Applied Quantum Mechanics

Final Exam, January 18, 2009
Fall Semester

- The exam is taken as take-home. Answers should be sent via email till Friday noon, January 30, 2009.
- Use of external references is allowed, but they should be cited in the final report.
- Groups of two-students may turn in one solution leaflet.

Fock-Darwin States in Quantum Wells

The single-particle Hamiltonian of an electron subject to a two-dimensional external potential $V_{ext}(\mathbf{r})$, with $\mathbf{r} = x\hat{x} + y\hat{y}$ and neglection of spin, is given by

$$\hat{H}_0 = \frac{1}{2m^*}\left[-i\hbar\nabla + \frac{e}{c}\mathbf{A}(\mathbf{r})\right]^2 + V_{ext}(\mathbf{r})$$

Fock and Darwin studied this system for the case of parabolic potential, that is $V_{ext}(\mathbf{r}) = \frac{1}{2}m^*{\omega_0}^2 r^2$.

a) If the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is constant along *z*-axis $\mathbf{B} = B_0\hat{z}$, show that the Hamiltonian is

$$\hat{H}_0 = -\frac{\hbar^2}{2m^*}\nabla^2 + \frac{1}{2}m^*\omega^2 r^2 + \frac{1}{2}\omega_c\hat{L}_z$$

where $\omega^2 = {\omega_0}^2 + \frac{1}{4}{\omega_c}^2$ with $\omega_c = eB_0/m^*c$ being the cyclotron frequency, and $\hat{L}_z$ is the *z* component of the angular momentum operator. Obtain the eigen-states and eigen-energies in polar coordinates (solutions are found in terms of associated Laguerre polynomials). Plot the typical energy spectra in terms of applied magnetic field ranging from zero to 5T ($\hbar\omega_0$=100meV, m^*=0.067m_0).

b) Extend this formulation to the case of an exciton, a two-fermion particle consisting of an electron and a hole, with opposite charges.
- Write down the Hamiltonian, with the effect of spin taken into account. You may ignore the center- of-mass angular moment.
- Considering that the total wave function of electron-hole pair is entangled, propose the solution in the form of an expansion. Is the total wavefunction antisymmetric?
- Formulate a numerical method to derive the eigen-states, and eigen-energies.

c) Solve this problem in the case of a three-dimensional confining potential of a disk-like quantum dot. The quantum dot is approximated as a long cylindrical structure having the diameter 100nm, and consists of a 20nm layer of GaAs sandwiched between thick $Al_{0.3}Ga_{0.7}As$ layers. The vacuum potential may be taken as infinite, so that the wavefunctions vanish on the boundary. You need to obtain only the ground state binding energy and wavefunction. Plot radial and axial distribution of electron and hole wavefunction. Obtain the dependence of ground state binding energy on the applied magnetic field.

M. Helle, *Few Electron Quantum Dot Molecules*, PhD Dissertation, Helsinki University of Technology (2006).
N. Nagosa, *Quantum Field Theory in Condensed Matter Physics*, Springer, Berlin (1999).

Quintic (Three-particle) Interactions in Condensates

For a system of identical Bosons with two- and three-particle interactions, write down the coordinate-representation of Hamiltonian.

a) Obtain the Hamiltonian in terms of standard Bosonic creation and annihilation operators.
b) Within the approximation of contact interaction, where particles do not interact unless they come into contact, obtain the Hamiltonian in terms of the field operators in the second quantization framework.
c) Employ the Bogoliubov approximation in the limit of large number of particles to derive the effective Hamiltonian.
d) Obtain the exact Heisenberg equation of motion in the limit of zero temperature, and therefore, the mean-field governing equation. Obtain the solution numerically when the condensate is confined in a spherical potential barrier.
e) What is the generalization for higher-order interactions? Write down the Bogoliubov-de Gennes equations for this system.
f) Derive the dispersion equation of free particles in such a condensate, if the spatial distribution of particles is uniform.
g) Show that the solvability condition for higher-order interactions is that the potential would be a function of the magnitude of mean-field wavefunction.

N. P. Proukakis and B. Jackson *J. Phys. B* **41** 203002 (2008)
E. Kenge et al. *J. Phys. B* **41** 205202 (2008)

Good Luck