EM Scattering

Homework assignment 2

Problem 1:

Consider a dielectric layer in free space. The layer has a relative dielectric constant of $\varepsilon_d = 2.25$, is infinite, and is parallel to the x-y plane. A plane wave coming from below the layer hits it at an angle equal to θ_i. The polarization of the incident wave is either TE^z where the electric field has no normal component (e.g., in the y-direction) or TM^z where the magnetic field has no normal component. The frequency of the wave is $f = 100\text{GHz}$.

1. Plot the magnitude of the reflection coefficient as function of θ_i in both TE and TM cases when the layer is 1 mm-thick
2. Plot the magnitude of the reflection coefficient as function of θ_i in both TE and TM cases when the layer is 5 mm-thick

Solution

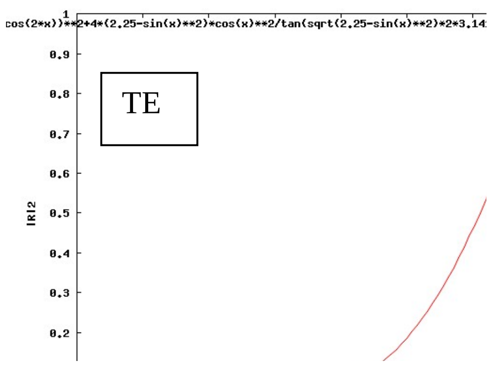
For the TE case:

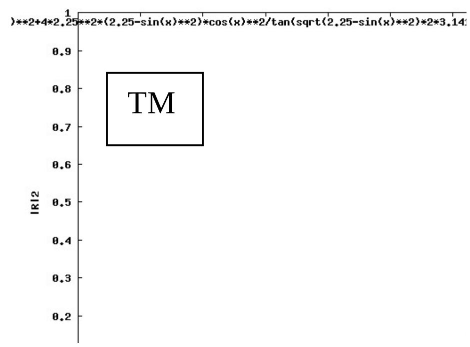
$$
\begin{aligned}
&R_{TE} = \frac{R_{TE}^{10}\left[1-\exp\left(-2jk_{1,z}d\right)\right]}{1-\left(R_{TE}^{10}\right)^2\exp\left(-2jk_{1,z}d\right)} = \frac{\left(k_{0,z}^2 - k_{1,z}^2\right)}{\left(k_{0,z}^2 + k_{1,z}^2\right) - 2jk_{0,z}k_{1,z}\cot\left(k_{1,z}d\right)} \\
&k_{0,z} = k_0\cos\theta_i \;,\; k_{1,z} = k_0\sqrt{\epsilon_d - \sin^2\theta_i} \;,\; \nu(\theta_i) \equiv \sqrt{\epsilon_d - \sin^2\theta_i} \\
&\rightarrow R_{TE} = \frac{\left(1-\epsilon_d\right)}{\epsilon_d + \cos\left(2\theta_i\right) - 2j\nu(\theta_i)\cos\theta_i\cot\left[\nu(\theta_i)k_0 d\right]} \\
&\rightarrow \left|R_{TE}\right|^2 = \frac{\left(1-\epsilon_d\right)^2}{\left[\epsilon_d + \cos\left(2\theta_i\right)\right]^2 + 4\left(\epsilon_d - \sin^2\theta_i\right)\cos^2\theta_i\cot^2\left[\nu(\theta_i)k_0 d\right]}
\end{aligned}
\tag{1.1}
$$

For the TM case:

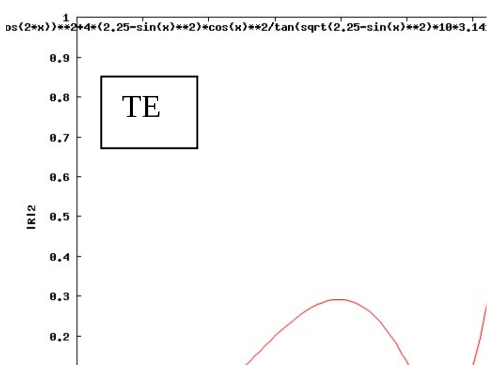
$$
\begin{aligned}
R_{TM} &= \frac{R_{TM}^{10}\left[1-\exp\left(-2jk_{1,z}d\right)\right]}{1-\left(R_{TM}^{10}\right)^2\exp\left(-2jk_{1,z}d\right)} = \frac{\epsilon_1^2 k_{0,z}^2 - \epsilon_0^2 k_{1,z}^2}{\epsilon_1^2 k_{0,z}^2 + \epsilon_0^2 k_{1,z}^2 - 2j\,\epsilon_0\epsilon_1 k_{0,z}k_{1,z}\cot\left(k_{1,z}d\right)} \\
&= \frac{\epsilon_d^2\cos^2\theta_i + \sin^2\theta_i - \epsilon_d}{\epsilon_d^2\cos^2\theta_i + \epsilon_d - \sin^2\theta_i - 2j\,\epsilon_d\nu(\theta_i)\cos\theta_i\cot\left[\nu(\theta_i)k_0 d\right]} \\
&= \frac{\left(\epsilon_d-1\right)\left[\epsilon_d\cos^2\theta_i - \sin^2\theta_i\right]}{\epsilon_d^2\cos^2\theta_i - \sin^2\theta_i + \epsilon_d - 2j\,\epsilon_d\nu(\theta_i)\cos\theta_i\cot\left[\nu(\theta_i)k_0 d\right]} \\
\rightarrow \left|R_{TM}\right|^2 &= \frac{\left(\epsilon_d-1\right)^2\left[\epsilon_d\cos^2\theta_i - \sin^2\theta_i\right]^2}{\left[\epsilon_d^2\cos^2\theta_i - \sin^2\theta_i + \epsilon_d\right]^2 + 4\epsilon_d^2\left(\epsilon_d - \sin^2\theta_i\right)\cos^2\theta_i\cot^2\left[\nu(\theta_i)k_0 d\right]}
\end{aligned} \tag{1.2}
$$

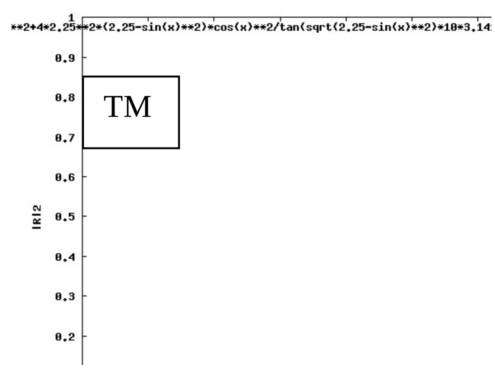
For a 1 mm-thick slab: $k_0 d = 2\pi d/\lambda = 2\pi/3$





For a 5 mm-thick slab: $k_0 d = 2\pi d/\lambda = 10\pi/3$





Problem 2:

A plane radio wave is incident on the ionosphere from the air. The relative dielectric constant of the ionosphere is $\varepsilon_d = 1 - \left(f_p / f \right)^2$ where f is the frequency of the wave and $f_p = 5\text{MHz}$ is the plasma frequency of the ionosphere. The ionosphere is assumed to be infinitely thick.

1. Compute the reflection coefficient for both TE and TM polarizations when the frequency of the wave is f = 3 MHz.
2. Repeat this calculation at the frequency of f = 6 MHz. Find the Brewster angle and the critical angle for total reflection.

Solution

TE and TM reflection coefficients:

$$R_{TE} = \frac{\cos\theta_i - \sqrt{\epsilon_d - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\epsilon_d - \sin^2\theta_i}},\ R_{TM} = \frac{\epsilon_d \cos\theta_i - \sqrt{\epsilon_d - \sin^2\theta_i}}{\epsilon_d \cos\theta_i + \sqrt{\epsilon_d - \sin^2\theta_i}} \tag{1.3}$$

1. At 3 MHz we have $\varepsilon_d = 1 - \left(f_p / f \right)^2 = -1.77$ so that

$$\begin{aligned} R_{TE} &= \frac{\cos\theta_i - \sqrt{\epsilon_d - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\epsilon_d - \sin^2\theta_i}} = \frac{\cos\theta_i + j\sqrt{\left|\epsilon_d\right| + \sin^2\theta_i}}{\cos\theta_i - j\sqrt{\left|\epsilon_d\right| + \sin^2\theta_i}} \rightarrow \left|R_{TE}\right| = 1 \\ R_{TM} &= \frac{\epsilon_d \cos\theta_i - \sqrt{\epsilon_d - \sin^2\theta_i}}{\epsilon_d \cos\theta_i + \sqrt{\epsilon_d - \sin^2\theta_i}} = \frac{\epsilon_d \cos\theta_i + j\sqrt{\left|\epsilon_d\right| + \sin^2\theta_i}}{\epsilon_d \cos\theta_i - j\sqrt{\left|\epsilon_d\right| + \sin^2\theta_i}} \rightarrow \left|R_{TM}\right| = 1 \end{aligned} \tag{1.4}$$

2. At 6 MHz we have $\varepsilon_d = 1 - \left(f_p / f \right)^2 = 0.306$. Now at an angle of incidence larger than $\arcsin\left(\sqrt{\varepsilon_d}\right) = 0.586\ \text{rad} = 33.6\ \text{deg}$ we have complete reflection so that $\left|R_{TE}\right| = \left|R_{TM}\right| = 1$. Below this angle

$$R_{TE} = \frac{\cos\theta_i - \sqrt{0.306 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{0.306 - \sin^2\theta_i}},\ R_{TM} = \frac{0.306\cos\theta_i - \sqrt{0.306 - \sin^2\theta_i}}{0.306\cos\theta_i + \sqrt{0.306 - \sin^2\theta_i}} \tag{1.5}$$

The Brewster angle is

$$\arcsin\left(\sqrt{\frac{\varepsilon_d}{\varepsilon_d + 1}}\right) = 28.95\ \text{deg} \tag{1.6}$$

Problem 3:

Consider the scalar wave equation $(\nabla^2 + k^2)\psi = 0$ in cylindrical coordinates, and a particular solution given by

$$\psi(\rho,\phi) = \exp(-jm\phi)\, J_m(k\rho)$$

where k is the wave number of the medium.

1. Find the corresponding $\boldsymbol{M}$ and $\boldsymbol{N}$ vectors
2. Sketch the real and imaginary parts of their components along the radial direction (as function of ρ) for m=1 and $\phi = 0$
3. If these vectors are to represent an electric field how does the polarization change as function of radial distance ρ? (Again take m=1 and $\phi = 0$)

Solution

1. We have $k_z = 0$. The M and N vectors are

$$\boldsymbol{M}_{m,k_z=0} = \exp(-jm\phi)\left[-\hat{\rho}\ \frac{jm}{k\rho} J_m(k\rho) - \hat{\boldsymbol{\phi}}\, J'_m(k\rho)\right] \tag{1.7}$$

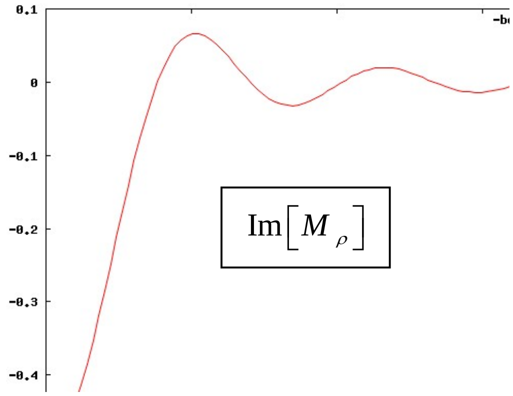
$$\boldsymbol{N}_{m,k_z=0} = \exp(-jm\phi) J_m(k\rho)\hat{\boldsymbol{z}} \tag{1.8}$$

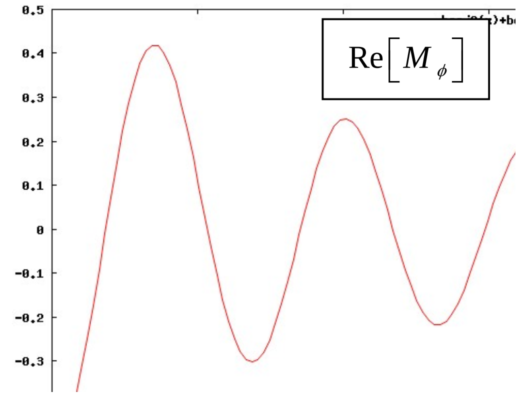
2. For m=1 and $\phi = 0$

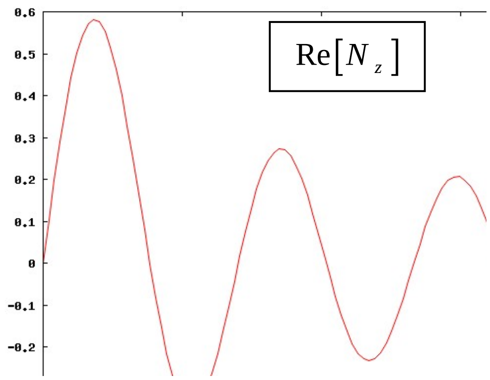
$$\boldsymbol{M}_{m,k_z=0}(\rho,\phi=0) = -\hat{\rho}\ \frac{j}{k\rho} J_1(k\rho) - \hat{\boldsymbol{\phi}}\, J'_1(k\rho) \tag{1.9}$$

$$\boldsymbol{N}_{m,k_z=0}(\rho,\phi=0) = J_1(k\rho)\hat{\boldsymbol{z}} \tag{1.10}$$

The radial (imaginary) and azimuthal (real) components of M and the z-component (real) of N are plotted below:







<u>Problem 4:</u>

We would like to solve the 2D, **scalar** scattering problem of a scalar wave incident upon an infinitely long circular cylinder whose radius is given by a and whose axis coincides with the z-axis. The incident wave is

$$\Psi_i(x,y) = A\exp\left(-jk_{i,x}x - jk_{i,y}y\right)$$

with $k_{i,x}^2 + k_{i,y}^2 = k^2$ where k is the wave number in the background medium. Like the incident wave, the scattered wave $\Psi_s(x,y)$ satisfies the 2D wave equation

$$\frac{\partial^2\Psi_s}{\partial x^2} + \frac{\partial^2\Psi_s}{\partial y^2} + k^2\Psi_s = 0$$

1. Using cylindrical coordinates, find the general solution of this equation in terms of a functional series. Take into account the behavior of the scattered wave at infinity.
2. The boundary condition on the surface of the cylinder is $\Psi = 0$ in which $\Psi = \Psi_i + \Psi_s$ is the total field. Using this boundary condition, solve the scattering problem by determining the coefficients of the series found above.

Solution

The scattered field may be written as

$$\Psi_s(\rho,\phi) = \sum_{m=-\infty}^{\infty} B_m \exp(-jm\phi) H_m^{(2)}(k\rho) \tag{1.11}$$

which satisfies the Helmholtz equation as well as the boundary condition at infinity. The plane wave may be expanded as

$$A\exp\left(-jk_{i,x}x - jk_{i,y}y\right) = A\sum_{m=-\infty}^{\infty}(-j)^m J_m(k\rho)\exp\left[-jm(\phi-\phi_i)\right] \tag{1.12}$$

On the surface of the cylinder $\rho = a$ it is required that

$$\sum_{m=-\infty}^{\infty}\exp(-jm\phi)\left[A(-j)^m J_m(ka)\exp(jm\phi_i) + B_m H_m^{(2)}(ka)\right] = 0 \tag{1.13}$$

As a result

$$B_m = -(-j)^m \frac{J_m(ka)}{H_m^{(2)}(ka)}\exp(jm\phi_i)A \tag{1.14}$$

The scattered field reads:

$$\Psi_s(\rho,\phi) = -A\sum_{m=-\infty}^{\infty}(-j)^m \frac{J_m(ka)}{H_m^{(2)}(ka)} H_m^{(2)}(k\rho)\exp\left[-jm(\phi-\phi_i)\right] \tag{1.15}$$

The scattered field in the far-field zone is:

$$\Psi_s(\rho,\phi) \sim -A\sqrt{\frac{2}{\pi k\rho}}\exp\left[-(jk\rho - j\pi/4)\right]\sum_{m=-\infty}^{\infty}\frac{J_m(ka)}{H_m^{(2)}(ka)}\exp\left[-jm(\phi-\phi_i)\right] \tag{1.16}$$