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**Problem 1:**

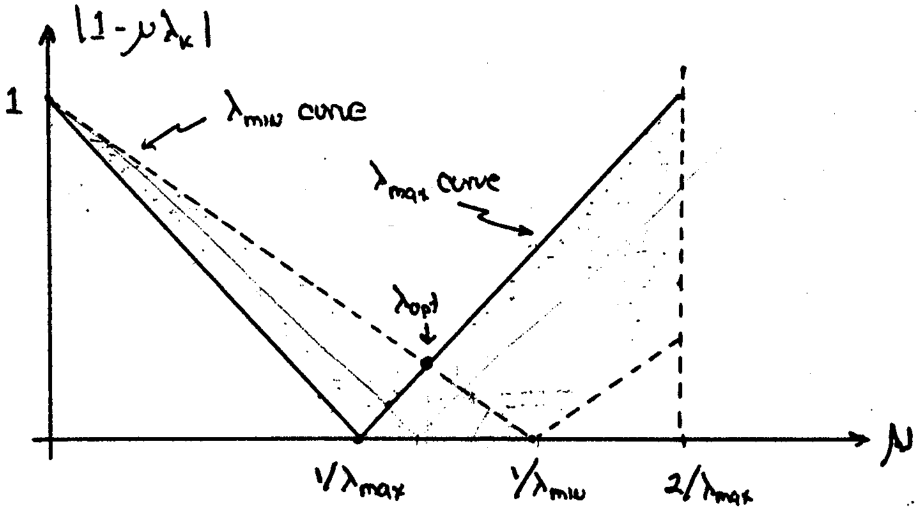
(a) We have seen that the dynamic behavior of the weight vector, $\mathbf{w}(n)$, in the steepest descent algorithm behaves as

$$\mathbf{w}(n) = \mathbf{w}_0 + \sum_{k=1}^{M} \mathbf{q}_k(1-\mu\lambda_k)^n v_k(0)$$

To maximize the rate of convergence, we want to choose $\mu$ so that we minimize the slowest decaying mode, i.e.,

$$\min_{\mu}\left[\max_{k}|1-\mu\lambda_k|\right]$$

Shown in the figure below is a plot of $|1-\mu\lambda_k|$ as a function of $\mu$ for $\lambda_{\min}$ and $\lambda_{\max}$.



Clearly, for a given value of $\mu$, the values $|1-\mu\lambda_k|$ will lie within the shaded region indicated in the figure. Therefore, to minimize the largest value of $|1-\mu\lambda_k|$ we choose $\mu = \mu_{\text{opt}}$ as shown, which is the solution to

$$1-\mu\lambda_{\max} = -(1-\mu\lambda_{\min})$$

Therefore,

$$\mu_{\text{opt}} = \frac{2}{\lambda_{\max}+\lambda_{\min}}$$

(b) The slowest decaying mode with $\mu_{\text{opt}}$ is the rate at which both the minimum and maximum eigenvalues converge, i.e.,

$$\left[1 - \frac{2\lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}\right]^n = \left(\frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}\right)^n$$

**Problem 2:**

(a) Evaluating the gradient vector we have

$$\nabla_{\mathbf{w}} J(\mathbf{w}) = 2E\{e(n)\nabla e(n)\} = -2E\{e(n)\mathbf{x}(n)\} = -2\mathbf{p} + 2\mathbf{R}\mathbf{w}$$

Thus,

$$\mathbf{w}(n+1) = \mathbf{w}(n) - \tfrac{1}{2}\mu\mathbf{R}^{-1}\left[2\mathbf{R}\mathbf{w}(n) - 2\mathbf{p}\right]$$

and we have

$$\mathbf{w}(n+1) = \mathbf{w}(n) - \mu\mathbf{w}(n) + \mu\mathbf{w}^o = (1-\mu)\mathbf{w}(n) + \mu\mathbf{w}^o$$

Thus, the Newton algorithm is stable for $0 < \mu < 2$.

(b) The convergence is the fastest when $\mu = 1$. Note, in fact, that when $\mu = 1$ the Newton iteration converges in one step to $\mathbf{w}^o$.

(c) The gradient approximation is

$$\nabla e^2(n) = -2e(n)\mathbf{x}(n)$$

Therefore the LMS-type algorithm is

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu e(n)\mathbf{R}^{-1}\mathbf{x}(n)$$

Comparing this to the LMS algorithm we see that the step direction is changed by $\mathbf{R}^{-1}$

(d) From (b) we see that

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu\mathbf{R}^{-1}\mathbf{x}(n)d(n) - \mu\mathbf{R}^{-1}\mathbf{x}(n)\mathbf{x}^T(n)\mathbf{w}(\mathbf{n})$$

Assuming that the signal $\mathbf{x}(n)$ is uncorrelated with the filter tap weight vector, $\mathbf{w}(n)$, then

$$E\{\mathbf{w}(n+1)\} = E\{\mathbf{w}(n)\} + \mu\mathbf{R}^{-1}\mathbf{p} - \mu\mathbf{R}^{-1}\mathbf{R}E\{\mathbf{w}(n)\}$$

and, therefore,

$$E\{\mathbf{w}(n+1)\} = (1-\mu)E\{\mathbf{w}(n)\} + \mu\mathbf{w}^o$$

**Problem 3:** Using the $n$th-order approximation to $\mathbf{R}^{-1}$

$$\mathbf{R}^{-1}(n) = \mu\sum_{k=0}^{n}[\mathbf{I} - \mu\mathbf{R}]^k$$

we have

$$\mathbf{R}^{-1}(n+1) = \mu\sum_{k=0}^{n+1}[\mathbf{I} - \mu\mathbf{R}]^k = \mu[\mathbf{I} - \mu\mathbf{R}]\sum_{k=0}^{n}[\mathbf{I} - \mu\mathbf{R}]^k + \mu\mathbf{I}$$

Therefore,

$$\mathbf{R}^{-1}(n+1) = [\mathbf{I} - \mu\mathbf{R}]\mathbf{R}^{-1}(n) + \mu\mathbf{I}$$

and

$$\mathbf{w}(n+1) = \mathbf{R}^{-1}(n+1)\mathbf{p} = [\mathbf{I} - \mu\mathbf{R}]\mathbf{R}^{-1}(n)\mathbf{p} + \mu\mathbf{p} = [\mathbf{I} - \mu\mathbf{R}]\mathbf{w}(n) + \mu\mathbf{p}$$

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## Problem 4

(a) With $n_0 = 1$ the delayed LMS adaptive filter update equation is

$$\begin{aligned}\mathbf{w}(n+1) &= \mathbf{w}(n) + \alpha e(n-1)\mathbf{x}(n-1)\\ &= \mathbf{w}(n) + \alpha\left[d(n-1) - \mathbf{w}^T(n-1)\mathbf{x}(n-1)\right]\mathbf{x}(n-1)\end{aligned}$$

Taking the expected value, assuming that the weight vector $\mathbf{w}(n)$ is uncorrelated with the data vector $\mathbf{x}(n)$, we have

$$E\{\mathbf{w}(n+1)\} = E\{\mathbf{w}(n)\} - \alpha\mathbf{R}_x E\{\mathbf{w}(n-1)\} + \alpha\mathbf{p}$$

Thus, the expected value of the weight vector satisfies a second-order difference equation. Diagonalizing the autocorrelation matrix and expressing this equation in terms of the rotated coefficient vector, $\mathbf{v}(n)$, we have, for the $k$th coefficient,

$$E\{v_k(n+1)\} = E\{v_k(n)\} - \alpha\lambda_k E\{v_k(n-1)\} + \alpha p_k$$

where $\lambda_k$ for $k = 1, \ldots, N$ are the eigenvalues of $\mathbf{R}_x$. Since the characteristic equation for $E\{v_k(n)\}$ is

$$1 - z^{-1} + \alpha\lambda_k z^{-2} = 0$$

in order for $E\{v_k(n)\}$ to converge it is necessary that the roots of the characteristic equation lie inside the unit circle. Since the roots of the characteristic equation are

$$z_k = \tfrac{1}{2}\left\{1 \pm \sqrt{1 - 4\alpha\lambda_k}\right\}$$

then the delayed LMS algorithm converges in the mean if

$$\alpha < 1/\lambda_{\max}$$

(b) With $\lambda_k = 1$ and $\alpha = 0.1$, the slowest decaying mode for the LMS algorithm behaves as

$$(1 - \alpha\lambda)^k = (0.9)^k$$

and the time constant is

$$\tau = \frac{1}{\alpha\lambda} = 10$$

For the delayed LMS, the roots of the characteristic equation are

$$z_k = \tfrac{1}{2}\left\{1 \pm \sqrt{1 - .4}\right\} = 0.887 \quad \text{and} \quad .1127$$

Therefore, the slowest decaying mode behaves as $(0.887)^k$ which is approximately the same as the LMS algorithm. Thus, the time constants for delayed LMS with $d = 1$ is about the same as LMS.