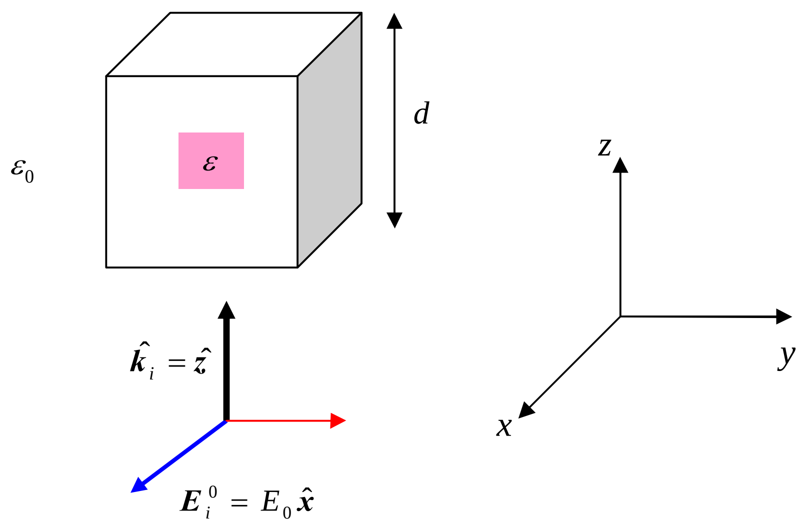
EM Scattering

Homework assignment 1

Problem 1:

An incident wave travels in free space (dielectric constant ε_0, permeability μ_0, wavenumber $k = \omega\sqrt{\varepsilon_0\mu_0}$) along the z-direction and impinges upon a cube of dielectric material with a constant permittivity of ε. The faces of the dielectric cube are parallel to the x-y, y-z, z-x planes respectively. The length of each side of the cube is d. The polarization of the incident electric field is along the x-axis. Use the Born approximation to calculate the scattered electric and magnetic fields in the far-field zone along the $\hat{\boldsymbol{k}}_s = \pm\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{k}}_s = \pm\hat{\boldsymbol{x}}$, and $\hat{\boldsymbol{k}}_s = \pm\hat{\boldsymbol{y}}$ directions.



Solution

In Born approximation we have for the far-zone scattering field:

$$\boldsymbol{E}_s^f(\boldsymbol{r}) = \frac{\exp(-jkr)}{r}\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \tag{1.1}$$

$$\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) = \left\{E_0\int_V \exp\left[j\left(\boldsymbol{k}_s - \boldsymbol{k}_i\right)\cdot\boldsymbol{r}\right]\delta\varepsilon_p(\boldsymbol{r})dV\right\}\left[\hat{\boldsymbol{e}}_i - \left(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right] \tag{1.2}$$

Where $\boldsymbol{k}_i = k\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{e}}_i = \hat{\boldsymbol{x}}$, $\delta\varepsilon_p(\boldsymbol{r}') = \varepsilon - \varepsilon_0$. Note that within this approximation no wave is scattered along the $\hat{\boldsymbol{k}}_s = \pm\hat{\boldsymbol{x}}$, i.e., along a direction parallel to the polarization of the incident plane wave. Along $\hat{\boldsymbol{k}}_s = \pm\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{k}}_s = \pm\hat{\boldsymbol{y}}$, the scattered wave is polarized along $\hat{\boldsymbol{x}} = \hat{\boldsymbol{e}}_i$ as $\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{k}}_s = 0$. The integral in (1.2) yields for $\hat{\boldsymbol{k}}_s = \pm\hat{\boldsymbol{y}}$

$$\begin{aligned}&(\varepsilon-\varepsilon_0)\int_V \exp\left[j\left(\pm k\hat{\boldsymbol{y}} - k\hat{\boldsymbol{z}}\right)\cdot\boldsymbol{r}\right]dV = (\varepsilon-\varepsilon_0)\int_V \exp\left(\pm jky - jkz\right)dV \\ &= \frac{4a(\varepsilon-\varepsilon_0)}{k^2}\sin^2\left(\frac{ka}{2}\right)\end{aligned} \tag{1.3}$$

In the case $\hat{\boldsymbol{k}}_s = \pm\hat{\boldsymbol{z}}$:

$$(\varepsilon-\varepsilon_0)\int_V \exp\left[j\left(\pm k\hat{\boldsymbol{z}} - k\hat{\boldsymbol{z}}\right)\cdot\boldsymbol{r}\right]dV = \begin{cases}(\varepsilon-\varepsilon_0)a^3, \ \hat{\boldsymbol{k}}_s = +\hat{\boldsymbol{z}} \\ (\varepsilon-\varepsilon_0)\dfrac{a^2}{k}\sin(ka), \hat{\boldsymbol{k}}_s = -\hat{\boldsymbol{z}}\end{cases} \tag{1.4}$$

Problem 2:

Repeat this calculation for a dielectric cylinder with the radius a, height h, and dielectric constant ε. Now find the scattered field in any direction $\hat{\boldsymbol{k}}_s$.

Solution

Let us again focus on the integral

$$\int_V \exp\left[j\left(\boldsymbol{k}_s - \boldsymbol{k}_i\right)\cdot\boldsymbol{r}\right]\delta\varepsilon_p(\boldsymbol{r})dV = (\varepsilon-\varepsilon_0)\int_V \exp\left[j\left(\boldsymbol{k}_s - k\hat{\boldsymbol{z}}\right)\cdot\boldsymbol{r}\right]dV \tag{1.5}$$

In cylindrical coordinates:

$$\boldsymbol{k}_s = \left(k_{s,\rho}\cos\phi_s, k_{s,\rho}\sin\phi_s, k_{s,z}\right), \boldsymbol{r} = \left(r\cos\varphi, r\sin\varphi, z\right) \tag{1.6}$$

So that the above integral becomes

$$\begin{aligned}&(\varepsilon-\varepsilon_0)\int_0^a\int_0^{2\pi}\int_{-h/2}^{h/2} \exp\left[jk_{s,\rho}r\cos\left(\varphi-\phi_s\right) + j\left(k_{s,z}-k\right)z\right]rdrd\varphi dz \\ &= (\varepsilon-\varepsilon_0)\frac{2\sin\left[\left(k_{s,z}-k\right)h/2\right]}{k_{s,z}-k}\int_0^a\int_0^{2\pi}\exp\left[jk_{s,\rho}r\cos\left(\varphi-\phi_s\right)\right]rdrd\varphi\end{aligned} \tag{1.7}$$

Note that

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left[jk_{s,\rho}r\cos\left(\varphi-\phi_s\right)\right]d\varphi = J_0\left(k_{s,\rho}r\right) \tag{1.8}$$

So that the integral yields

$$\begin{aligned}&\left(\varepsilon-\varepsilon_0\right)\frac{4\pi\sin\left[\left(k_{s,z}-k\right)h/2\right]}{k_{s,z}-k}\int_0^a J_0\left(k_{s,\rho}r\right)rdr = \left(\varepsilon-\varepsilon_0\right)\frac{4\pi\sin\left[\left(k_{s,z}-k\right)h/2\right]}{\left(k_{s,z}-k\right)k_{s,\rho}^2}\int_0^{k_{s,\rho}a} J_0\left(u\right)udu \\ &= \left(\varepsilon-\varepsilon_0\right)\frac{4\pi\sin\left[\left(k_{s,z}-k\right)h/2\right]}{\left(k_{s,z}-k\right)k_{s,\rho}^2}\left(uJ_1\left(u\right)\Big|_0^{k_{s,\rho}a}\right) = \left(\varepsilon-\varepsilon_0\right)\frac{4\pi a\sin\left[\left(k_{s,z}-k\right)h/2\right]}{\left(k_{s,z}-k\right)k_{s,\rho}}J_1\left(k_{s,\rho}a\right)\end{aligned} \tag{1.9}$$

Note that in spherical coordinates $k_{s,z} = k\cos\theta_s, k_{s,\rho} = k\sin\theta_s$. As a result, in spherical coordinates we have the result given by

$$\left(\varepsilon-\varepsilon_0\right)\frac{4\pi a\sin\left[\left(\cos\theta_s-1\right)kh/2\right]}{k^2\left(\cos\theta_s-1\right)\sin\theta_s}J_1\left(ka\sin\theta_s\right) \tag{1.10}$$

There is no dependence on ϕ_s here but remember that we have to include the vector

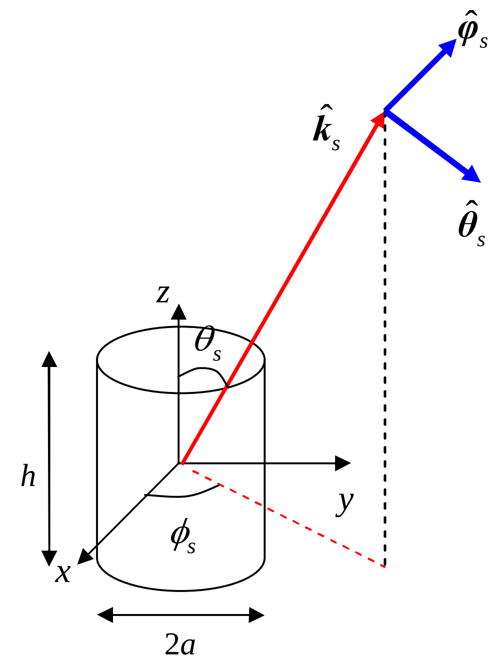
$$\hat{\boldsymbol{e}}_i-\left(\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s = \hat{\boldsymbol{x}}-k_{s,x}\hat{\boldsymbol{k}}_s = \hat{\boldsymbol{x}}-\sin\theta_s\cos\phi_s\hat{\boldsymbol{k}}_s \tag{1.11}$$

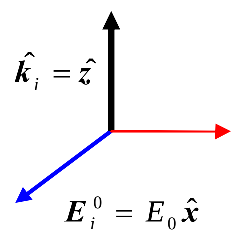
The final result is

$$\begin{aligned}\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right) &= E_0\frac{4\pi a\left(\varepsilon-\varepsilon_0\right)\sin\left[\left(\cos\theta_s-1\right)kh/2\right]}{k^2\left(\cos\theta_s-1\right)\sin\theta_s}J_1\left(ka\sin\theta_s\right)\left[\hat{\boldsymbol{x}}-\sin\theta_s\cos\phi_s\hat{\boldsymbol{k}}_s\right] \\ &= E_0\frac{4\pi a\left(\varepsilon-\varepsilon_0\right)\sin\left[\left(\cos\theta_s-1\right)kh/2\right]}{k^2\left(\cos\theta_s-1\right)\sin\theta_s}J_1\left(ka\sin\theta_s\right)\left[\cos\theta_s\cos\phi_s\hat{\boldsymbol{\theta}}_s-\sin\phi_s\hat{\boldsymbol{\phi}}_s\right]\end{aligned} \tag{1.12}$$

Where we have used

$$\left(\cos\theta_s\cos\phi_s,\cos\theta_s\sin\phi_s,-\sin\theta_s\right)=\hat{\boldsymbol{\theta}}_s,\ \left(-\sin\phi_s,\cos\phi_s,0\right)=\hat{\boldsymbol{\phi}}_s \tag{1.13}$$





Problem 3:

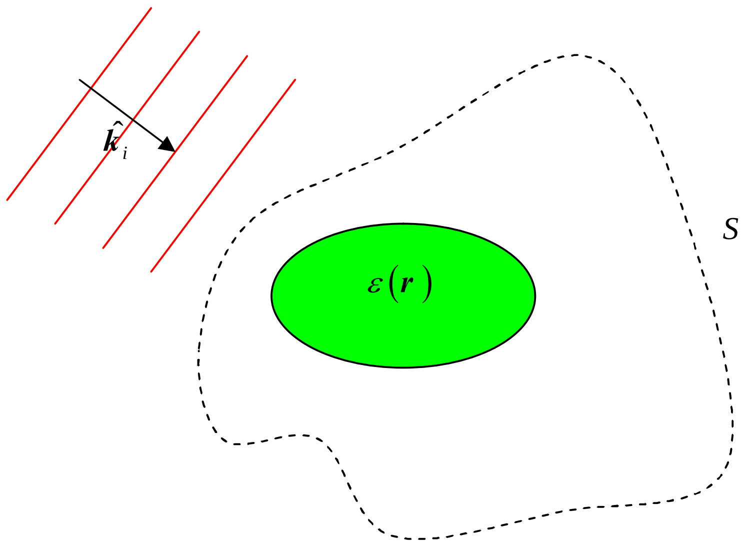
Consider a plane wave traveling in free space. The wave is incident upon a certain dielectric object of arbitrary shape with a dielectric constant $\varepsilon(\boldsymbol{r})$ generating equivalent currents inside the object. Let $\boldsymbol{E}_s, \boldsymbol{H}_s$ denote the scattered field, i.e., the field induced by these equivalent currents. In the far field zone the scattered electric field is given by:

$$\boldsymbol{E}_s^f(\boldsymbol{r}) = \frac{\exp(-jkr)}{r}\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)$$

Where $k = \omega\sqrt{\varepsilon_0\mu_0}$, $\hat{\boldsymbol{k}}_s$ denotes the direction of scattering and $\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)$ is the vector scattering amplitude (see the powerpoint slides for a description). Using the reciprocity theorem of electromagnetic theory show that

$$e_0 \cdot Q_\perp\left(\hat{k}_s, \hat{k}_i\right) = \frac{k\eta}{4\pi j}\left\{ e_0 \cdot \oint_S \left[\hat{n} \times E_s(r)\right] \exp\left(jk_s \cdot r\right) dS - h_0 \cdot \oint_S \left[H_s(r) \times \hat{n}\right] \exp\left(jk_s \cdot r\right) dS \right\}$$

Where e_0 is any arbitrary constant vector with $\hat{k}_s \cdot e_0 = 0$, and $h_0 = -(1/\eta)\hat{k}_s \times e_0$, $\eta = \sqrt{\mu_0 / \varepsilon_0}$, and S is any arbitrary closed surface surrounding the object. (See figure below)



Solution

The reciprocity theorem relates two different solutions of the Maxwell equations in the same medium. It states that

$$\oint_S \left(E^1 \times H^2 - E^2 \times H^1\right) \cdot ds = \int_V \left(E^2 \cdot J^1 - E^1 \cdot J^2\right) dV \qquad (1.14)$$

Here E^1, H^1 and E^2, H^2 denote the field generates by the volume sources J^1 and J^2, respectively. Let $E^1, H^1 = E_s, H_s$ be the scattered field generated by the equivalent currents inside the object $J^1(r) = j\omega\delta\varepsilon(r)E(r)$. Let the second field be that of a plane wave propagating in the direction of $-\hat{k}_s$. Thus

$$E^2 = E_0 \exp\left(jk_s \cdot r\right), H^2 = H_0 \exp\left(jk_s \cdot r\right), H_0 = -\frac{1}{\eta}\left(k_s \times E_0\right) \qquad (1.15)$$

This field has no sources inside the surface S surrounding the object. Hence, applying the reciprocity theorem, it follows that

$$\oint_S \left(\boldsymbol{E}_s \times \boldsymbol{H}^2 - \boldsymbol{E}^2 \times \boldsymbol{H}_s \right) \cdot d\boldsymbol{s} = \int_V \boldsymbol{E}^2 \cdot \boldsymbol{J}^1 dV \tag{1.16}$$

Which leads to

$$\begin{aligned} &\oint_S \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r} \right)\left(\boldsymbol{E}_s \times \boldsymbol{H}_0 - \boldsymbol{E}_0 \times \boldsymbol{H}_s \right) \cdot d\boldsymbol{s} = \boldsymbol{E}_0 \cdot \int_V \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r} \right) \boldsymbol{J}^1 dV \\ &= j\omega \boldsymbol{E}_0 \cdot \int_V \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r} \right) \delta\varepsilon\left(\boldsymbol{r} \right) \boldsymbol{E}\left(\boldsymbol{r} \right) dV = j\omega \frac{4\pi\epsilon_0}{k^2} \boldsymbol{E}_0 \cdot \boldsymbol{Q}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i \right) = \frac{4\pi j}{k\eta} \boldsymbol{E}_0 \cdot \boldsymbol{Q}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i \right) \end{aligned} \tag{1.17}$$

Note that $\boldsymbol{E}_0 \cdot \hat{\boldsymbol{k}}_s = 0$ as the electric field should be perpendicular to the propagation direction in a plane-wave solution. Thus $\boldsymbol{E}_0 \cdot \boldsymbol{Q}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i \right) = \boldsymbol{E}_0 \cdot \boldsymbol{Q}_\perp \left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i \right)$. Rewriting the left hand side of (1.17) we obtain

$$\oint_S \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r} \right)\left[\boldsymbol{H}_0 \cdot \left(\hat{\boldsymbol{n}} \times \boldsymbol{E}_s \right) - \boldsymbol{E}_0 \cdot \left(\boldsymbol{H}_s \times \hat{\boldsymbol{n}} \right) \right] ds = \frac{4\pi j}{k\eta} \boldsymbol{E}_0 \cdot \boldsymbol{Q}_\perp \left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i \right) \tag{1.18}$$

The above relation holds for any constant vector $\boldsymbol{E}_0$, as long as $\boldsymbol{E}_0 \cdot \hat{\boldsymbol{k}}_s = 0$. If we put $\hat{\boldsymbol{k}}_s = -\hat{\boldsymbol{k}}_i$ and replace $\boldsymbol{E}_0$ with the constant vector $\boldsymbol{E}_i^0$ of the incident wave we obtain the result:

$$\begin{aligned} \boldsymbol{E}_i^0 \cdot \boldsymbol{Q}_\perp \left(-\hat{\boldsymbol{k}}_i, \hat{\boldsymbol{k}}_i \right) = \frac{k\eta}{4\pi j} \Bigg\{ &\boldsymbol{e}_0 \cdot \oint_S \left[\hat{\boldsymbol{n}} \times \boldsymbol{E}_s\left(\boldsymbol{r} \right) \right] \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r} \right) dS - \\ &\boldsymbol{H}_i^0 \cdot \oint_S \left[\boldsymbol{H}_s\left(\boldsymbol{r} \right) \times \hat{\boldsymbol{n}} \right] \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r} \right) dS \Bigg\} \end{aligned} \tag{1.19}$$