Review

- Capacitors/Inductors
  - Voltage/current relationship
  - Stored Energy
- 1st Order Circuits
  - RL / RC circuits
  - Steady State / Transient response
  - Natural / Step response

Lecture #5

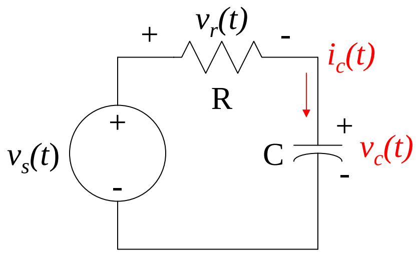
OUTLINE

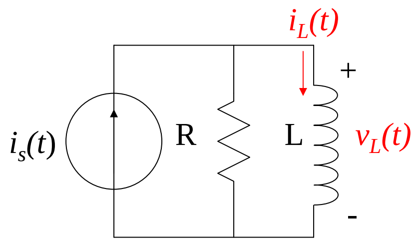
- **Chap 4**
  - **RC and RL Circuits with General Sources**
    - Particular and complementary solutions
    - Time constant
  - **Second Order Circuits**
    - The differential equation
    - Particular and complementary solutions
    - The natural frequency and the damping ratio
- **Chap 5**
  - **Types of Circuit Excitation**
  - **Why Sinusoidal Excitation?**
  - **Phasors**
  - **Complex Impedances**

Reading

Chap 4, Chap 5 (skip 5.7)

First Order Circuits





KVL around the loop:

$v_r(t) + v_c(t) = v_s(t)$

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

KCL at the node:

$$\frac{v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} v(x)dx = i_s(t)$$

$$\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = i_s(t)$$

Complete Solution

- Voltages and currents in a 1st order circuit satisfy a differential equation of the form

$$x(t) + \tau\frac{dx(t)}{dt} = f(t)$$

  - *f(t)* is called the forcing function.
- The complete solution is the sum of particular solution (forced response) and complementary solution (natural response).

$$x(t) = x_p(t) + x_c(t)$$

  - Particular solution satisfies the forcing function
  - Complementary solution is used to satisfy the initial conditions.
  - The initial conditions determine the value of *K*.

$$x_p(t) + \tau\frac{dx_p(t)}{dt} = f(t)$$

$$x_c(t) + \tau\frac{dx_c(t)}{dt} = 0$$

$$x_c(t) = Ke^{-t/\tau}$$

Homogeneous equation

The Time Constant

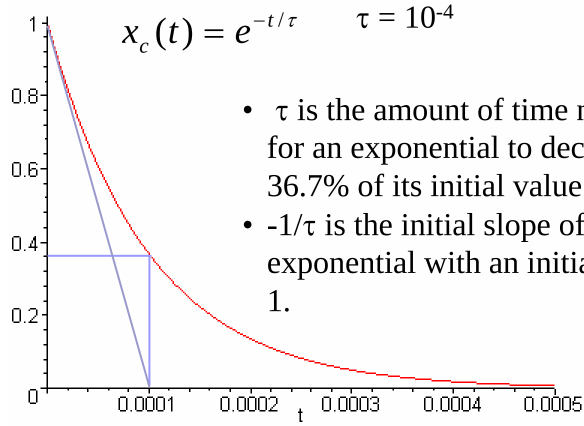
- The complementary solution for any 1st order circuit is

$$x_c(t) = Ke^{-t/\tau}$$

- For an RC circuit, $\tau = RC$
- For an RL circuit, $\tau = L/R$

What Does $X_c(t)$ Look Like?

$$x_c(t) = e^{-t/\tau} \qquad \tau = 10^{-4}$$



- τ is the amount of time necessary for an exponential to decay to 36.7% of its initial value.
- $-1/\tau$ is the initial slope of an exponential with an initial value of 1.

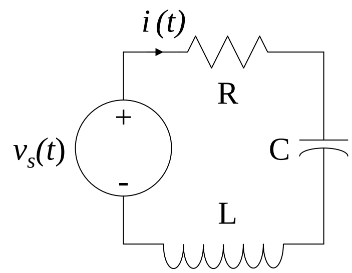
The Particular Solution

- The particular solution $x_p(t)$ is usually a weighted sum of $f(t)$ and its first derivative.
- If $f(t)$ is constant, then $x_p(t)$ is constant.
- If $f(t)$ is sinusoidal, then $x_p(t)$ is sinusoidal.

2nd Order Circuits

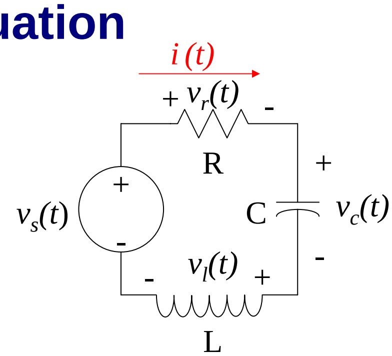
- Any circuit with a single capacitor, a single inductor, an arbitrary number of sources, and an arbitrary number of resistors is a circuit of order 2.
- Any voltage or current in such a circuit is the solution to a 2nd order differential equation.

A 2nd Order RLC Circuit



- Application: Filters
  - A bandpass filter such as the IF amp for the AM radio.
  - A lowpass filter with a sharper cutoff than can be obtained with an RC circuit.

The Differential Equation



KVL around the loop:

$$v_r(t) + v_c(t) + v_l(t) = v_s(t)$$

$$Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(x)dx + L\frac{di(t)}{dt} = v_s(t)$$

$$\frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) + \frac{d^2 i(t)}{dt^2} = \frac{1}{L}\frac{dv_s(t)}{dt}$$

The Differential Equation

The voltage and current in a second order circuit is the solution to a differential equation of the following form:

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t)$$

$$x(t) = x_p(t) + x_c(t)$$

$X_p(t)$ is the particular solution (forced response) and $X_c(t)$ is the complementary solution (natural response).

The Particular Solution

- The particular solution $x_p(t)$ is usually a weighted sum of $f(t)$ and its first and second derivatives.
- If $f(t)$ is constant, then $x_p(t)$ is constant.
- If $f(t)$ is sinusoidal, then $x_p(t)$ is sinusoidal.

The Complementary Solution

The complementary solution has the following form:

$$x_c(t) = Ke^{st}$$

K is a constant determined by initial conditions.

s is a constant determined by the coefficients of the differential equation.

$$\frac{d^2Ke^{st}}{dt^2} + 2\alpha\frac{dKe^{st}}{dt} + \omega_0^2 Ke^{st} = 0$$

$$s^2Ke^{st} + 2\alpha sKe^{st} + \omega_0^2 Ke^{st} = 0$$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Characteristic Equation

- To find the complementary solution, we need to solve the characteristic equation:

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

$$\alpha = \zeta\omega_0$$

- The characteristic equation has two roots-call them s_1 and s_2.

$$x_c(t) = K_1e^{s_1t} + K_2e^{s_2t}$$

$$s_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1}$$

Damping Ratio and Natural Frequency

$$\zeta = \frac{\alpha}{\omega_0}$$

damping ratio

$$s_1 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1}$$

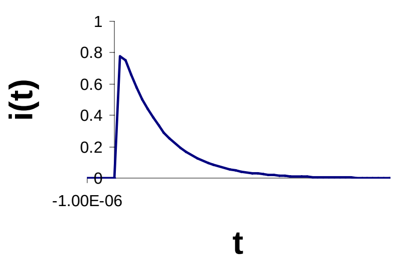
$$s_2 = -\zeta\omega_0 - \omega_0\sqrt{\zeta^2 - 1}$$

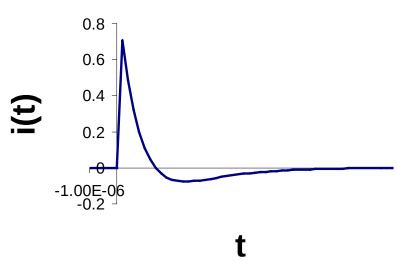
- The damping ratio determines what type of solution we will get:
  - Exponentially decreasing ($\zeta > 1$)
  - Exponentially decreasing sinusoid ($\zeta < 1$)
- The natural frequency is ω_0
  - It determines how fast sinusoids wiggle.

Overdamped : Real Unequal Roots

- If $\zeta > 1$, s_1 and s_2 are real and not equal.

$$i_c(t) = K_1 e^{\left(-\zeta\omega_0 + \omega_0\sqrt{\zeta^2-1}\right)t} + K_2 e^{\left(-\zeta\omega_0 - \omega_0\sqrt{\zeta^2-1}\right)t}$$



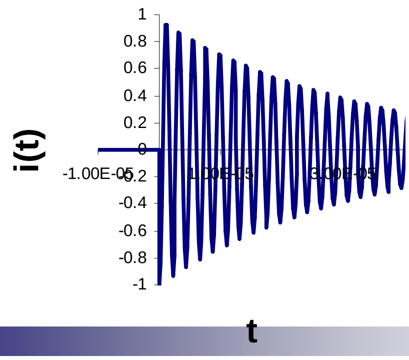


Underdamped: Complex Roots

- If $\zeta < 1$, s_1 and s_2 are complex.
- Define the following constants:

$$\alpha = \zeta\omega_0 \qquad \omega_d = \omega_0\sqrt{1-\zeta^2}$$

$$x_c(t) = e^{-\alpha t}\left(A_1 \cos\omega_d t + A_2 \sin\omega_d t\right)$$



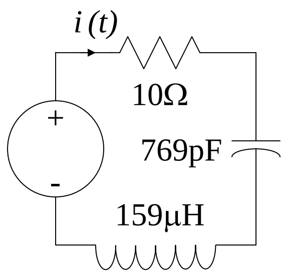
Critically damped: Real Equal Roots

- If $\zeta = 1$, s_1 and s_2 are real and equal.

$$x_c(t) = K_1 e^{-\varsigma\omega_0 t} + K_2 t e^{-\varsigma\omega_0 t}$$

Example

For the example, what are ζ and ω_0?



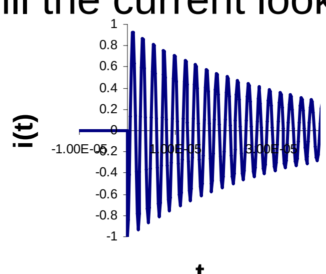
$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC} i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

$$\frac{d^2 x_c(t)}{dt^2} + 2\zeta\omega_0 \frac{dx_c(t)}{dt} + \omega_0^2 x_c(t) = 0$$

$$\omega_0^2 = \frac{1}{LC} \ , \ 2\zeta\omega_0 = \frac{R}{L} \ , \ \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

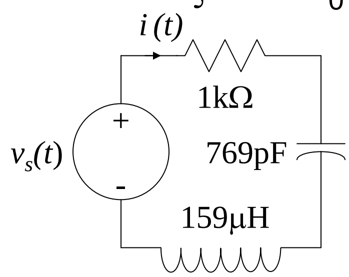
Example

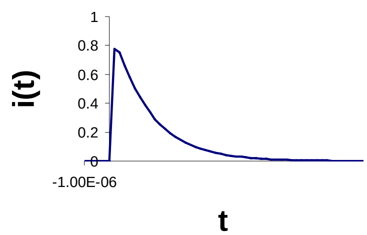
- $\zeta = 0.011$
- $\omega_0 = 2\pi 455000$
- Is this system over damped, under damped, or critically damped?
- What will the current look like?



Slightly Different Example

- Increase the resistor to 1kΩ
- What are ζ and ω_0?

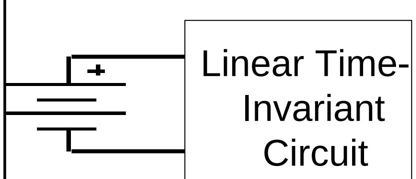




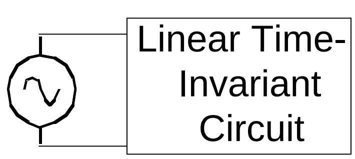
$\zeta = 2.2$

$\omega_0 = 2\pi 455000$

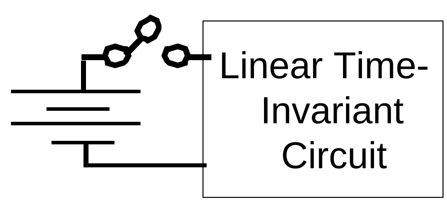
Types of Circuit Excitation



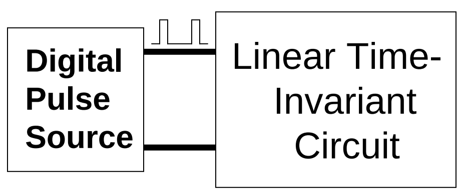
Steady-State Excitation (DC Steady-State)



Sinusoidal (Single-Frequency) Excitation →AC Steady-State



Step Excitation **OR**



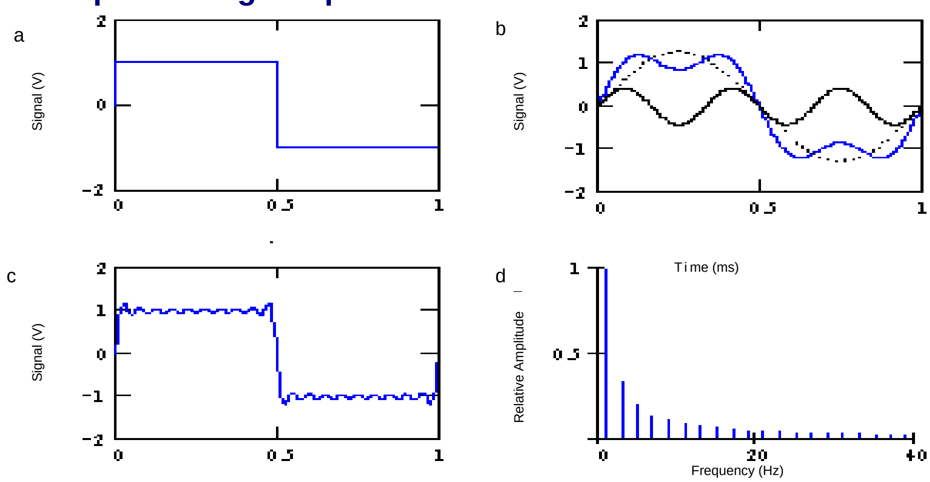
Transient Excitations

Why is Single-Frequency Excitation Important?

- Some circuits are driven by a single-frequency sinusoidal source.
- Some circuits are driven by sinusoidal sources whose frequency changes slowly over time.
- You can express any periodic electrical signal as a sum of single-frequency sinusoids – so you can analyze the response of the (linear, time-invariant) circuit to each individual frequency component and then sum the responses to get the total response.

- This is known as Fourier Transform and is tremendously important to all kinds of engineering disciplines!

Representing a Square Wave as a Sum of Sinusoids



(a)Square wave with 1-second period. (b) Fundamental component (dotted) with 1-second period, third-harmonic (solid black) with1/3-second period, and their sum (blue). (c) Sum of first ten components. (d) Spectrum with 20 terms.

Steady-State Sinusoidal Analysis

- Also known as AC steady-state
- Any steady state voltage or current in a linear circuit with a sinusoidal source is a sinusoid.
  - This is a consequence of the nature of particular solutions for sinusoidal forcing functions.
- All AC steady state voltages and currents have the same frequency as the source.
- In order to find a steady state voltage or current, all we need to know is its magnitude and its phase relative to the source
  - We already know its frequency.
- Usually, an AC steady state voltage or current is given by the particular solution to a differential equation.

The Good News!

- We do not have to find this differential equation from the circuit, nor do we have to solve it.
- Instead, we use the concepts of phasors and complex impedances.
- Phasors and complex impedances convert problems involving differential equations into circuit analysis problems.

Phasors

- A phasor is a complex number that represents the magnitude and phase of a sinusoidal voltage or current.
- Remember, for AC steady state analysis, this is all we need to compute-we already know the frequency of any voltage or current.

Complex Impedance

- Complex impedance describes the relationship between the voltage across an element (expressed as a phasor) and the current through the element (expressed as a phasor).
- Impedance is a complex number.
- Impedance depends on frequency.
- Phasors and complex impedance allow us to use Ohm's law with complex numbers to compute current from voltage and voltage from current.

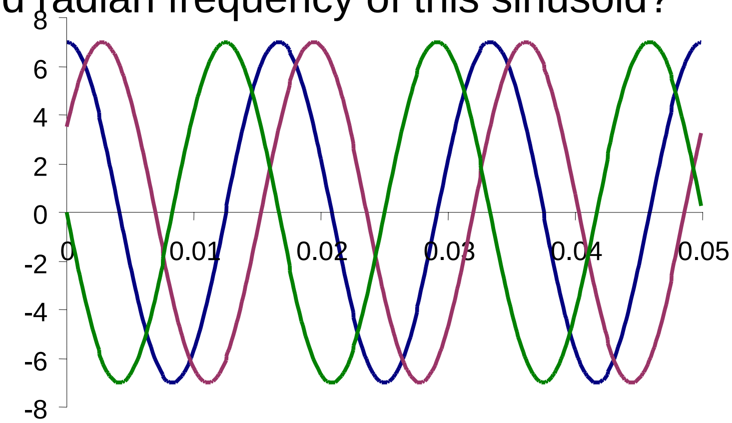
Sinusoids

$$v(t) = V_M \cos(\omega t + \theta)$$

- Amplitude: V_M
- Angular frequency: $\omega = 2\pi f$
  - Radians/sec
- Phase angle: θ
- Frequency: $f = 1/T$
  - Unit: 1/sec or Hz
- Period: T
  - Time necessary to go through one cycle

Phase

What is the amplitude, period, frequency, and radian frequency of this sinusoid?



Phasors

- A phasor is a complex number that represents the magnitude and phase of a sinusoid:

$$X_M \cos(\omega t + \theta) \quad \text{Time Domain}$$

$$\Updownarrow$$

$$\mathbf{X} = X_M \angle \theta \quad \text{Frequency Domain}$$