• Image Degradation Model (Linear/Additive)

\[
g(x, y) = f(x, y) \star h(x, y) + \eta(x, y)
\]

\[
G(u, v) = F(u, v) H(u, v) + N(u, v)
\]
Digital Image Processing

Image Restoration and Reconstruction

- **Source of noise**
  - Objects Impurities
  - Image acquisition (digitization)
  - Image transmission

- **Spatial properties of noise**
  - Statistical behavior of the gray-level values of pixels
  - Noise parameters, correlation with the image

- **Frequency properties of noise**
  - Fourier spectrum
    - White noise (a constant Fourier spectrum)
Digital Image Processing

Image Restoration and Reconstruction

- Some Noises Distribution:

\[ p(z) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(z-\bar{z})^2}{2\sigma^2}} \quad \text{Gaussian} \]

\[ p(z) = 0.607 \frac{2^{1/2}}{b} e^{-\frac{z^2}{2b}} \quad \text{Rayleigh} \]

\[ p(z) = K e^{-(z-a-1)/(b-1)} \quad \text{Gamma} \]

\[ p(z) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq z \leq b \\ 0 & \text{otherwise} \end{cases} \quad \text{Uniform} \]

\[ p(z) = \begin{cases} P_b & \text{for } a \leq z \leq b \\ P_a & \text{for } z < a \\ 0 & \text{otherwise} \end{cases} \quad \text{Impulse} \]

FIGURE 5.2 Some important probability density functions.
• Test Pattern
  – Histogram has three spikes (Impulse)
  – For two independent random variables $(x, y)$

$$z = x + y \Rightarrow p_z(z) = p_x(x) * p_y(y)$$
Digital Image Processing

Image Restoration and Reconstruction

- Noisy Images (1):

Gaussian  Rayleigh  Gamma
Digital Image Processing

Image Restoration and Reconstruction

- Noisy Images (2):

Exponential  Uniform  Salt & Pepper
• Periodic\textsuperscript{1} Noise:
  – Electronic Devices

\textsuperscript{1} Disturbance
• Estimation of Noise Parameters
  – Periodic noise
    • Observe the frequency spectrum
  – Random noise with PDFs
    • Case 1: Imaging system is available
      – Capture image of “flat” environment
    • Case 2: Single noisy image is available
      – Take a strip from constant area
      – Draw the histogram and observe it
      – Estimate PDF parameters or measure the mean and variance
• Noise Estimation:
  - Shape: Histogram of a subimage (Background)

**FIGURE 5.6** Histograms computed using small strips (shown as inserts) from (a) the Gaussian, (b) the Rayleigh, and (c) the uniform noisy images in Fig. 5.4.
• Noise Estimation:
  - Parameters:
    Maximum Likelihood method: \( \{ z_i \}_{i=1}^{N} \Rightarrow \theta = \arg \max \left\{ \prod_{i=1}^{N_S} p(z_i, \theta) \right\} \)
  - Momentum
    \[
    \mu = \sum_{z_i \in S} z_i p(z_i)
    \]
    \[
    \sigma^2 = \sum_{z_i \in S} (z_i - \mu)^2 p(z_i)
    \]
    Solve: \( \{ \theta \} \rightarrow \) PDF Parameters
  - Power Spectrum Estimation:
    \[
    |\tilde{N}(u,v)|^2 = E \left\{ |\Im \{ \eta(x,y) \}|^2 \right\} = \frac{1}{N} \sum_{i=1}^{N} |\Im \{ \eta_i(x,y) \}|^2
    \]
    \( \eta_i(x,y) \): Flat (No object) Subimages
Digital Image Processing

Image Restoration and Reconstruction

• Noise-only spatial filter:

\[ g(x, y) = f(x, y) + \eta(x, y) \Leftrightarrow G(u, v) = F(u, v) + N(u, v) \]

• Mean Filters:
  – A \( m \times n \) mask centered at \((x, y)\): \( S_{xy} \)
  – An algebraic operation on the mask pixels!
• Mean Filter:
  
  - Arithmetic mean:
    \[ \hat{f}(x, y) = \frac{1}{mn} \sum_{(s, t) \in S_{xy}} g(s, t) \]
  
  - Noise Reduction (uncorrelated, zero mean) vs. Blurring
  
  - Geometric mean:
    \[ \hat{f}(x, y) = \left( \prod_{(s, t) \in S_{xy}} (g(s, t)) \right)^{\frac{1}{mn}} \]
  
  - Same smoothing with less detail degradation:
• **Mean Filter:**
  - Harmonic mean:
    \[
    \hat{f}(x, y) = \frac{mn}{\sum_{(s, t) \in S_{xy}} \frac{1}{g(s, t)}}
    \]
  - Salt Reduction (Pepper will increase)/ Good for Gaussian like
  - Contra-harmonic mean:
    \[
    \hat{f}(x, y) = \frac{\sum_{(s, t) \in S_{xy}} (g(s, t))^{Q+1}}{\sum_{(s, t) \in S_{xy}} g(s, t)^Q}
    \]
  - \(Q>0\) for pepper and \(Q<0\) for salt noise
Digital Image Processing

Image Restoration and Reconstruction

Example (1):
- Additive Noise
- Mean
  - Arithmetic
  - Geometric

Original

Noisy

More-Less Blurring

Arithmetic

Geometric
• Example (2):
  – Impulsive
  – Contra-harmonic
  – Correct use

Noisy (Pepper)  Noisy (Salt)

\[ Q = \pm 1.5 \]
Example (3):
- Impulsive Noise
- Contra-harmonic
- Wrong use

Q=-1.5

Q=+1.5
• Noise-only spatial filter:

\[ g(x, y) = f(x, y) + \eta(x, y) \]

• Mean Filters:
  – A \( m \times n \) mask centered at \((x, y)\): \( S_{xy} \)
  – An Ordering/Ranking operation on mask members!
Image Restoration and Reconstruction

- **Median**: Most Familiar OS filter:
  \[
  \hat{f}(x, y) = \text{median}\{g(s, t)\}
  \]
  - Little Blurring/Single/Double valued noises

- **Max/min**:
  \[
  \hat{f}(x, y) = \text{Max}\{g(s, t)\}, \quad \hat{f}(x, y) = \text{min}\{g(s, t)\}
  \]
  - Find Brightest/Darkest Points (Pepper/Salt Reducer)
**Midpoint:**

- OS+Mean Properties (Gaussian, Uniform)

\[ \hat{f}(x, y) = \frac{1}{2} \left[ \max_{(s,t) \in S_{xy}} \{g(s,t)\} + \min_{(s,t) \in S_{xy}} \{g(s,t)\} \right] \]
Digital Image Processing

Image Restoration and Reconstruction

- **Alpha Trimmed mean:**
  - Delete \((d/2)\) lowest and highest samples

\[
\hat{f}(x, y) = \frac{1}{mn - d} \sum_{(s, t) \in S_{xy}} g_r(s, t)
\]

- \(d = 0 \rightarrow \) Mean Filter
- \(d = mn-1 \rightarrow \) Median Filter
- Others: Combinational Properties
Example (1):
- Impulsive Noise
- Median Filter
- 1-2-3- passes
- Less Blurring
Digital Image Processing

Image Restoration and Reconstruction

• Example (2):
  – Max and min Filters:

![3x3 Max Filter](image1)

![3x3 Min Filter](image2)
Example (3):

- Noise:
  - Uniform (a) and Impulsive (b)
- Mask size:
  - 5×5
- Filters
  - Arithmetic mean (c)
  - Geometric mean (d)
  - Median (e)
  - Alpha Trimmed (f)
Adaptive, local noise reduction:
- If $\sigma^2_\eta$ is small, return $g(x, y)$
- If $\sigma^2_L > \sigma^2_\eta$, return value close to $g(x, y)$
- If $\sigma^2_L \approx \sigma^2_\eta$, return the arithmetic mean $m_L$

$$\hat{f}(x, y) = g(x, y) - \frac{\sigma^2_\eta}{\sigma^2_L(x, y)} [g(x, y) - m_L(x, y)]$$

- Non-stationary noise or poor estimation:

$$\max \left[ \frac{\sigma^2_\eta}{\sigma^2_L(x, y)} \right] = 1$$
Digital Image Processing

Image Restoration and Reconstruction

- Example:
  - $N(0, 100)$

- Filters:
  - Arithmetic Mean
  - Geometric Mean
  - Local

Original Noisy  A. Mean

Original Noisy  A. Mean

G- Mean  Local
Image Restoration and Reconstruction

Adaptive Median:

\[ A_1 = Z_{med} - Z_{min}, \quad A_2 = Z_{med} - Z_{max} \]

if \((A_1 > 0 \text{ and } A_2 < 0)\) then

\[
\begin{align*}
B_1 &= Z_{xy} - Z_{min}, \\
B_2 &= Z_{xy} - Z_{max}
\end{align*}
\]

\[
\begin{align*}
&\text{if } (B_1 > 0 \text{ and } B_2 < 0) \text{ then } \hat{Z}_{xy} = Z_{xy} \text{ else } \hat{Z}_{xy} = Z_{med}
\end{align*}
\]

else

\[
\begin{align*}
\text{increase Mask size:} & \\
(\leq S_{max}): & \text{Check "A" condition.} \\
(> S_{max}): & \hat{Z}_{xy} = Z_{med}
\end{align*}
\]

\[
\begin{align*}
Z_{min} &= \text{Minimum intensity in } S_{xy} \\
Z_{max} &= \text{Maximum intensity in } S_{xy} \\
Z_{med} &= \text{Median of intensity in } S_{xy} \\
Z_{xy} &= \text{Intensity value at } (x, y) \\
S_{Max} &= \text{Maximum allowed size of } S_{xy}
\end{align*}
\]
- **Example:**
  - Noise: Salt and Pepper (Pa= Pb=0.25)
Image Restoration and Reconstruction

- Band Reject Filter (Periodic Noises)
  - Ideal, Butterworth, and Gaussian
Digital Image Processing

Image Restoration and Reconstruction

• Band Reject Filter (Periodic Noises)

\[
H_f(u, v) = \begin{cases} 
1 & D(u, v) < D_0 - \frac{W}{2} \\
0 & D_0 - \frac{W}{2} \leq D(u, v) \leq D_0 + \frac{W}{2} \\
1 & D(u, v) > D_0 + \frac{W}{2}
\end{cases}
\]

\[
H_{B_n}(u, v) = \frac{1}{1 + \left[ \frac{D(u, v)W}{D^2(u, v) - D_0^2} \right]^{2n}}
\]

\[
H_G(u, v) = 1 - \exp\left( -\frac{1}{2} \left[ \frac{D^2(u, v) - D_0^2}{D(u, v)W} \right]^2 \right)
\]
Digital Image Processing

Image Restoration and Reconstruction

- Example:

Noisy

Spectrum

BBRF(1)

DeNoised
• Band Pass Filter (Extract Periodic Noises)
  – Analysis of Spectrum in different bands
    \[ H_{bp}(u,v) = 1 - H_{br}(u,v) \]
  – Noise Pattern:
• Notch Reject Filter:

\[ NPF = 1 - NRF \]

INRF

GNRF

BNRF

NPF = 1 - NRF
• Notch Filter (Single Frequency Removal):

\[
D_1(u,v) = \left[ (u - M/2 - u_0)^2 + (v - N/2 - v_0)^2 \right]^{1/2}
\]

\[
D_2(u,v) = \left[ (u - M/2 + u_0)^2 + (v - N/2 + v_0)^2 \right]^{1/2}
\]

\[
H_I(u,v) = \begin{cases} 
0 & \text{if } D_1(u,v) \leq D_0 \text{ and } D_2(u,v) \leq D_0 \\
1 & \text{otherwise}
\end{cases}
\]

\[
H_B(u,v) = \frac{1}{1 + \left( \frac{D_0^2}{D_1(u,v)D_2(u,v)} \right)}
\]

\[
H_G(u,v) = 1 - \exp \left( -\frac{1}{2} \left( \frac{D_1(u,v)D_2(u,v)}{D_0^2} \right) \right)
\]
Digital Image Processing

Image Restoration and Reconstruction

• Example

• Horizontal Pattern in space
• Vertical Pattern in Frequency

Spectrum

Notch in Vertical lines

Notch Passed Spectrum Vertical

Notch Rejected Spectrum

E. Fatemizadeh, Sharif University of Technology, 2012
Digital Image Processing

Image Restoration and Reconstruction

- Several Single Frequency Noises:

Images  Spectrum
• Multiple Notch will disturb the image:
  – Extract interference pattern using Notch Pass
    \[ N(u, v) = H_{NP}(u, v)G(u, v) \Rightarrow \eta(x, y) = \mathcal{F}^{-1}\{H_{NP}(u, v)G(u, v)\} \]
  – Now Perform Filter Function in Spatial Domain as An Adaptive-Local Point Process:
    \[ \hat{f}(x, y) = g(x, y) - w(x, y)\eta(x, y) \]
  – Optimization Criteria: minimum variance of filtered images
    \[ \min\{\sigma_f^2\} \]
**Optimum Notch Filter:**

\[
\sigma_f^2(x, y) = \frac{1}{(2d + 1)^2} \sum_{s=-d}^{+d} \sum_{t=-d}^{+d} \left[ \hat{f}(x + s, y + t) - \bar{f}(x, y) \right]^2
\]

\[
\bar{f}(x, y) = \frac{1}{(2d + 1)^2} \sum_{s=-d}^{+d} \sum_{t=-d}^{+d} \hat{f}(x + s, y + t)
\]

\[
\therefore \sigma_f^2(x, y) = \frac{1}{(2d + 1)^2} \sum_{s=-d}^{+d} \sum_{t=-d}^{+d} \left\{ \left[ g(x + s, y + t) - w(x + s, y + t)\eta(x + s, y + t) \right] - \left[ \bar{g}(x, y) - \bar{w}(x, y)\bar{\eta}(x, y) \right] \right\}^2
\]

Smoothness of \( w(x, y) \):

\[
w(x + s, y + t) \approx w(x, y), \quad w(x, y)\eta(x, y) \approx w(x, y)\bar{\eta}(x, y)
\]
• **Optimum Notch Filter:**

\[
\sigma^2_j(x, y) = \frac{1}{(2d+1)^2} \sum_{s=-d}^{+d} \sum_{t=-d}^{+d} \left\{ g(x+s, y+t) - w(x, y) \eta(x+s, y+t) - \cdots \left[ \bar{g}(x, y) - w(x, y) \bar{\eta}(x, y) \right] \right\}^2
\]

\[
\frac{\partial \sigma^2_j(x, y)}{\partial w(x, y)} = 0 \Rightarrow w(x, y) = \frac{g(x, y) \eta(x, y) - \bar{g}(x, y) \bar{\eta}(x, y)}{\eta^2(x, y) - \bar{\eta}^2(x, y)}
\]

• Computed for each pixel using surrounded mask
Digital Image Processing

Image Restoration and Reconstruction

- Result:

Former images Spectrum (without fftshift)
Digital Image Processing

Image Restoration and Reconstruction

Result:

\[ N(u,v) \quad \eta(x,y) \]
Digital Image Processing

Image Restoration and Reconstruction

• Result:

Filtered Image
Image Restoration and Reconstruction

• Linear Degradation:

\[
g(x, y) = H\{f(x, y)\} + \eta(x, y)
\]

L-System:

\[
H\{f(x, y)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha, \beta) H\{\delta(x - \alpha, y - \beta)\} d\alpha d\beta
\]

\[
h(x, y, \alpha, \beta) = H\{\delta(x - \alpha, y - \beta)\}
\]

LSI-System:

\[
H\{f(x, y)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha, \beta) h(x - \alpha, y - \beta) d\alpha d\beta
\]

\[
g(x, y) = f(x, y) \star h(x, y) + \eta(x, y)
\]

\[
G(u, v) = F(u, v) H(u, v) + N(u, v)
\]
Degradation Estimation:

- Image Observation:
  - Look at the image and ...

- Experiments:
  - Acquire image using well defined object (pinhole)

- Modeling:
  - Introduce certain model for certain degradation using physical knowledge.
Degradation (Using Experimental PSF)

\[ H_s(u,v) = \frac{G(u,v)}{A_{PSF}} \]
Digital Image Processing

Image Restoration and Reconstruction

• Atmospheric Turbulence:

![Atmospheric Turbulence Images](NASA)
• Modeling of turbulence in atmospheric images:

\[ H(u, v) = \exp \left( -k \left( u^2 + v^2 \right)^{5/6} \right) \]

**FIGURE 5.25**
Illustration of the atmospheric turbulence model.
(a) Negligible turbulence.
(b) Severe turbulence, \( k = 0.0025 \).
(c) Mild turbulence, \( k = 0.001 \).
(d) Low turbulence, \( k = 0.00025 \).
(Original image courtesy of NASA.)
Digital Image Processing

Image Restoration and Reconstruction

• Motion Blurring:
  – Camera Limitation;
  – Object movement.
Motion Blurring Modeling:

\[
g(x, y) = \int_{0}^{T} f(x - x_0(t), y - y_0(t)) dt
\]

\[
G(u, v) = \iint_{-\infty}^{\infty} \left( \int_{0}^{T} f(x - x_0(t), y - y_0(t)) dt \right) e^{-j2\pi(ux + vy)} dxdy
\]

\[
G(u, v) = F(u, v) \int_{0}^{T} e^{-j2\pi(ux_0(t) + vy_0(t))} dt = F(u, v) H(u, v)
\]
Linear one/Two dimensional motion blurring:

\[ x_0(t) = \frac{at}{T}, \quad t_{\text{Max}} = T \Rightarrow H(u,v) = \frac{T}{\pi u a} \sin(\pi u a) e^{-j\pi u a} \]

\[ x_0(t) = \frac{at}{T}, \quad y_0(t) = \frac{bt}{T} \Rightarrow H(u,v) = \frac{T}{\pi (u a + v b)} \sin(\pi (u a + v b)) e^{-j\pi (u a + v b)} \]
• Motion Blurring Example (1): 
  – \( a=b=0.1 \) and \( T=1 \) 
  – Original (Left) and Notion Blurred (Right)
• Motion Blurring Example (2):
• Uniform Motion Blurring (?):

\[
\begin{array}{ccc}
\frac{1}{9} & \times & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\quad \begin{array}{ccc}
\frac{1}{25} & \times & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\quad \begin{array}{ccc}
\frac{1}{49} & \times & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]
Digital Image Processing

Image Restoration and Reconstruction

• Out of Focus Blurring:

\[ h(x, y) = \frac{1}{\pi R^2} \text{circ} \left( \frac{\sqrt{x^2 + y^2}}{R} \right) \]

\[ \text{circ}(r) = \begin{cases} 
1 & r \leq 1 \\
0 & r > 1 
\end{cases} \]
Performance Metrics in Image Restoration:

- Definition
  - \( f(x, y) \): Clean Image
  - \( n(x, y) \): zero mean uncorrelated noise with variance \( \sigma^2 \)
  - \( g(x, y) \): Blurred Image, \( f(x, y) \star h(x, y) \)
  - \( \hat{f}(x, y) \): Restored Image

- SNR (Signal to Noise Ratio), Image Quality:
  \[
  SNR = 10 \log_{10} \frac{\sum_{x,y} f^2(x, y)}{\sigma^2}
  \]

- BSNR (Blurred SNR)
  \[
  BSNR = 10 \log_{10} \frac{\sum_{x,y} g^2(x, y)}{\sigma^2}
  \]
Digital Image Processing

Image Restoration and Reconstruction

- Performance Metrics in Image Restoration:
  - ISNR (Improvement in SNR):
    \[
    \text{ISNR} = 10 \log_{10} \frac{\sum_{x,y} (f(x,y) - g(x,y))^2}{\sum_{x,y} (f(x,y) - \hat{f}(x,y))^2}
    \]
Digital Image Processing

Image Restoration and Reconstruction

• Inverse Filtering:
  – Without Noise:

\[
\hat{F}(u,v) = \frac{G(u,v)}{\hat{H}(u,v)} = \frac{F(u,v)H(u,v)}{\hat{H}(u,v)} \approx F(u,v)
\]

⇒ Problem of division by zero or NaN!
Inverse Filtering:

- With Noise:

\[
\hat{F}(u,v) = \frac{G(u,v)}{\hat{H}(u,v)} = \frac{F(u,v)H(u,v) + N(u,v)}{\hat{H}(u,v)} \approx F(u,v) + \frac{N(u,v)}{\hat{H}(u,v)}
\]

⇒ Problem of division by zero!

⇒ Impossible to recover even if H(.,.) is known!!
• Pseudo Inverse (Constrained) Filtering:
  – Set infinite (large) value to zero;

\[
F(u, v) = \begin{cases} 
  \frac{G(u, v)}{\hat{H}(u, v)} & |\hat{H}(u, v)| \geq H_{THR} \\
  0 & |\hat{H}(u, v)| < H_{THR}
\end{cases}
\]
Example:
- Full Band (a)
- Radius 50 (b)
- Radius 70 (c)
- Radius 85 (d)
Digital Image Processing

Image Restoration and Reconstruction

- Threshold Effect:

Ghost
Digital Image Processing

Image Restoration and Reconstruction

• Phase Problem:
Wiener Filtering in 2D case:

\[ g(x, y) = s(x, y) + \eta(x, y) \Rightarrow G(u, v) = S(u, v) + N(u, v) \]

\[ \hat{F}(u, v) = W(u, v).G(u, v) \]

\[ E(u, v) = F(u, v) - \hat{F}(u, v) = F(u, v) - W(u, v).G(u, v) \]

\[
E\left\{ \left| E(u, v) \right|^2 \right\} = E\left\{ (F - WG)(F - WG)^* \right\} \\
= E\left\{ FF^* + WGG^*W^* - WGF^* - FW^*G^* \right\} \\
= E\left\{ |F|^2 + W|G|^2 W^* - WGF^* - W^*G^*F \right\}
\]
Wiener Filtering in 2D case:

\[
E \left\{ |E(u, v)|^2 \right\} = P_{FF} + WP_{GG} W^* - W^* P_{FG} - WP_{GF}
\]

\[
\frac{\partial E \left\{ |E(u, v)|^2 \right\}}{\partial W} = 0 \Rightarrow W(u, v) = \frac{P_{FG}(u, v)}{P_{GG}(u, v)}
\]

\[
P_{XX}(u, v) = E \left\{ |X|^2 \right\} : \text{Spectral Estimation}
\]

\[
P_{XY}(u, v) = E \left\{ XY^* \right\} : \text{Cross Spectral Estimation}
\]

\[
P_{XY}(u, v) = P_{YX}^*(u, v)
\]
**Wiener Filtering in 2D case:**

- **Special Cases:**
  - **Noise Only:**

\[
g(x, y) = f(x, y) + \eta(x, y) \Leftrightarrow G(u, v) = F(u, v) + N(u, v)
\]

Uncorrelated "zero mean noise" and "image":

\[
P_{FG}(u, v) = E \left\{ F \left( F + N^* \right) \right\} = E \left\{ |F|^2 \right\} + E \{ F \} E \{ N^* \} = P_{FF}
\]

\[
P_{GG}(u, v) = E \left\{ (F + G) \left( F + N^* \right) \right\} = E \left\{ |F|^2 \right\} + E \left\{ |N|^2 \right\} + E \{ F \} E \{ N^* \} + E \{ F^* \} E \{ N \} = P_{FF} + P_{NN}
\]

\[
W(u, v) = \frac{P_{FF}(u, v)}{P_{FF}(u, v) + P_{NN}(u, v)}
\]
Digital Image Processing

Image Restoration and Reconstruction

• Degradation plus Noise:

\[ g(x, y) = f(x, y) * h(x, y) + \eta(x, y) \]

\[ G(u,v) = F(u,v) H(u,v) + N(u,v) \]

Uncorrelated Noise and Image:

\[ W(u,v) = \frac{P_{FF} H^*}{P_{FF} |H|^2 + P_{NN}} = \frac{H^*}{|H|^2 + \frac{P_{NN}}{P_{FF}}} \]

\[ = \frac{1}{H} \frac{|H|^2}{|H|^2 + \frac{P_{NN}}{P_{FF}}} = \frac{1}{H} \frac{|H|^2}{|H|^2 + \left(\frac{P_{FF}}{P_{NN}}\right)^{-1} SNR} \]
• Degradation plus Noise:
  – White Noise

\[ W(u, v) = \frac{1}{H} \frac{|H|^2}{|H|^2 + K} \]

Select Interactively
• Wiener Filter is known as:
  – Wiener-Hopf
  – Minimum Mean Square Error
  – Least Square Error

• Problems with Wiener:
  – $P_{FF}$
  – $P_{NN}$
• Phase in Wiener Filter:

\[
W = \frac{1}{H} \frac{|H|^2}{|H|^2 + \frac{P_{NN}}{P_{FF}}} \Rightarrow \left\{ \begin{array}{l}
|W| = \frac{|H|}{|H|^2 + \frac{P_{NN}}{P_{FF}}} \\
\angle W = -\angle H = \angle \left(\frac{1}{H}\right)
\end{array} \right.
\]

• No Phase compensation!
• Wiener Filter vs. Inverse Filter:

\[ W = \frac{H^*}{|H|^2 + \frac{P_{NN}}{P_{FF}}} \implies \lim_{P_{NN} \to \infty} W = \frac{H^*}{|H|^2} = \begin{cases} \frac{1}{H} & H \neq 0 \\ 0 & H = 0 \end{cases} \]
• Other Wiener Related Filter:

\[ W_S = \left( \frac{1}{H} \right)^\alpha \left( \frac{H^*}{|H|^2 + \beta \frac{P_{NN}}{P_{FF}}} \right)^{1-\alpha} \], \quad 0 \leq \alpha \leq 1

\[ W_{0.5} = \sqrt{\frac{1}{|H|^2 + \frac{P_{NN}}{P_{FF}}}} \exp\left( -j\angle\left( \frac{1}{H} \right) \right) \]
• Nonlinear Filter:

\[
\hat{F}(u, v) = \left| G(u, v) \right|^\alpha e^{j\angle G(u, v)} = \left| G(u, v) \right|^{\alpha-1} G(u, v)
\]

\[\begin{align*}
\alpha &\ll 1 \quad \text{HP like} \\
\alpha &\gg 1 \quad \text{LP like}
\end{align*}\]

\[
\hat{F}(u, v) = \begin{cases} 
\log\left(\left| G(u, v) \right| \right) e^{j\angle G(u, v)} & |G(u, v)| > G_{\min} \\
0 & \text{O.W.}
\end{cases}
\]
Image Restoration and Reconstruction

- Example:

- Full Inverse
- Pseudo Inverse
- Wiener
Digital Image Processing

Image Restoration and Reconstruction

**Example:**
- Motion Blurring + Noise

![Degraded Image](image1)
![Inverse Image](image2)
![Wiener Image](image3)

Noise Decrease
• **Algebraic Approach**

  – **A Matrix Review:**

  • Matrix Diagonalization\(^1\): Each square matrix \( A \) (with \( n \) linearly independent eigenvectors) can be decomposed into the very special form:

\[
A = PDP^{-1}
\]

• **\( P \):** Matrix composed of the eigenvectors of \( A \).

• **\( D \):** Diagonal matrix constructed from the corresponding eigenvalues of \( A \).

\(^1\) [http://mathworld.wolfram.com/MatrixDiagonalization.html](http://mathworld.wolfram.com/MatrixDiagonalization.html)
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Image Restoration and Reconstruction

- **Algebraic Approach:**
  - **Matrix Representation of Circular Convolution:**

    \[
    y(n) = f(n) * g(n) = \sum_{j=0}^{N-1} f(j) g(n-j), \ \text{Periodicity: } g(i) = g(i \pm N)
    \]

    In Matrix Form:

    \[
    \begin{bmatrix}
    y_0 \\
    y_1 \\
    y_2 \\
    \vdots \\
    y_{N-1}
    \end{bmatrix} =
    \begin{bmatrix}
    g_0 & g_{N-1} & g_{N-2} & \cdots & g_1 \\
    g_1 & g_0 & g_{N-1} & \cdots & g_2 \\
    g_2 & g_1 & g_0 & \cdots & g_3 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    g_{N-1} & g_{N-2} & g_{N-3} & \cdots & g_0
    \end{bmatrix}
    \begin{bmatrix}
    f_0 \\
    f_1 \\
    f_2 \\
    \vdots \\
    f_{N-1}
    \end{bmatrix}
    \]

    \[C: \ \text{A Circulant Matrix!}\]

    \[C(k, j) = g(k - j)\]
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Image Restoration and Reconstruction

- **Algebraic Approach:**
  - Diagonalization of Circular Convolution:

\[
C = WD W^* = WD W^{-1}, \quad W^* = W^{-1}
\]

\[
W(m, n) = e^{\frac{j \cdot 2\pi \cdot m \cdot n}{N}}
\]

\[
D = \begin{bmatrix}
G(0) & 0 & \ldots & 0 \\
0 & G(1) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & G(N-1)
\end{bmatrix}
\]

\[
G(k) = DFT\left(g(n)\right)_K
\]
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Image Restoration and Reconstruction

- **Algebraic Approach:**
  - Diagonalization of Circular Convolution:

  \[ f * g = \left( W D W^* \right) f = W D \begin{pmatrix} W^* f \end{pmatrix} \]

  - Extension to 2D case is similar!
**Digital Image Processing**

**Image Restoration and Reconstruction**

- **Algebraic Approach to Noise Reduction:**
  - Least Square Approach:
    - **Matrix Representation of Degradation Model:**
      \[
g = Hf + n \Rightarrow n = g - Hf
\]
      \[
\hat{f} = \arg \min_{f} \left\{ \|n\|^2 \right\} = \arg \min_{f} \left\{ \|g - Hf\|^2 \right\}
\]
      \[
\frac{\partial \|g - Hf\|^2}{\partial f} = \frac{\partial (g - Hf)(g - Hf)^T}{\partial f} = 0
\]
      \[
\frac{\partial (g - Hf)(g - Hf)^T}{\partial f} = -2H^T(g - Hf) = 0
\]
      \[
\therefore \hat{f} = \left( H^TH \right)^{-1} H^T g
\]
      - **Pseudo Inverse (Left):**
      \[
\left( H^T H \right)^{-1} H^T H = I
\]
• Least Square Approach:
  – If $H$ is square and $H^{-1}$ exists then: $\hat{f} = H^{-1}g$
  – This is Inverse Filter!

\[
\hat{f} = (H^T H)^{-1} H^T g = H^{-1}g
\]
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Image Restoration and Reconstruction

- **Constrained Least Square (CLS) Approach:**
  
  \[ \hat{f} = \arg \min_{f} \left\{ \|Qf\|^2 \right\} \text{ Const. to } \|g - Hf\|^2 = \|n\|^2 \]

- **Q** is a linear operator (High pass version of image, Laplacian and etc.)
  
  - Using Lagrange Multiplier:
    
    \[ J(f) = \|Qf\|^2 + \lambda \left( \|g - Hf\|^2 - \|n\|^2 \right) \]
    
    \[ \frac{\partial J(f)}{\partial f} = 2Q^TQf - 2\lambda H^T(g - Hf) \]
    
    \[ \therefore \hat{f} = \left( H^TH + \frac{1}{\lambda} Q^TQ \right)^{-1} H^Tg = \left( H^TH + \gamma Q^TQ \right)^{-1} H^Tg \]
    
    \[ \lambda : \text{Lagrange Multiplier which should be be satisfy: } \|g - Hf\|^2 = \|n\|^2 \]
• CLS in frequency domain:
  - Diagonalization:

\[
H = WD_H W^{-1} = WD_H W^* \\
H^T = WD_H^* W^{-1} = WD_H^* W^*
\]

\[
\Rightarrow \left\{ \begin{array}{l}
H^T H = W |D_H|^2 W^* \\
Q^T Q = W |D_Q|^2 W^*
\end{array} \right.
\]

\[
\left( H^T H + \frac{1}{\lambda} Q^T Q \right) f = H^T g \Rightarrow W \left( |D_H|^2 + \gamma |D_Q|^2 \right) W^* f = WD_H^* W^* g
\]

Multiply by \( W^{-1} \):

\[
\left( |D_H|^2 + \gamma |D_Q|^2 \right) W^* f = D_H^* W^* g \Rightarrow \left( |H(u, v)|^2 + \gamma |C(u, v)|^2 \right) \hat{F}(u, v) = H^*(u, v) G(u, v)
\]

\( C(u, v) \): Fourier Transform of \( Q \) (Linear Operator!)

\[
\Rightarrow \hat{F}(u, v) = \frac{H^*(u, v)}{|H(u, v)|^2 + \gamma |C(u, v)|^2} G(u, v)
\]
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Image Restoration and Reconstruction

• CLS in frequency domain:

\[
\hat{F}(u, v) = \frac{H^*(u, v)}{|H(u, v)|^2 + \gamma |C(u, v)|^2} G(u, v)
\]

\[
\hat{f} = \left( H^T H + \frac{1}{\lambda} Q^T Q \right)^{-1} H^T g
\]
Image Restoration and Reconstruction

- Wiener Filter vs. CLS filter:

\[
\begin{align*}
\mathbf{R}_f &= E\{\mathbf{f}\mathbf{f}^T\} \\
\mathbf{R}_n &= E\{\mathbf{n}\mathbf{n}^T\} \\
\mathbf{Q}^T\mathbf{Q} &= \mathbf{R}_f^{-1}\mathbf{R}_n
\end{align*}
\]

\[
\hat{F}(u,v) = \frac{H^*(u,v)}{|H(u,v)|^2 + \gamma |C(u,v)|^2} G(u,v)
\]

\[
\hat{f} = \left(\mathbf{H}^T\mathbf{H} + \frac{1}{\lambda} \mathbf{Q}^T\mathbf{Q}\right)^{-1} \mathbf{H}^T g
\]

\(\gamma = 0 \iff \text{Inverse Filter}\)

\(\gamma = 1 \iff \text{Wiener Filter (General form)}\)
For Laplacian as a measure of smoothness:

\[ P(x, y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \]

\[
Q = \begin{bmatrix}
C_0 & C_{M-1} & C_{M-2} & \ldots & C_1 \\
C_1 & C_0 & C_{M-1} & \ldots & C_2 \\
C_2 & C_1 & C_0 & \ldots & C_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_{M-1} & C_{M-2} & C_{M-3} & \ldots & C_0
\end{bmatrix}
\]

\[
C_j = \begin{bmatrix}
P_e(j,0) & P_e(j,N-1) & \ldots & P_e(j,1) \\
\vdots & \vdots & \ddots & \vdots \\
P_e(j,N-1) & \ldots & \ldots & P_e(j,0)
\end{bmatrix}
\]
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Image Restoration and Reconstruction

• How to estimate $\gamma$:

Residual Vector: $r = g - H\hat{f}$

It can be shown that $\varphi(r) = |r|^2 = r^Tr$ is monotonically increasing with $\gamma$.

We seek for $\gamma$ such that $|r|^2 = |n|^2 \pm \delta$, $\delta$ is our accuracy

1. Initial Guess on $\gamma$

2. Calculate $|r|^2 = |g - H\hat{f}|^2$

3. If criteria $\left(|r|^2 = |n|^2 \pm \delta\right)$ is not reached:
   
   3.a if $\left(|r|^2 < |n|^2 - \delta\right)$ increase $\gamma$.

   3.b if $\left(|r|^2 > |n|^2 + \delta\right)$ decrease $\gamma$.

4. Re-calculate new filter and repeat until criteria is metted.
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Image Restoration and Reconstruction

• How to estimate γ:

  – Computational Tips:

    We need $\|r\|^2$ and $\|n\|^2$.

    $$R(u, v) = G(u, v) - H(u, v) \hat{F}(u, v) \Rightarrow \|r\|^2 = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} r^2(x, y)$$

    $$\sigma_n^2 = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left| n(x, y) - m_n \right|^2$$

    $$m_n = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} n(x, y)$$

    $$\|n\|^2 = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} n^2(x, y)$$

    $\therefore \|n\|^2 = MN \left( \sigma_n^2 + m_n^2 \right)$

    Need only $\sigma_n^2$ and $m_n^2$. 

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Image Restoration and Reconstruction

• Example:
  - CLS

High Noise    Mid Noise    Low Noise
Digital Image Processing

Image Restoration and Reconstruction

- Example:

Correct Noise Parameter

Incorrect Noise Parameter
**Computation Efforts:**

\[ g = Hf + n \Rightarrow n = g - Hf \]

\[ \hat{f} = \left( H^T H \right)^{-1} H^T g \]

H is size of MN × MN (65536 × 65536)!!!

**Gradient Descent Methods of function minimization:**

\[
\min \Phi(x) \underbrace{\stackrel{GD}{\longrightarrow}}_{x_{(k+1)}} x_{(k)} = x_{(k)} - \mu \left. \frac{\partial \Phi(x)}{\partial x} \right|_{(k)} = x_{(k)} - \mu \nabla \Phi \left( x_{(k)} \right)
\]

Convergence Condition: \( 0 \leq \mu \leq \frac{2}{\max \left\{ |\lambda| \right\}} \)

\( \lambda \): Eigenvalue of \( E \{ xx^T \} \)
Iterative LS or CLS (Tikhonov-Miller):

**LS:**
\[ g = Hf + n \Rightarrow n = g - Hf \]
\[ \Phi(f) = \frac{1}{2} \| g - Hf \|^2 \Rightarrow \frac{\partial \| g - Hf \|^2}{\partial f} = \nabla \Phi = -H^T (g - Hf) \]

1. \( f_{(0)} = \mu H^T g \)

2. \( f_{(k+1)} = f_{(k)} + \mu H^T (g - Hf_{(k)}) = \mu H^T g + (I - \mu H^T H)f_{(k)} = f_{(0)} + (I - \mu H^T H)f_{(k)} \)

**CLS:**
\[ J(f) = \frac{1}{2} \left( \|Qf\|^2 + \lambda \left( \|g - Hf\|^2 - \|n\|^2 \right) \right) \Rightarrow \frac{\partial J(f)}{\partial f} = Q^T Qf - \lambda H^T (g - Hf) \]

1. \( f_{(0)} = \mu H^T g \)

2. \( f_{(k+1)} = f_{(k)} + \mu \left( \lambda H^T (g - Hf_{(k)}) - Q^T Qf_{(k)} \right) = \mu \lambda H^T g + \left( I - \mu \left( \lambda HH^T + Q^T Q \right) \right)f_{(k)} \)
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Image Restoration and Reconstruction

- Example (Uniform Noise):
  - Blurred by a 7×7 Disc (BSNR=20dB)
  - Restored Using Generalized Inverse Filter with $T=10^{-3}$ (ISNR=-32.9dB)
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Image Restoration and Reconstruction

• Example (Gaussian Noise):
  – Blurred by a 5×Gaussian ($\sigma^2=1$) (BSNR=20dB)
  – Restored Using Generalized Inverse Filter with $T=10^{-3}$ (ISNR=-36.6dB)
Example (Uniform Noise):

- Blurred by a 7×7 Disc (BSNR=20dB)
- Restored Using Generalized Inverse Filter with $T=10^{-1}$ (ISNR=+0.61 dB)
Example (Gaussian Noise):

- Blurred by a 5xGaussian ($\sigma^2=1$) (BSNR=20dB)
- Restored Using Generalized Inverse Filter with $T=10^{-1}$ (ISNR=-1.8 dB)
• Example (Uniform Noise):
  – Blurred by a 7×7 Disc (BSNR=20dB)
  – Restored Using CLS Filter with $\Upsilon=1$ (ISNR=+2.5 dB)
• Example (Gaussian Noise):
  – Blurred by a $5 \times$ Gaussian ($\sigma^2=1$) (BSNR=20dB)
  – Restored Using CLS Filter with $\gamma=1$ (ISNR=+1.3 dB)
Example (Uniform Noise):
- Blurred by a 7×7 Disc (BSNR=20dB)
- Restored Using CLS Filter with $\gamma=10^{-4}$ (ISNR=-21.9 dB)
Digital Image Processing

Image Restoration and Reconstruction

- Example (Gaussian Noise):
  - Blurred by a $5 \times$ Gaussian ($\sigma^2=1$) (BSNR=20dB)
  - Restored Using CLS Filter with $\gamma=10^{-4}$ (ISNR=-22.1 dB)
• Example:
• Wiener Filter Example (input):
  – Original (Left) and degraded with SNR = 7dB (Right)
Wiener Filter Example (output):
- Known Noisy Image (Left) and Known Clear Image (Right)
Iterative Wiener Filter:

- Wiener in Matrix form (Spatial Filter):

\[ g = Hf + n \]

\[ W = R_f H^T \left( HR_f H^T + R_n \right)^{-1} \quad T \text{ is Hermitian Transform!} \]

\[ \hat{f} = Wf \]

- \( R_f \) : Autocorrelation of \( f \)
- \( R_n \) : Autocorrelation of \( n \)
Iterative Wiener Filter:

- Conditions:
  1. The original image and noise are statistically uncorrelated.
  2. The power spectral density (or autocorrelation) of the original image and noise are known.
  3. Both original image and the noise are zero mean.
Iterative Wiener Filter:

- Iterative Algorithm:

1. \( R_f (0) = R_g \)
2. \( W(i+1) = R_f (i) H^T \left( HR_f (i) H^T + R_n \right)^{-1} \)
3. \( \hat{f}(i+1) = W(i+1)g \)
4. \( R_f (i+1) = E \left\{ \hat{f}(i+1) \hat{f}^T (i+1) \right\} \)
5. Repeat 2,3,4 until convergence.
Digital Image Processing

Image Restoration and Reconstruction

- Example Iterative Wiener Filter:

  original
  degraded (blur filter, SNR=40dB)

  Iterative 1st
  Iterative 25th
  Additive 25th
Adaptive Wiener Filter:
- Image are Non Stationary!
- Need Adaptive WF which is locally optimal.
- Assume small region which image are stationary

Image Model in each region:
\[ f(x, y) = \mu_f(x, y) + \sigma_f \eta(x, y) \]
\[ \eta : \text{zero-mean white noise with unit variance!} \]
\[ \mu_f, \sigma_f : \text{Constant over each region.} \]

Noisy Image:
\[ g(x, y) = f(x, y) + v(x, y), \quad \sigma_v : \text{Constant over each region} \]
Local Wiener Filter in each region:

\[ W_a(u, v) = \frac{P_{ff}}{P_{ff} + P_{vv}} = \frac{\sigma_f^2}{\sigma_f^2 + \sigma_v^2} \]

\[ w_a(x, y) = \frac{\sigma_f^2}{\sigma_f^2 + \sigma_v^2} \delta(x, y) \]

\[ \hat{f}(x, y) = (g(x, y) - \mu_f) * w_a(x, y) + \mu_f \]

\[ \hat{f}(x, y) = (g(x, y) - \mu_f) \frac{\sigma_f^2}{\sigma_f^2 + \sigma_v^2} + \mu_f \]

\( \mu_f(x, y) \): Low-pass filtered on noisy image.

\( \mu_f(x, y) = \mu_g(x, y) \), Zero mean assumption

\( g(x, y) - \mu_f(x, y) \): Hi-pass filtered on noisy image.

\[ \therefore \hat{f}(x, y) = HP(x, y) \frac{\sigma_f^2(x, y)}{\sigma_f^2(x, y) + \sigma_v^2} + LP(x, y) \]
Parameter Estimation:

\[ \sigma_g^2 = \text{Local Noisy Image Variance} \]
\[ \sigma_v^2 = \text{Variance in a smooth image region or background} \]
\[ \sigma_f^2(x, y) = \text{Max}\left(\sigma_g^2(x, y) - \sigma_v^2, 0\right) \]
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Image Restoration and Reconstruction

Example (1):

- **example simulations**
- **noiseless image** $f(m,n)$
- **noise (variance = 100)** $v(m,n)$
- **noisy signal** $g(m,n)$
Example (1):
Results:

\[
\begin{align*}
LP(m, n) + \frac{\sigma_f^2(m, n)}{\sigma_f^2(m, n) + \sigma_v^2} HP(m, n) \\
\frac{\sigma_f^2(m, n)}{\sigma_f^2(m, n) + \sigma_v^2} HP(m, n)
\end{align*}
\]
Results:

Another example with higher noise (variance = 225)

\[
\begin{align*}
g(m, n) & \quad LP(m, n) \\
\frac{\sigma_f^2(m, n)}{\sigma_f^2(m, n) + \sigma_v^2} HP(m, n)
\end{align*}
\]
Spatial Transform:

\[
\begin{bmatrix}
  x \\
  y \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  x' \\
  y' \\
\end{bmatrix}
\]

\[r(x, y) = f(x, y)\]
\[s(x, y) = g(x', y')\]

tiepoints

original

_distorted_
• An example of Spatial Transform:

\[
x' = c_1 x + c_2 y + c_3 xy + c_4
\]

\[
y' = c_5 x + c_6 y + c_7 xy + c_8
\]
• Gray Level Interpolation:
  – Main Problem: Non-Integer coordination!
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Image Restoration and Reconstruction

• Gray Level Interpolation:
  – Nearest neighbor
  – Bilinear interpolation
    • Use 4 nearest neighbors
  – Cubic convolution interpolation
    • Fit a surface over a large number of neighbors
• Example (1):

Original

Tiepoints after distortion

Nearest neighbor

Bilinear

restored

restored
Digital Image Processing

Image Restoration and Reconstruction

- Example (2):

original

![Original Image](image1.png)

distorted

![Distorted Image](image2.png)

Difference

![Difference Image](image3.png)

restored

![Restored Image](image4.png)
Matlab Image Restoration Command:
- `deconvblind`: Restore image using blind deconvolution
- `deconvlucy`: Restore image using accelerated Richardson-Lucy algorithm
- `deconvreg`: Restore image using Regularized filter
- `deconvwnr`: Restore image using Wiener filter
- `wiener2`: Perform 2-D adaptive noise-removal filtering
- `edgetaper`: Taper the discontinuities along the image edges