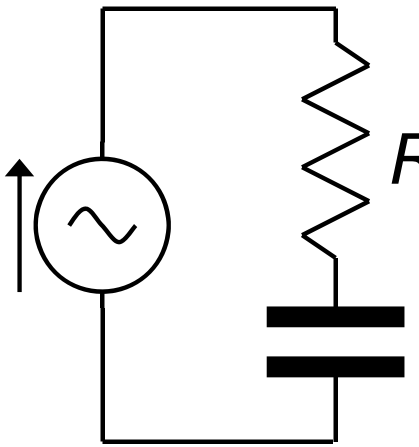
$$P = \frac{V_{rms}^2}{|Z|} \cos\varphi$$

$$P = I_{rms}^2 |Z| \cos\varphi$$

# Example 1



Determine the average power dissipated in the circuit:



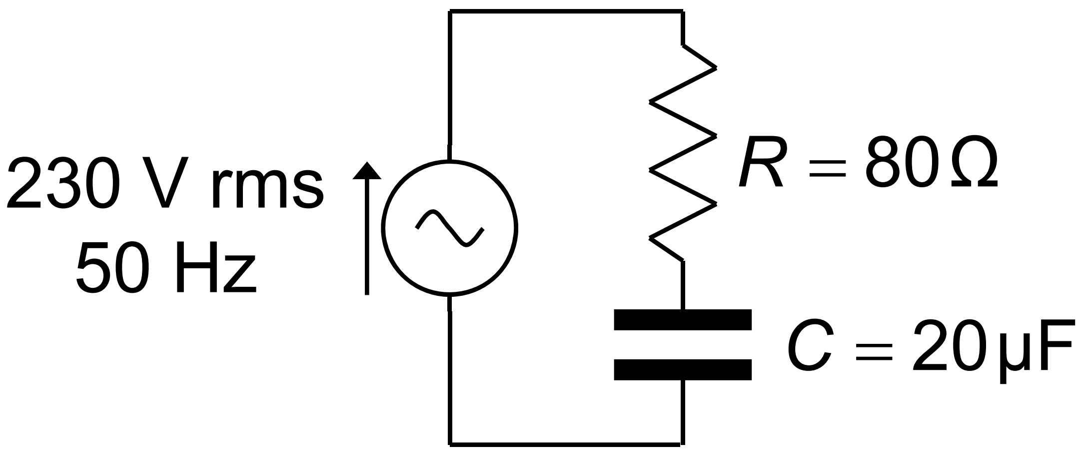

$$Z = R + \frac{1}{j\omega C}$$

$$= 80 + \frac{1}{j2\pi \times 50 \times 20 \times 10^{-6}}$$

$$= 80 - j159.2\ \Omega$$

$$= 178.1 \angle -1.105\ (\text{-}63.3^\circ)\ \Omega$$

# Example 1



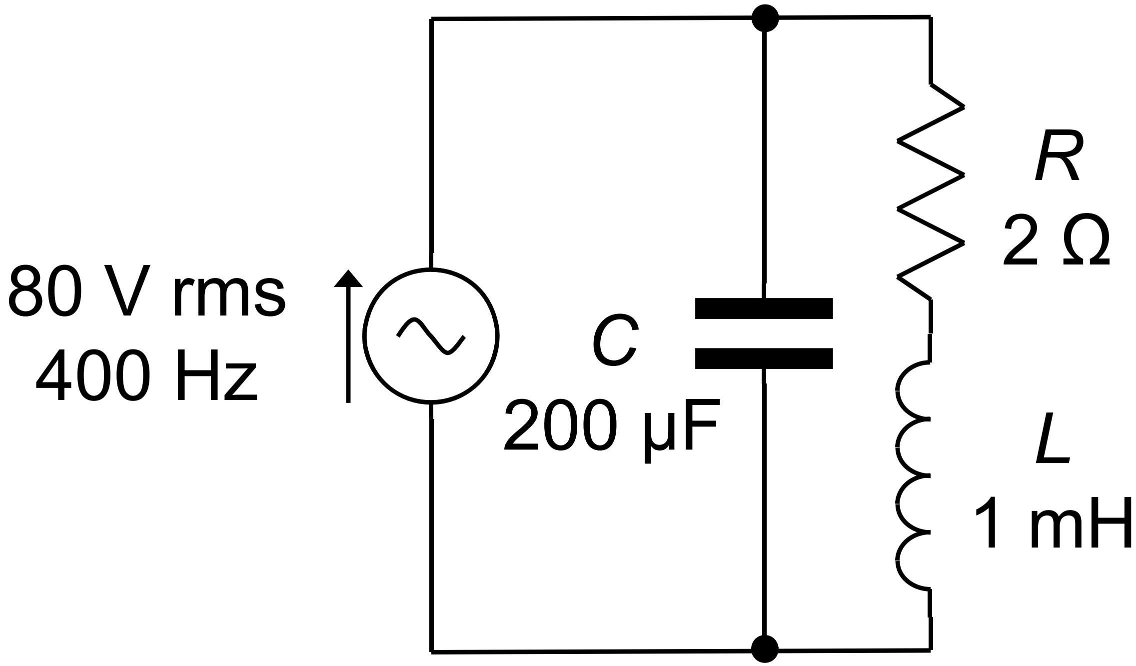

$$P = \frac{V_{rms}^2}{|Z|}\cos\varphi$$

$$= \frac{230^2}{178.1}\cos -1.105$$

$$= 133.4\ \text{W}$$

# Example 2

Determine the average power dissipated in the circuit:

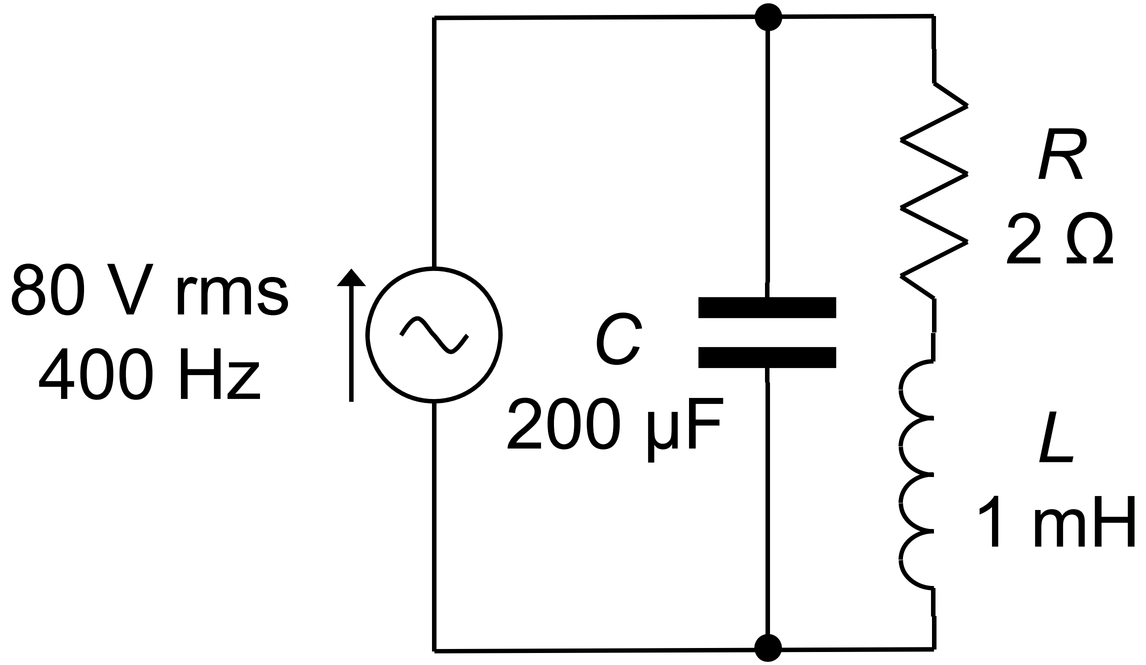



The driving-point impedance of this circuit at 400 Hz (calculated previously) is:

$$Z = 3.091\,\Omega \angle -0.9282$$

# Example 2

$$Z = 3.091\,\Omega\,\angle -0.9282$$




$$P = \frac{V_{rms}^2}{|Z|}\cos\phi$$

$$= \frac{80^2}{3.091}\cos -0.9283$$

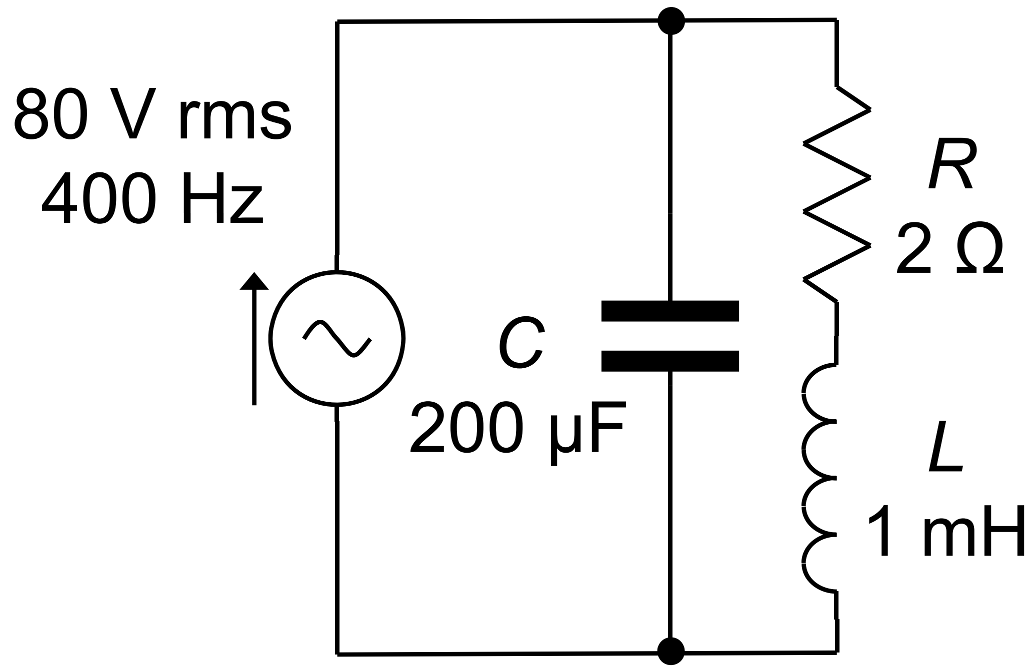
$$= 1241\,\text{W}$$

# Example 2

Determine the average power dissipated in the circuit

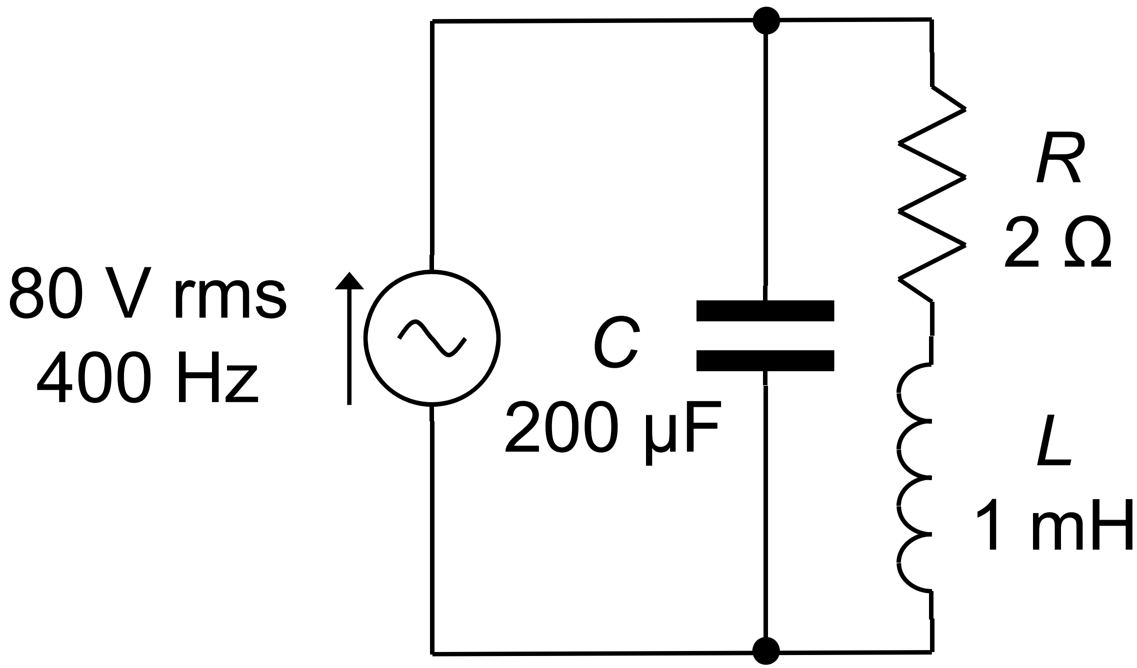
Since no power is dissipated in the capacitor we only need to calculate the power in the inductor-resistor leg

$$
\begin{aligned}
Z_{LR} &= R + j\omega L \\
&= 2 + j2\pi \times 400 \times 10^{-3} \\
&= 2 + j2.513 \\
&= 3.212\ \angle 0.8986
\end{aligned}
$$



# Example 2

$$Z_{LR} = 3.212\ \angle 0.8986$$




$$P = \frac{V_{rms}^2}{|Z|}\cos\varphi$$

$$= \frac{80^2}{3.212}\cos 0.8986$$

$$= 1241\ \mathrm{W}$$

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# Lecture 9

## Power Factor
## Three-Phase Electric Power

# True and Apparent Power

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The apparent power $P_a$ in a circuit is:

$$P_a = V_{rms} I_{rms}$$

Apparent power is measured in VA

The true power $P$ dissipated in a circuit is:

$$P = V_{rms} I_{rms} \cos\varphi$$

True power is measured in W

# Power Factor

The power factor is the ratio of the true power to the apparent power:

$$pf = \frac{P}{P_a} = \frac{V_{rms} I_{rms} \cos\varphi}{V_{rms} I_{rms}} = \cos\varphi$$

where ø is the phase difference between voltage and current.

It does not matter whether ø is phase of the current with respect to the voltage, or voltage with respect to the current, since:

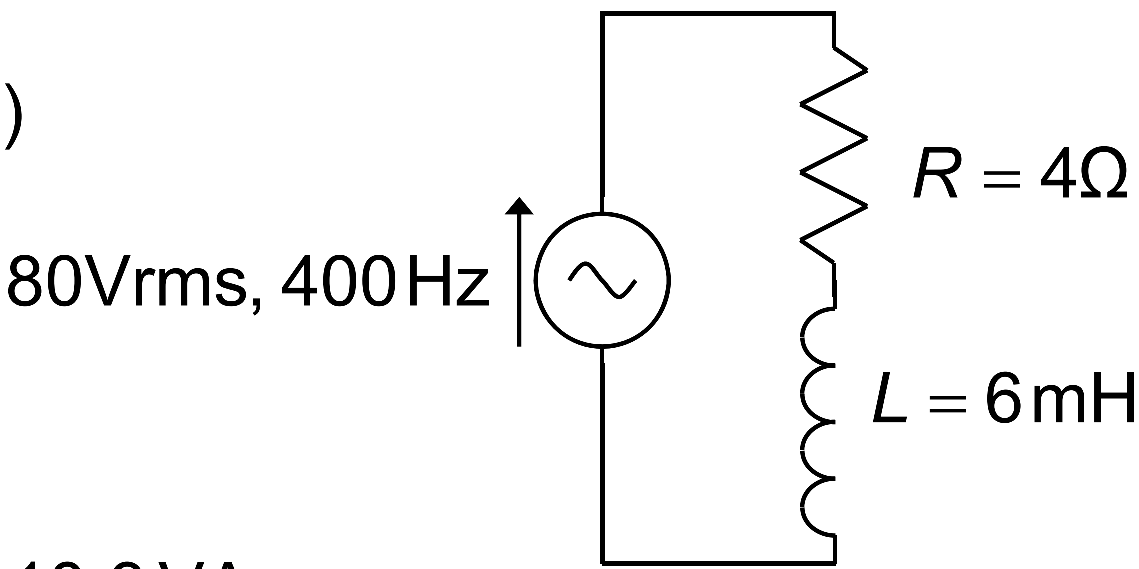
$$\cos\varphi = \cos - \varphi$$

# Example 1

Determine the power factor, apparent power and true power power dissipated in the circuit:

$$Z = 4 + j15.08\,\Omega$$

$$= 15.60\,\Omega \angle 1.312\,(75.1^\circ)$$




$$pf = \cos 1.312 = .2559$$

$$P_a = V_{rms} I_{rms} = \frac{V_{rms}^2}{|Z|} = 410.3\,\text{VA}$$

$$P = pf \times P_a = 105.0\text{ W}$$

# Power Factor Correction

Most industrial loads have a poor (*pf* << 1) power factor

Examples are induction motors and inductor-ballast lighting

Power factor can be corrected by connecting a reactance in parallel with the load

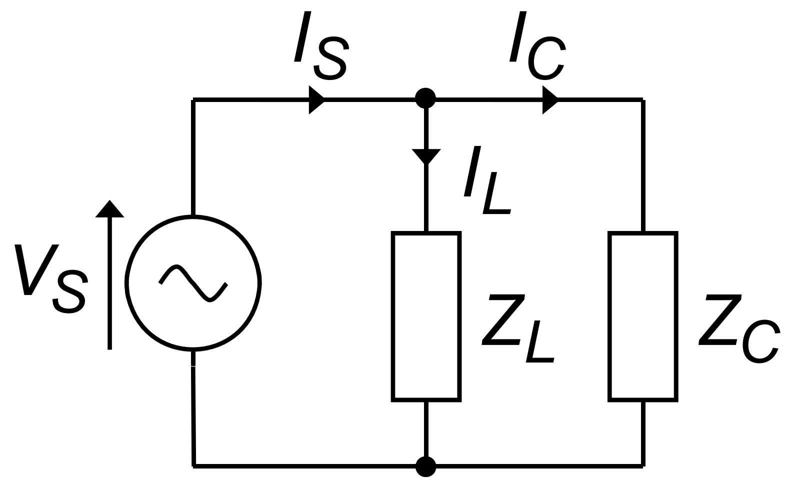
This reduces the apparent power and the rms current without affecting the load

This is obviously desirable because it reduces the current rating of the power wiring and supply

# Power Factor Correction

Power factor is normally corrected by connecting a reactive element $Z_C$ in parallel with the load $Z_L$ :

Supply current: $I_S$
Load current: $I_L$
Correction current: $I_C$




A unity overall power factor will be obtained provided that $V_S$ and $I_S$ are in phase:

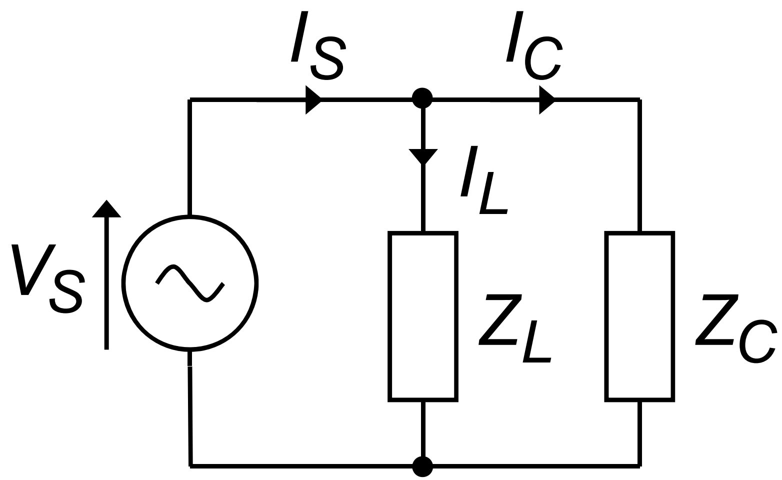
$$\frac{I_S}{V_S} = G\angle 0 = G + j0 \quad \text{(real)}$$

# Power Factor Correction

$$\frac{I_S}{V_S} = \frac{I_L}{V_S} + \frac{I_C}{V_S} = G + j0$$

$$\frac{1}{Z_L} + \frac{1}{Z_C} = G + j0$$

$$\left[\frac{1}{Z_L}\right]_{imag} = -\left[\frac{1}{Z_C}\right]_{imag}$$

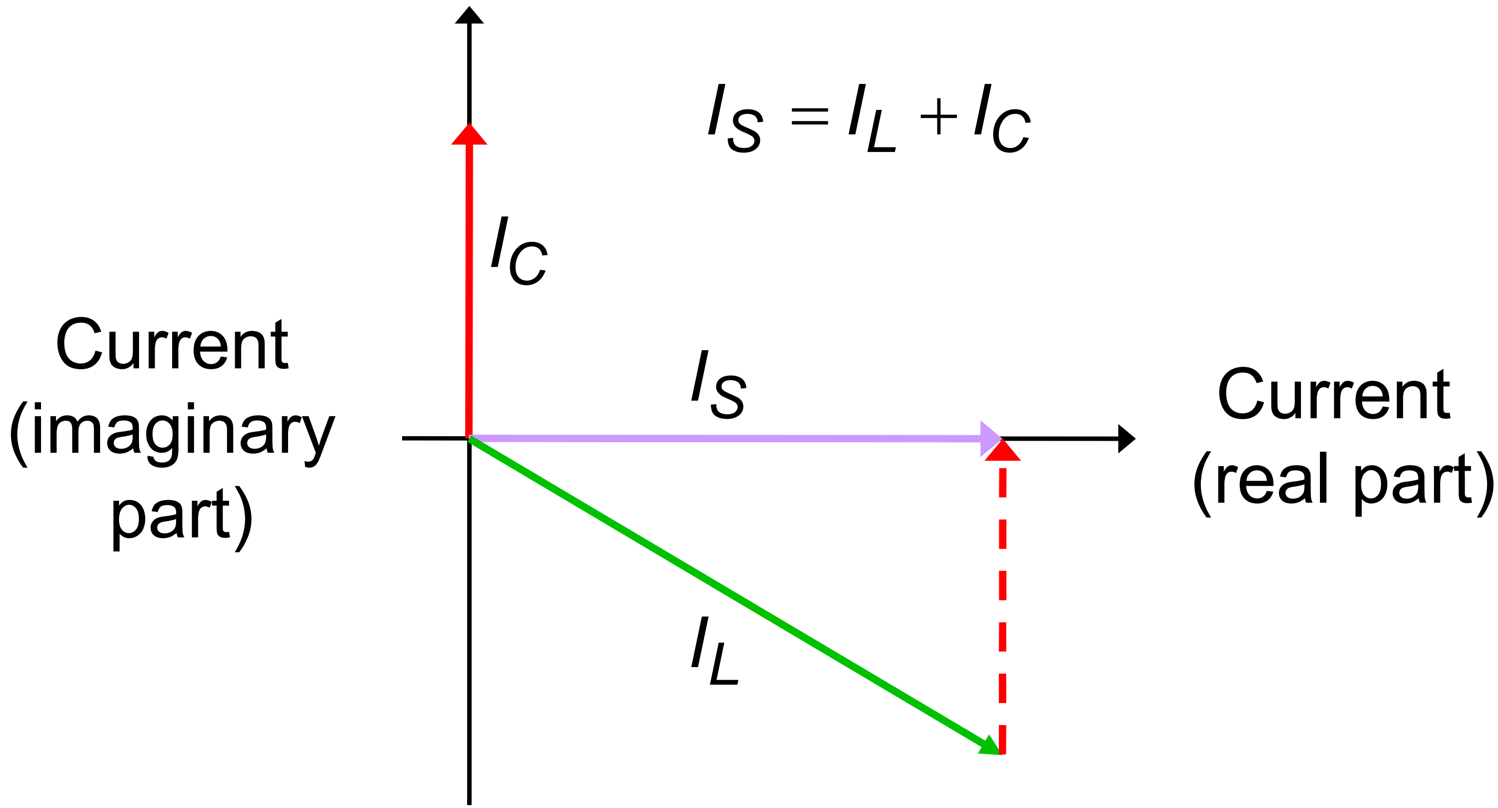



If $I_L$ leads $V_S$ then an inductor is used for correction

If $I_L$ lags $V_S$ then a capacitor is used for correction

# Power Factor Correction

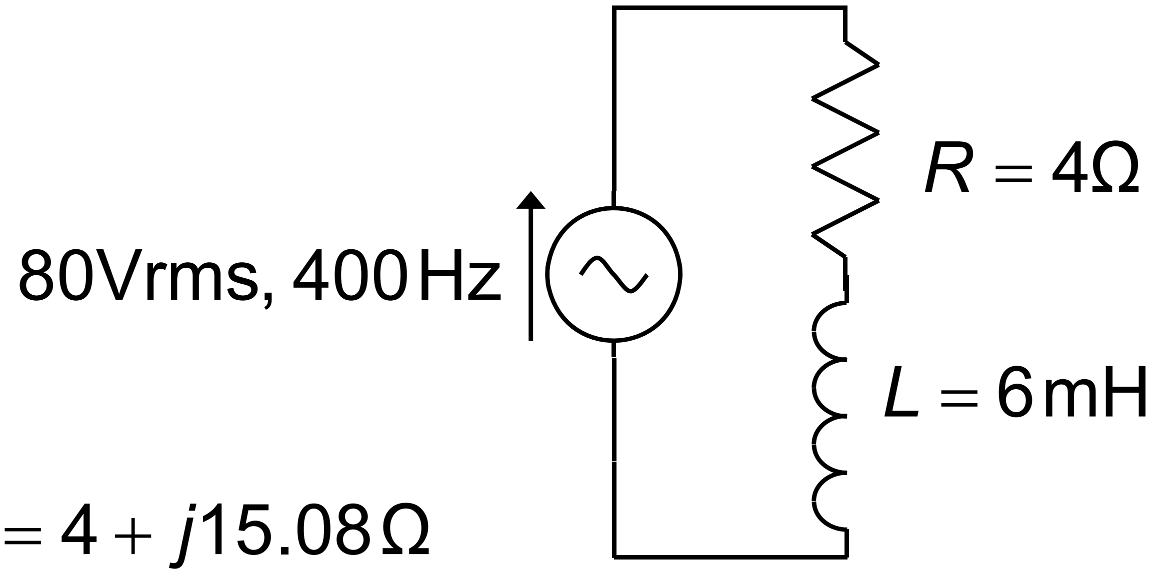
Correction of a lagging power factor load with a capacitor:




Note that the magnitude of the supply current $I_S$ is less than that of the load $I_L$

# Example 2

Choose a suitable power factor correction component for the circuit:

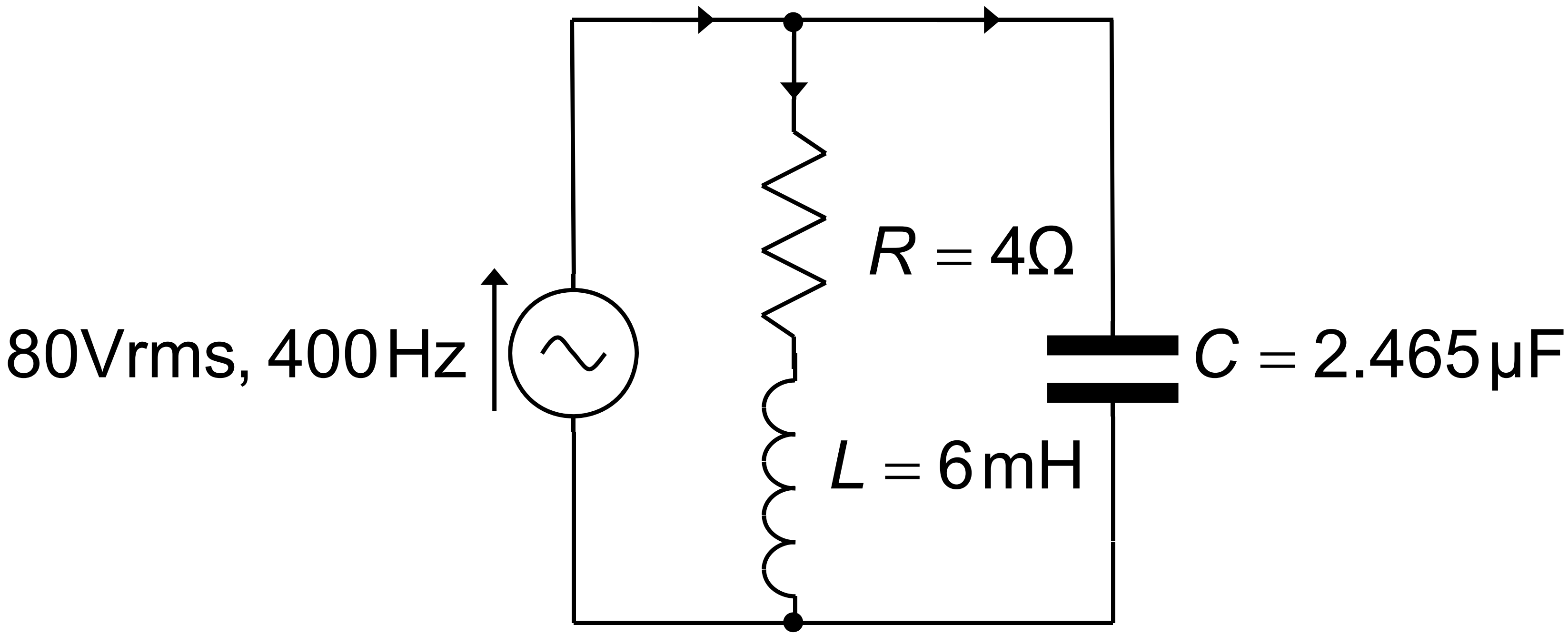



$$Z_L = 4 + j15.08\,\Omega$$

$$\frac{1}{Z_L} = \frac{4 - j15.08}{4^2 + 15.08^2} = 0.01643 - j0.06195$$

Thus: $$\frac{1}{Z_C} = +j0.06195$$

# Example 2




$$\frac{1}{Z_C} = +j0.06195 = j\omega C$$

$$C = \frac{0.06195}{2\pi \times 400} = 2.465\,\mu\text{F}$$

# Example 2

$$I_L = \frac{80}{Z_L}$$

$$= 80(0.01643 - j0.06195)$$

$$= 1.314 - j4.956$$

$$I_C = \frac{80}{Z_C}$$

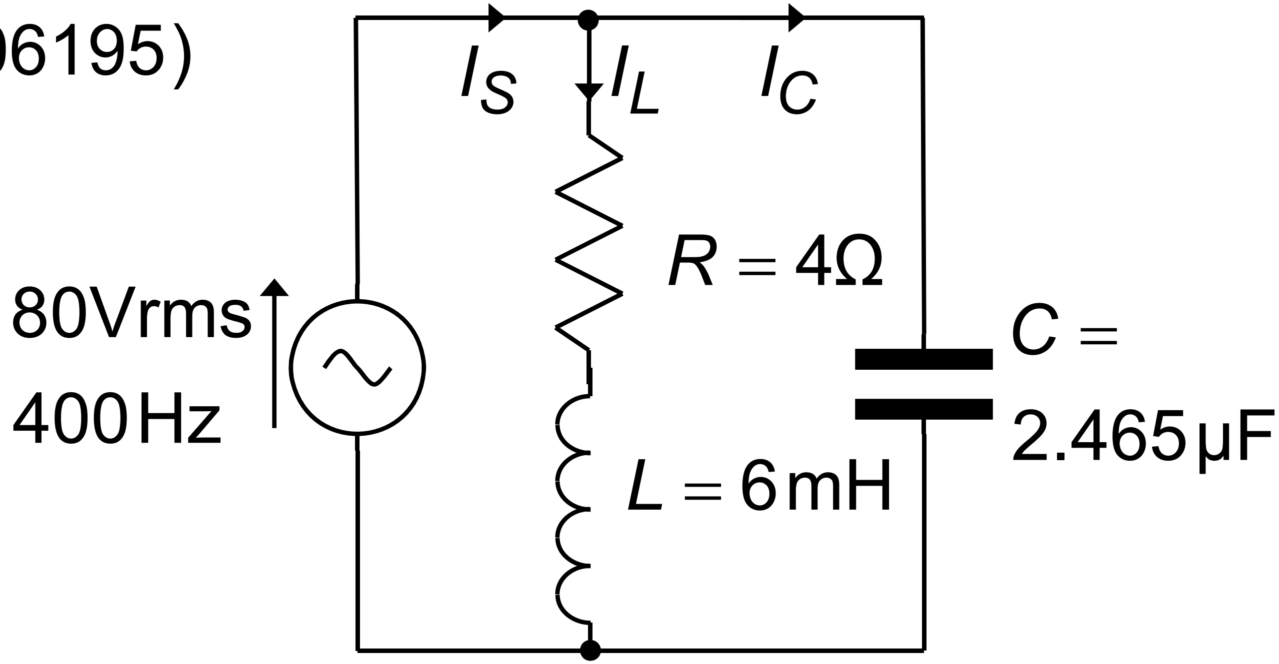
$$= 80 \times j0.06195$$

$$= j4.956$$

$$I_S = I_L + I_C$$

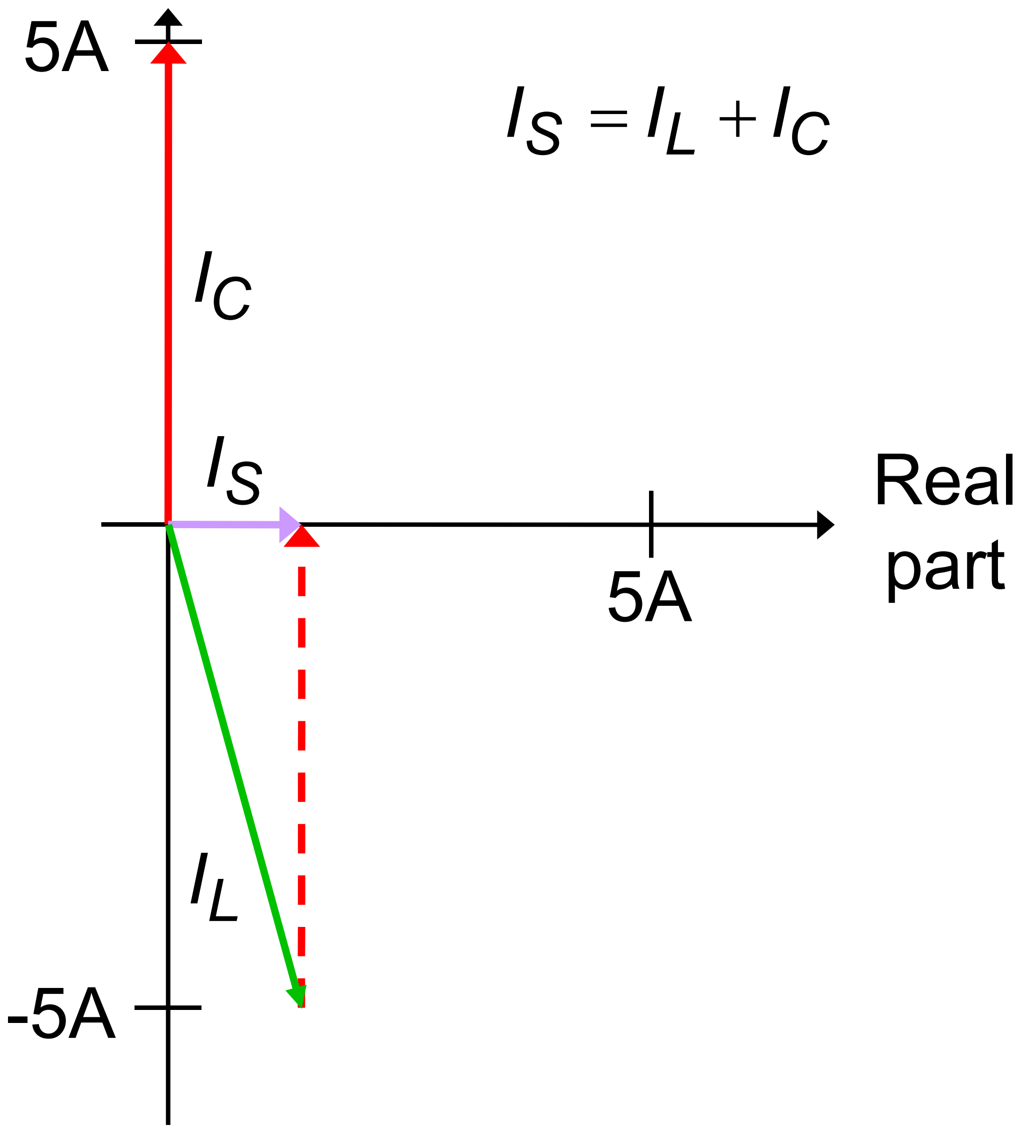
$$= 1.314 - j4.956 + j4.956$$

$$= 1.314$$



# Example 2





# Example 3

An electric motor operating from the 50 Hz mains supply has a lagging current with a power factor of .80

The rated motor current is 6 A at 230 V so that the magnitude of $1/Z_L$ is:

$$\left|\frac{1}{Z_L}\right| = \frac{I_L}{V_S} = \frac{6}{230} = 0.02609$$

and the phase of $1/Z_L$ is:

$$\angle\left\{\frac{1}{Z_L}\right\} = \cos^{-1} 0.8 = \pm 0.6435$$

Since the current lags the voltage the negative phase is used

# Example 3

$$\frac{1}{Z_L} = 0.02609\angle -0.6435 = 0.02087 - j0.01565$$

$$\frac{1}{Z_C} = +j0.01565 = j\omega C$$

$$C = \frac{0.01565}{2\pi \times 50} = 49.82\mu\text{F}$$

Before correction:

$$P_a = 230 \times 6 = 1380$$

$$P = pf \times P_a = 0.8 \times 1380$$

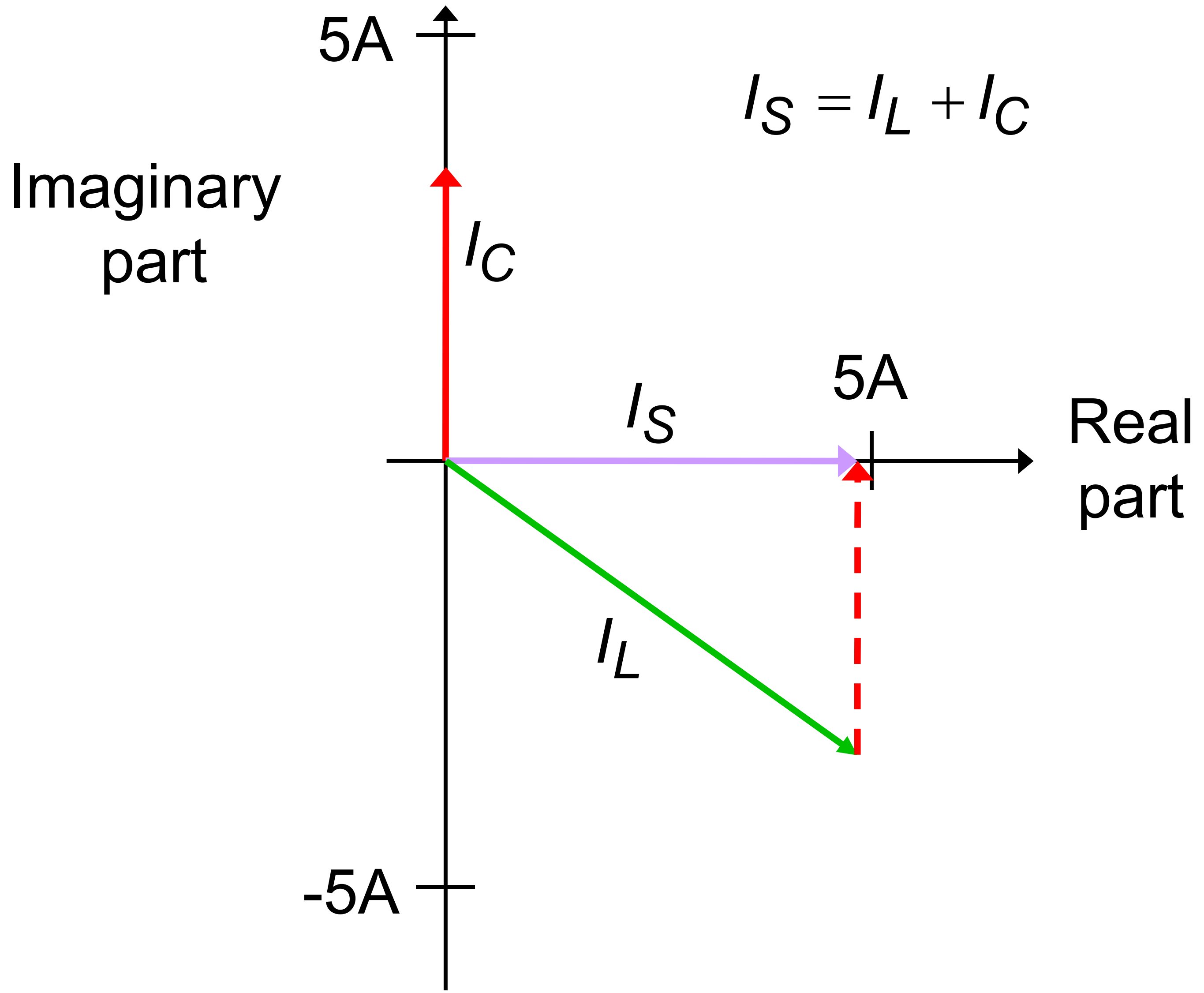
$$= 1104$$

After correction:

$$P = P_a = 1104$$

$$I_S = \frac{P}{V_S} = \frac{1104}{230} = 4.8$$

# Example 3



# Three-Phase Electric Power

Most ac power transmission systems use a three-phase system

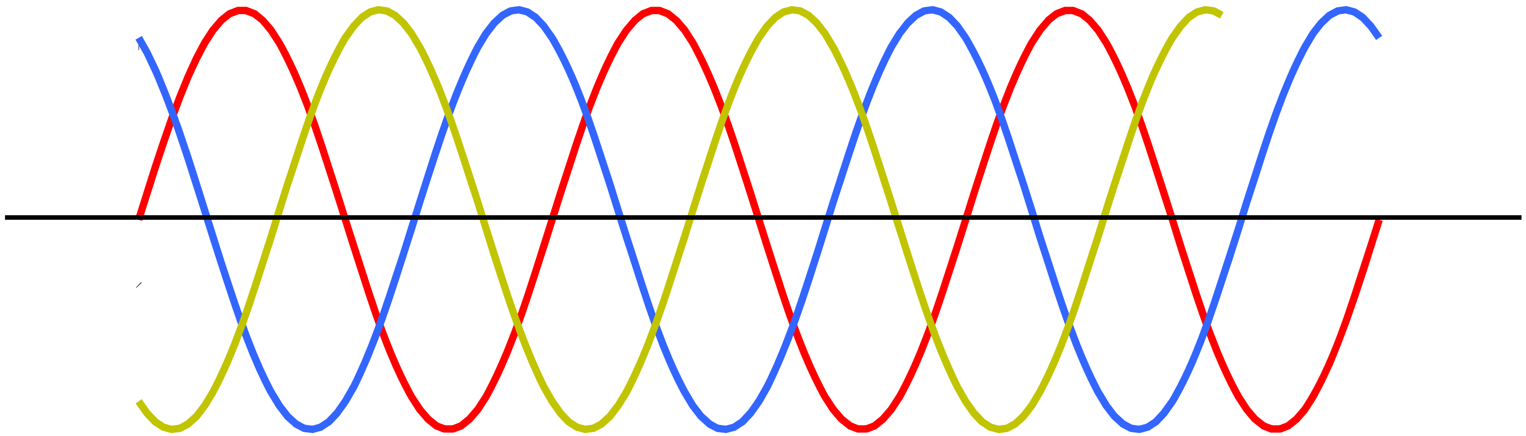
Three-phase is also used to power large motors and other heavy industrial loads

Three-phase consists of three sinusoids with phases $2\pi/3$ (120º) apart

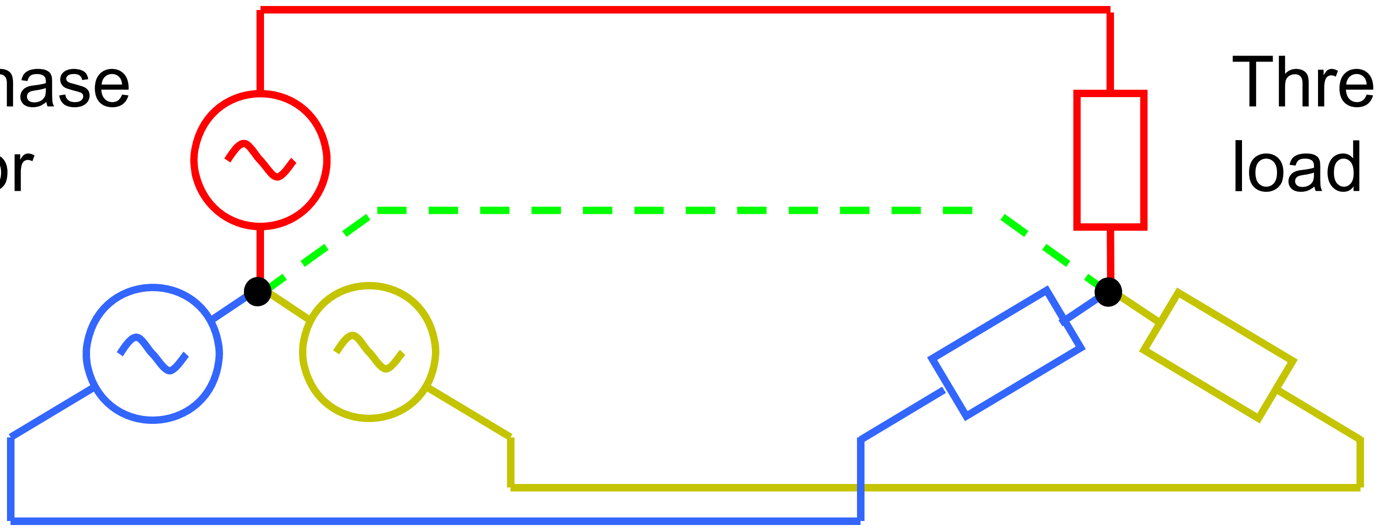
This allows more power to be transmitted down a given number of conductors than single phase

A three-phase transmission system consists of conductors for the three phases and sometimes a conductor for neutral

# Three-Phase Electric Power



Three-phase generator



Three-phase load

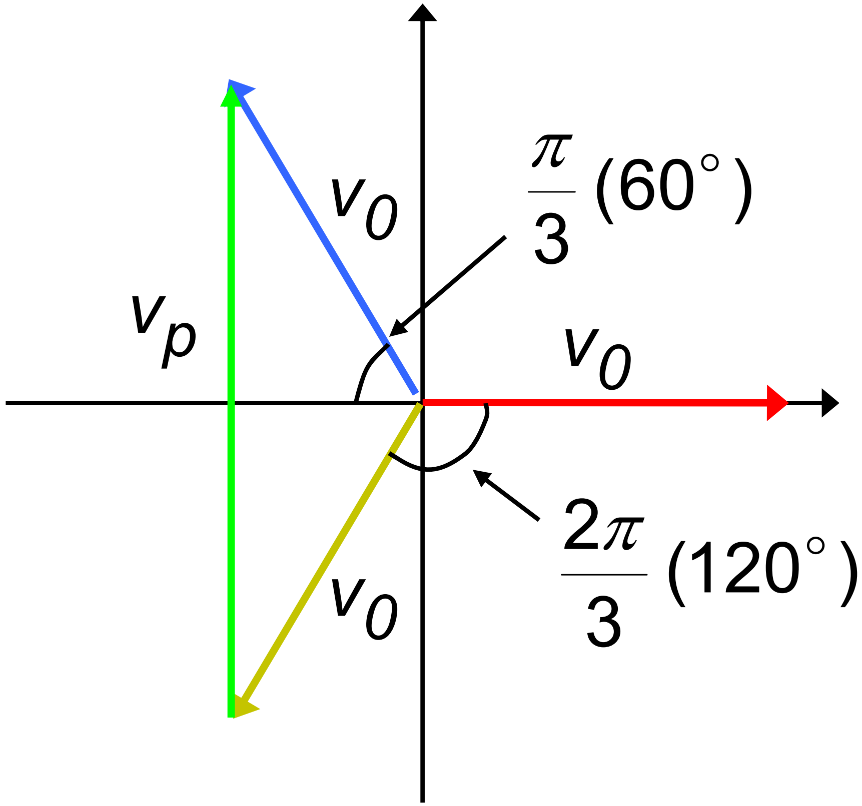
# Three-Phase Electric Power

Phase-to-neutral voltage $v_0$
Phase-to-phase voltage $v_p$

$$v_p = 2v_0 \sin\frac{\pi}{3}$$

$$= 2v_0 \frac{\sqrt{3}}{2}$$

$$= v_0\sqrt{3}$$



# Three-Phase Electric Power

UK domestic supply uses three -phase with a phase-to-neutral voltage $v_0$ of 230 V rms (325 V peak)

This corresponds to a phase-to-phase voltage $v_p$ of 400 V rms (563 V peak)

Each property is supplied with one phase and neutral

If the phases are correctly balanced (similar load to neutral on each) then the overall neutral current is zero

The UK electricity distribution network operates at 275 kV rms and 400 kV rms

# Lecture 10

## Energy Storage

# Energy Storage

Reactive components (capacitors and inductors) do not dissipate power when an ac voltage or current is applied

Power is dissipated only in resistors

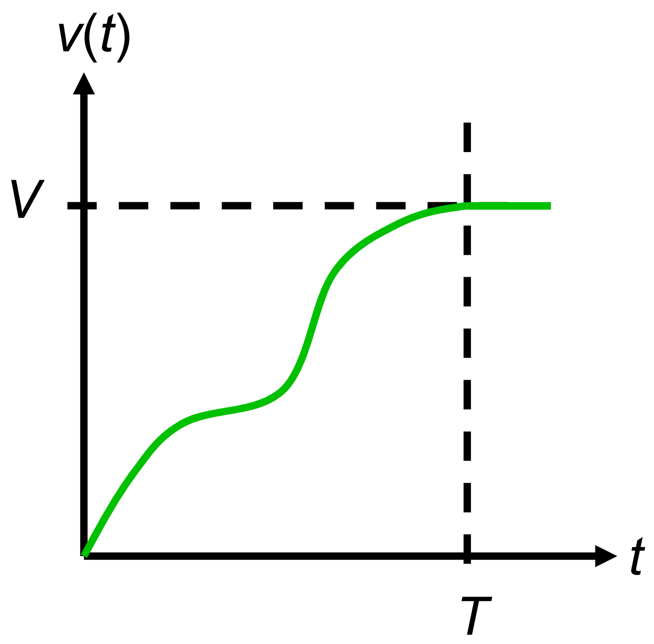
Instead reactive components store energy

During an ac cycle reactive components alternately store energy and then release it

Over a complete ac cycle there is no net change in energy stored, and therefore no power dissipation

# Energy Storage

The voltage across a capacitor is increased from zero to *V* producing a stored energy *E*:

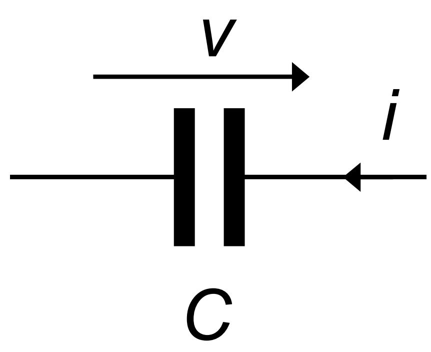



$$E = \int_0^T v(t)\,i(t)\,dt$$

$$= \int_0^T v(t)\,C\frac{dv}{dt}\,dt$$

$$= C\int_0^V v\,dv$$

$$E = \frac{1}{2}CV^2$$




$$i = C\frac{dv}{dt}$$

# Energy Storage

Example: calculate the energy storage in an electronic flash capacitor of 1000 μF charged to 400 V

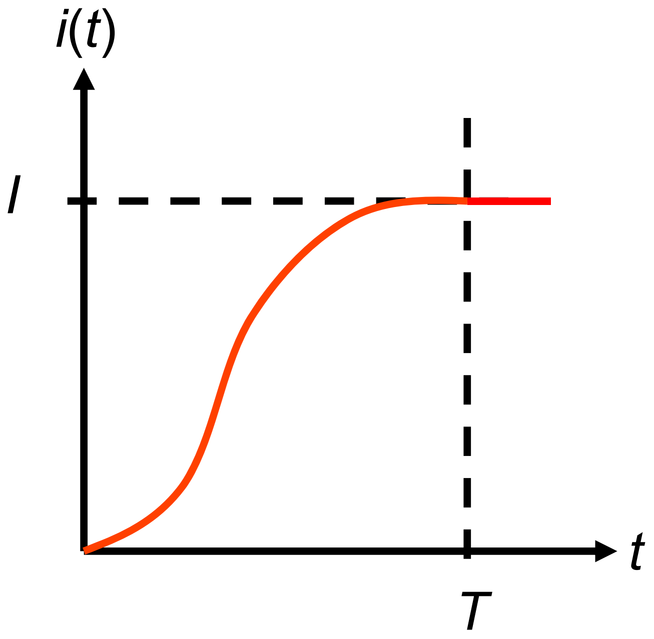
$$E = \frac{1}{2}CV^2$$

$$= \frac{1}{2} \times 1000 \times 10^{-6} \times 400^2$$

$$= 80 \text{ J}$$

# Energy Storage

The current in an inductor is increase from zero to *I* producing a stored energy *E*:

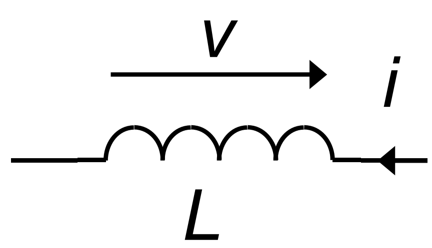



$$E = \int_0^T v(t)\,i(t)\,dt$$

$$= \int_0^T L\frac{di}{dt}\,i(t)\,dt$$

$$= L\int_0^I i\,di$$

$$E = \frac{1}{2}LI^2$$




$$v = L\frac{di}{dt}$$

# Energy Storage

Example: calculate the energy storage in a 2 mH inductor carrying a current of 10 A

$$E = \frac{1}{2} L i^2$$

$$= \frac{1}{2} \times 2 \times 10^{-3} \times 10^2$$

$$= 0.1 \text{ J}$$

# AC Circuit Analysis