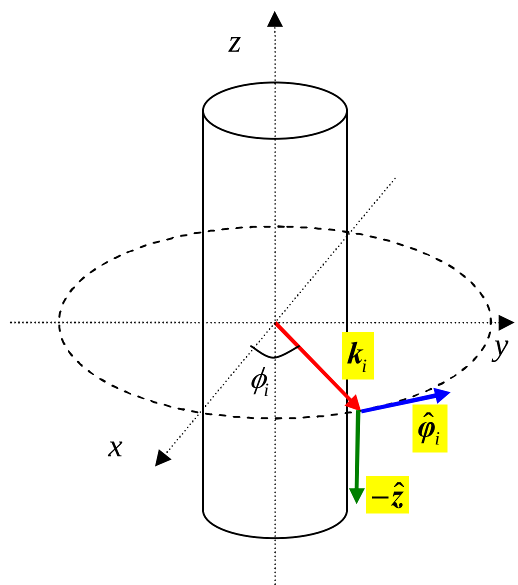
EM Scattering

Homework assignment 3

Problem 1:

A plane wave is normally incident on a perfectly conducting, infinitely long cylinder of radius a. For both the TE_z and TM_z case find the total current (integral over ϕ) on the cylinder. The dielectric constant and permeability of the surrounding medium are ε_0, μ_0.



Solution

We are interested in normal incidence where the wave vector $\boldsymbol{k}_i$ of the incident wave has no z-component. The unit vectors defining the polarization of the incident wave are

$$\hat{\boldsymbol{h}}_i = \frac{\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_i}{\left|\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_i\right|} = \hat{\boldsymbol{\varphi}}_i, \; \hat{\boldsymbol{v}}_i = \hat{\boldsymbol{h}}_i \times \hat{\boldsymbol{k}}_i = \hat{\boldsymbol{\varphi}}_i \times \hat{\boldsymbol{k}}_i = -\hat{\boldsymbol{z}} \tag{1.1}$$

The current density induced on the surface of the cylinder is given by $\boldsymbol{J}_s = \hat{\boldsymbol{n}} \times \boldsymbol{H}$ where $\boldsymbol{H}$ is the *total* magnetic field (incident plus scattered) and $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\rho}}$ is the unit vector normal to the surface. First consider an incident TE_z wave with $\boldsymbol{E}_i^0 = E_0 \hat{\boldsymbol{h}}_i = E_0 \hat{\boldsymbol{\varphi}}_i$. We have to find the components of the magnetic field tangential to the cylindrical surface.

For the scattered magnetic field we have (see power-point slides, but note that there is an error in the final formulas for the components of the magnetic field in both the TE and TM cases corresponding to a factor j):

$$H_{s,\phi}^{TE} = 0,$$

$$H_{s,z}^{TE} = -\left(\boldsymbol{E}_i^0 \cdot \hat{\boldsymbol{h}}_i\right)\left(\frac{k_{i,\rho}}{\eta k}\right)\exp\left(-jk_{i,z}z\right)\sum_{m=-\infty}^{\infty}\frac{(-j)^m J_m'\left(k_{i,\rho}R\right)}{H_m^{(2)'}\left(k_{i,\rho}R\right)}H_m^{(2)}\left(k_{i,\rho}\rho\right)\exp\left[-jm\left(\phi-\phi_i\right)\right] \quad (1.2)$$

$$= -\frac{E_0}{\eta}\sum_{m=-\infty}^{\infty}\frac{(-j)^m J_m'\left(kR\right)}{H_m^{(2)'}\left(kR\right)}H_m^{(2)}\left(k\rho\right)\exp\left[-jm\left(\phi-\phi_i\right)\right]$$

For the incident wave we have:

$$\boldsymbol{H}_i\left(\boldsymbol{r}\right) = \frac{1}{\eta}\hat{\boldsymbol{k}}_i \times \boldsymbol{E}_i\left(\boldsymbol{r}\right) = \frac{E_0}{\eta}\left(\hat{\boldsymbol{k}}_i \times \hat{\boldsymbol{h}}_i\right)\exp\left(-j\boldsymbol{k}_i\cdot\boldsymbol{r}\right) = \frac{E_0}{\eta}\hat{\boldsymbol{z}}\exp\left(-j\boldsymbol{k}_i\cdot\boldsymbol{r}\right)$$
$$= \frac{E_0}{\eta}\hat{\boldsymbol{z}}\sum_{m=-\infty}^{\infty}(-j)^m J_m\left(k\rho\right)\exp\left[-jm\left(\phi-\phi_i\right)\right] \quad (1.3)$$

The total field has a z-component only and is in the direction:

$$H_z^{TE} = \frac{E_0}{\eta}\sum_{m=-\infty}^{\infty}\left[J_m\left(k\rho\right) - \frac{J_m'\left(kR\right)}{H_m^{(2)'}\left(kR\right)}H_m^{(2)}\left(k\rho\right)\right](-j)^m\exp\left[-jm\left(\phi-\phi_i\right)\right] \quad (1.4)$$

On the surface of the cylinder

$$H_z^{TE}\left(R,\phi\right) = \frac{E_0}{\eta}\sum_{m=-\infty}^{\infty}\left[\frac{J_m\left(kR\right)H_m^{(2)'}\left(kR\right) - J_m'\left(kR\right)H_m^{(2)}\left(kR\right)}{H_m^{(2)'}\left(kR\right)}\right](-j)^m\exp\left[-jm\left(\phi-\phi_i\right)\right] \quad (1.5)$$

Now, note that

$$J_m\left(kR\right)H_m^{(2)'}\left(kR\right) - J_m'\left(kR\right)H_m^{(2)}\left(kR\right) = -j\left[J_m\left(kR\right)Y_m'\left(kR\right) - J_m'\left(kR\right)Y_m\left(kR\right)\right]$$
$$= \frac{-2j}{\pi kR} \quad (1.6)$$

so that

$$H_z^{TE}\left(R,\phi\right) = \frac{E_0}{\eta}\left(\frac{-2j}{\pi kR}\right)\sum_{m=-\infty}^{\infty}\frac{(-j)^m\exp\left[-jm\left(\phi-\phi_i\right)\right]}{H_m^{(2)'}\left(kR\right)} \quad (1.7)$$

The surface current density is

$$\boldsymbol{J}_s^{TE} = \hat{\boldsymbol{\rho}} \times \boldsymbol{H}^{TE} = -\hat{\boldsymbol{\varphi}}H_z^{TE}\left(R,\phi\right) \quad (1.8)$$

Obviously, there is no current along the cylinder (z-direction).

Now, consider the TM case where $\boldsymbol{E}_i^0 = E_0\hat{\boldsymbol{v}}_i = -E_0\hat{\boldsymbol{z}}$. The resulting scattered magnetic field is horizontal, but we are only interested in its ϕ component given by

$$
\begin{aligned}
H_{s,\phi}^{TM} &= -\left(\boldsymbol{E}_i^0 \cdot \hat{\boldsymbol{v}}_i\right)\frac{j}{\eta}\exp\left(-jk_{i,z}z\right) \\
&\qquad \sum_{m=-\infty}^{\infty}\frac{(-j)^m J_m\left(k_{i,\rho}R\right)}{H_m^{(2)}\left(k_{i,\rho}R\right)}H_m^{(2)\prime}\left(k_{i,\rho}\rho\right)\exp\left[-jm\left(\phi-\phi_i\right)\right] \\
&= -\frac{jE_0}{\eta}\sum_{m=-\infty}^{\infty}\frac{(-j)^m J_m\left(kR\right)}{H_m^{(2)}\left(kR\right)}H_m^{(2)\prime}\left(k\rho\right)\exp\left[-jm\left(\phi-\phi_i\right)\right]
\end{aligned}
\tag{1.9}
$$

The incident magnetic field is now

$$
\begin{aligned}
\boldsymbol{H}_i\left(\boldsymbol{r}\right) &= \frac{1}{\eta}\hat{\boldsymbol{k}}_i \times \boldsymbol{E}_i\left(\boldsymbol{r}\right) = \frac{E_0}{\eta}\left(\hat{\boldsymbol{k}}_i \times \hat{\boldsymbol{v}}_i\right)\exp\left(-j\boldsymbol{k}_i\cdot\boldsymbol{r}\right) = \frac{E_0}{\eta}\hat{\boldsymbol{\varphi}}\exp\left(-j\boldsymbol{k}_i\cdot\boldsymbol{r}\right) \\
&= -j\frac{E_0}{\eta}\frac{k}{k_{i,\rho}}\sum_{m=-\infty}^{\infty}(-j)^m \boldsymbol{M}_{m,k_{i,z}}^J(\rho,\phi,z)\exp\left(jm\phi_i\right) \\
&= j\frac{E_0}{\eta}\sum_{m=-\infty}^{\infty}(-j)^m\left[\hat{\rho}\;\frac{jm}{k\rho}J_m\left(k_{i,\rho}\rho\right) + \hat{\boldsymbol{\phi}}J_m'\left(k_{i,\rho}\rho\right)\right]\exp\left[-jm\left(\phi-\phi_i\right)\right]
\end{aligned}
\tag{1.10}
$$

The total field has the ϕ component given by

$$
H_\phi^{TM} = \frac{jE_0}{\eta}\sum_{m=-\infty}^{\infty}\left[J_m{}'\left(k\rho\right) - \frac{J_m\left(kR\right)H_m^{(2)\prime}\left(k\rho\right)}{H_m^{(2)}\left(kR\right)}\right](-j)^m\exp\left[-jm\left(\phi-\phi_i\right)\right] \tag{1.11}
$$

On the surface of the cylinder

$$
\begin{aligned}
H_\phi^{TM}\left(R,\phi\right) &= \frac{jE_0}{\eta}\sum_{m=-\infty}^{\infty}\left[\frac{J_m{}'\left(kR\right)H_m^{(2)}\left(kR\right) - J_m\left(kR\right)H_m^{(2)\prime}\left(kR\right)}{H_m^{(2)}\left(kR\right)}\right](-j)^m\exp\left[-jm\left(\phi-\phi_i\right)\right] \\
&= \frac{jE_0}{\eta}\left(\frac{-2}{\pi kR}\right)\sum_{m=-\infty}^{\infty}\frac{(-j)^m\exp\left[-jm\left(\phi-\phi_i\right)\right]}{H_m^{(2)}\left(kR\right)}
\end{aligned}
\tag{1.12}
$$

The induced current density is

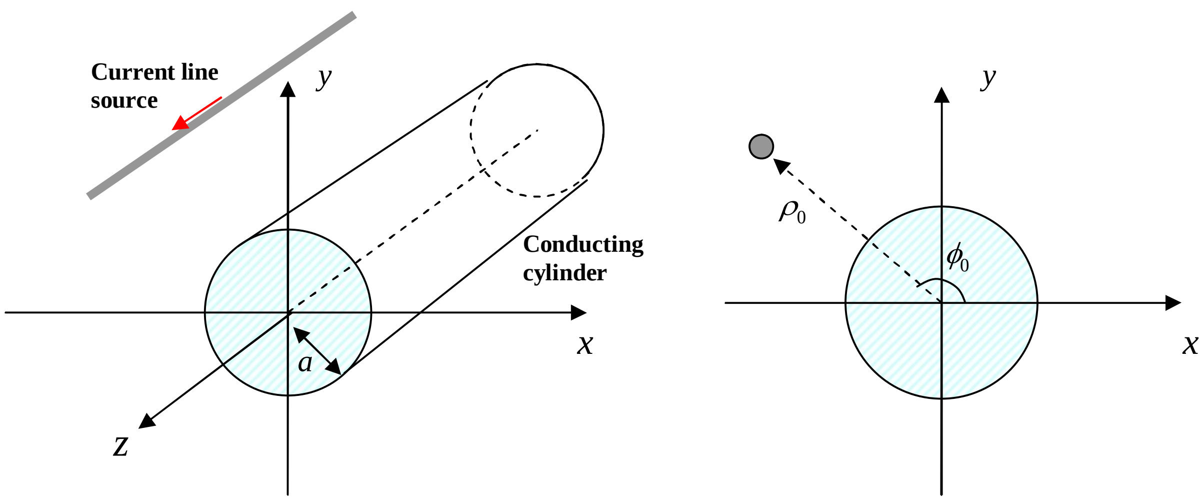
$$
\boldsymbol{J}_s^{TM} = \hat{\boldsymbol{\rho}}\times\boldsymbol{H}^{TM} = \hat{\boldsymbol{z}}H_\phi^{TM}\left(R,\phi\right) \tag{1.13}
$$

The total current flowing on the cylinder in the z-direction is

$$I=\int_0^{2\pi} J_z^{TM} R d\phi = \int_0^{2\pi} H_\phi^{TM}(R,\phi) R d\phi = \frac{-2jE_0}{\pi k\eta} \sum_{m=-\infty}^{\infty} \frac{(-j)^m}{H_m^{(2)}(kR)} \int_0^{2\pi} \exp\left[-jm(\phi-\phi_i)\right] d\phi$$
$$= \frac{-2jE_0}{\pi k\eta} \frac{2\pi}{H_0^{(2)}(kR)} = \frac{-4jE_0}{\omega\mu_0 H_0^{(2)}(kR)} \tag{1.14}$$

Problem 2:

We would like to solve the TM scattering problem for an infinite, perfectly conducting cylinder (radius a) using a technique based on introducing an infinitely long line of electric current (line source). The line current is parallel to the cylinder and has the cylindrical coordinates ρ_0, ϕ_0. To simplify the situation, we assume the current along the line source to be constant (I) so that the variations of the field along the cylinder (along the z-axis) are neglected. The dielectric constant and permeability of the outside medium is ϵ, respectively, μ.



1. **Write down the equation for the longitudinal component of the electric field E_z in cylindrical coordinates.**

Solution

Assuming uniformity in the z-direction the Maxwell equations are decoupled into two sets: a TE solution where $E_z = 0, H_z \neq 0$ and a TM solution where $H_z = 0, E_z \neq 0$. The electric current source excited the TM mode, but not the TE mode. The equations for the TM solution are

$$\frac{\partial E_z}{\rho \partial \phi} = -j\omega\mu H_\rho$$

$$\frac{\partial E_z}{\partial \rho} = j\omega\mu H_\phi$$

$$\frac{\partial}{\rho \partial \rho}\left(\rho H_\phi\right) - \frac{\partial H_\rho}{\rho \partial \phi} = j\omega\epsilon E_z + J_z^i$$

Combining these equations leads to

$$\frac{\partial}{\rho \partial \rho}\left(\rho \frac{\partial E_z}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + k^2 E_z = j\omega\mu J_z^i$$

With $k^2 = \omega^2 \epsilon \mu$.

2. What is the solution of this equation for the electric current line source in the absence of the conducting cylinder?

Solution

The above equation is in fact the Helmholtz equation for a delta-function current distribution. In the absence of the cylinder, it's solution is a Hankel function of the 2nd kind. Although this is a very well-known result, we would like to derive this once again because then intermediate results can be used later.

A line source is described by a delta function, i.e., $J_z^i = I\delta(x - x_0)\delta(y - y_0)$. In cylindrical coordinates, it is written as

$$J_z^i = \frac{I}{\rho_0}\delta(\rho - \rho_0)\delta(\phi - \phi_0)$$

So that we have to solve the equation

$$\frac{\partial}{\rho \partial \rho}\left(\rho \frac{\partial E_z}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + k^2 E_z = \frac{j\omega\mu I}{\rho_0}\delta(\rho - \rho_0)\delta(\phi - \phi_0)$$

Note that away from the source, the general solution of this equation can be written as a combination of functions of the type $\exp\left(-jm\phi\right)\mathcal{Z}_m\left(k\rho\right)$ where $\mathcal{Z}_m\left(k\rho\right)$ is a Bessel function whose type depends on the boundary conditions of the problem.

In the region inside the circle passing though the source ($\rho < \rho_0$) we can write the longitudinal component of the electric field as

$$E_z(\rho,\phi) = \sum_{m=-\infty}^{\infty} c_m J_m(k\rho)\exp(-jm\phi) \qquad \rho<\rho_0$$

This choice of the Bessel function ensures the finiteness of the field at the center of coordinates. Outside the circle we have to satisfy the radiation condition so that

$$E_z(\rho,\phi) = \sum_{m=-\infty}^{\infty} b_m H_m^{(2)}(k\rho)\exp(-jm\phi) \qquad \rho > \rho_0$$

Next, we have to match these fields on the circle $\rho = \rho_0$. First, note that the electric field itself has to be continuous. Second, it follows from the original equation that

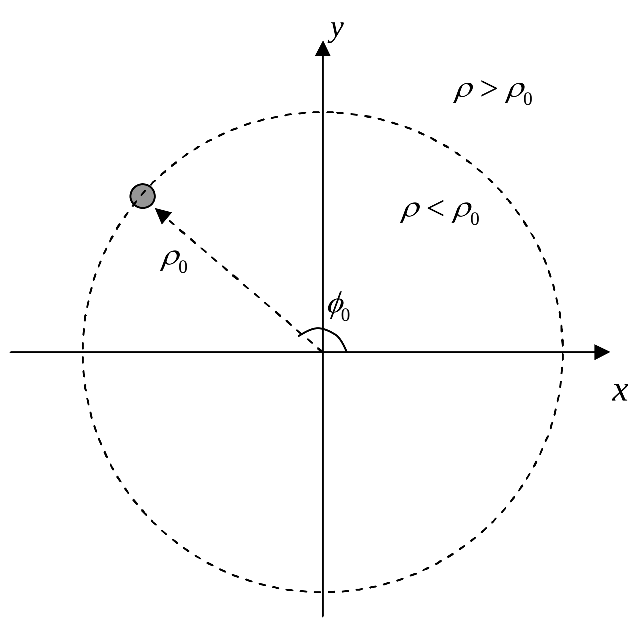
$$\left.\frac{\partial E_z}{\partial \rho}\right|_{\rho\downarrow\rho_0} - \left.\frac{\partial E_z}{\partial \rho}\right|_{\rho\uparrow\rho_0} = \frac{j\omega\mu I}{\rho_0}\delta(\phi-\phi_0)$$

The first condition leads to

$$\sum_{m=-\infty}^{\infty} c_m J_m(k\rho_0)\exp(-jm\phi) = \sum_{m=-\infty}^{\infty} b_m H_m^{(2)}(k\rho_0)\exp(-jm\phi)$$

Since it has to be satisfied for all angles, it follows that

$$c_m J_m(k\rho_0) = b_m H_m^{(2)}(k\rho_0)$$



The second condition yields

$$\sum_{m=-\infty}^{\infty}\left[b_m H_m^{(2)\prime}\left(k\rho_0\right)-c_m J_m'\left(k\rho_0\right)\right]\exp\left(-jm\phi\right)=\frac{j\omega\mu I}{k\rho_0}\delta(\phi-\phi_0)$$

which leads to

$$b_m H_m^{(2)\prime}\left(k\rho_0\right)-c_m J_m'\left(k\rho_0\right)=\frac{j\omega\mu I}{2\pi k\rho_0}\exp\left(jm\phi_0\right)$$

Combining the above results yields

$$\left[\frac{J_m\left(k\rho_0\right)}{H_m^{(2)}\left(k\rho_0\right)}H_m^{(2)\prime}\left(k\rho_0\right)-J_m'\left(k\rho_0\right)\right]c_m=\frac{j\omega\mu I}{2\pi k\rho_0}\exp\left(jm\phi_0\right)\rightarrow$$

$$c_m=\frac{j\omega\mu I}{2\pi k\rho_0}\frac{H_m^{(2)}\left(k\rho_0\right)\exp\left(jm\phi_0\right)}{J_m\left(k\rho_0\right)H_m^{(2)\prime}\left(k\rho_0\right)-J_m'\left(k\rho_0\right)H_m^{(2)}\left(k\rho_0\right)}$$

Note that

$$J_m\left(k\rho_0\right)H_m^{(2)\prime}\left(k\rho_0\right)-J_m'\left(k\rho_0\right)H_m^{(2)}\left(k\rho_0\right)=j\left[J_m'\left(k\rho_0\right)Y_m\left(k\rho_0\right)-J_m\left(k\rho_0\right)Y_m'\left(k\rho_0\right)\right]=\frac{2}{j\pi k\rho_0}$$

Which results in

$$c_m=-\frac{\omega\mu I}{4}H_m^{(2)}\left(k\rho_0\right)\exp\left(jm\phi_0\right)\rightarrow b_m=-\frac{\omega\mu I}{4}J_m\left(k\rho_0\right)\exp\left(jm\phi_0\right)$$

The resulting field is

$$E_z^f\left(\rho,\phi\right)=-\frac{\omega\mu I}{4}\sum_{m=-\infty}^{\infty}H_m^{(2)}\left(k\rho_0\right)J_m\left(k\rho\right)\exp\left[-jm\left(\phi-\phi_0\right)\right]\qquad \rho<\rho_0$$

$$E_z^f\left(\rho,\phi\right)=-\frac{\omega\mu I}{4}\sum_{m=-\infty}^{\infty}J_m\left(k\rho_0\right)H_m^{(2)}\left(k\rho\right)\exp\left[-jm\left(\phi-\phi_0\right)\right]\qquad \rho>\rho_0$$

Using the following addition theorem for the Bessel functions:

$$H_0^{(2)}\left(\sqrt{u^2+v^2-2uv\cos\beta}\right)=\sum_{m=-\infty}^{\infty}H_m^{(2)}(u)J_m(v)\cos\left(m\beta\right)\qquad u>v$$

We find the well known result:

$$E_z^f(\boldsymbol{r}) = -\frac{\omega\mu I}{4} H_0^{(2)}\left(k\left|\boldsymbol{r}-\boldsymbol{r}_0\right|\right)$$

3. What is the solution in the presence of the conducting cylinder?

Solution

The method of solution is quite similar to the previous case, but now we have to ensure the boundary condition on the surface of the cylinder. The expression for the electric field outside the circle passing through the source remains the same:

$$E_z(\rho,\phi) = \sum_{m=-\infty}^{\infty} b_m H_m^{(2)}(k\rho)\exp(-jm\phi) \qquad \rho > \rho_0$$

But, inside the circle, let us write

$$E_z(\rho,\phi) = \sum_{m=-\infty}^{\infty} \left[c_m J_m(k\rho) + d_m H_m^{(2)}(k\rho)\right]\exp(-jm\phi) \qquad a < \rho < \rho_0$$

To have a vanishing longitudinal electric field on the surface of the cylinder it is required that

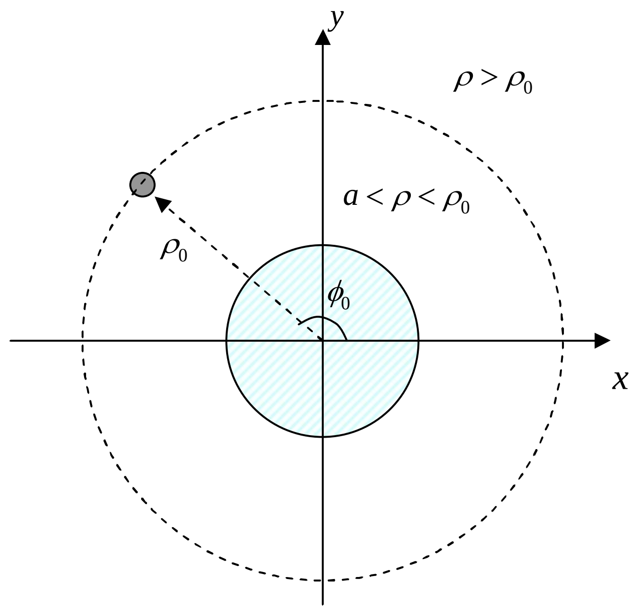
$$c_m J_m(ka) + d_m H_m^{(2)}(ka) = 0 \rightarrow d_m = -\frac{J_m(ka)}{H_m^{(2)}(ka)} c_m$$

The field inside the circle can now be written as

$$E_z(\rho,\phi) = \sum_{m=-\infty}^{\infty} c_m \left[J_m(k\rho) - \frac{J_m(ka)}{H_m^{(2)}(ka)} H_m^{(2)}(k\rho)\right]\exp(-jm\phi) \qquad a < \rho < \rho_0$$

We continue as in the previous problem by matching the fields at the source:

$$c_m\left[J_m(k\rho_0) - \frac{J_m(ka)}{H_m^{(2)}(ka)} H_m^{(2)}(k\rho_0)\right] = b_m H_m^{(2)}(k\rho_0)$$



$$b_m H_m^{(2)\prime}(k\rho_0) - c_m\left[J_m'(k\rho_0) - \frac{J_m(ka)}{H_m^{(2)}(ka)}H_m^{(2)\prime}(k\rho_0)\right] = \frac{j\omega\mu I}{2\pi k\rho_0}\exp(jm\phi_0)$$

We then find

$$c_m\left[J_m(k\rho_0)\frac{H_m^{(2)\prime}(k\rho_0)}{H_m^{(2)}(k\rho_0)} - J_m'(k\rho_0)\right] = \frac{j\omega\mu I}{2\pi k\rho_0}\exp(jm\phi_0)$$

So that

$$c_m = -\frac{\omega\mu I}{4}H_m^{(2)}(k\rho_0)\exp(jm\phi_0)$$

The field inside the circle passing through the source is now

$$E_z(\rho,\phi) = -\frac{\omega\mu I}{4}\sum_{m=-\infty}^{\infty}\left[J_m(k\rho) - \frac{J_m(ka)}{H_m^{(2)}(ka)}H_m^{(2)}(k\rho)\right]H_m^{(2)}(k\rho_0)\exp\left[-jm(\phi-\phi_0)\right] \qquad a<\rho<\rho_0$$

Note that this result can be written as

$$E_z(\rho,\phi) = E_z^f(\rho,\phi) + \frac{\omega\mu I}{4}\sum_{m=-\infty}^{\infty}\frac{J_m(ka)}{H_m^{(2)}(ka)}H_m^{(2)}(k\rho)H_m^{(2)}(k\rho_0)\exp\left[-jm(\phi-\phi_0)\right] \qquad a<\rho<\rho_0$$

Where $E_z^f(\rho,\phi)$ is the solution in the absence of the cylinder, found in the previous section.

4. **How can we find the solution to the problem of the scattering of an incident plane wave with a wave vector parallel to the x-y plane by using the solution of problem 3?**

Solution

Let us move the source to a point far away from the origin. In the absence of the cylinder, the field seen near the origin would then behave as a plane wave. To see this explicitly, consider the solution when no cylinder is present:

$$E_z^f(\rho,\phi) = -\frac{\omega\mu I}{4}\sum_{m=-\infty}^{\infty} H_m^{(2)}(k\rho_0) J_m(k\rho)\exp\left[-jm(\phi-\phi_0)\right] \qquad \rho<\rho_0$$

If the source is move far away ($\rho_0 \to \infty$) along a line making a constant angle ϕ_0 with the x-axis, then using the asymptotic expression for the Hankel function we will have

$$E_z(\rho,\phi) = -\frac{\omega\mu I}{4}\sqrt{\frac{2}{\pi k\rho_0}}\exp\left(-jk\rho_0 + j\frac{\pi}{4}\right)\sum_{m=-\infty}^{\infty} j^m J_m(k\rho)\exp\left[-jm(\phi-\phi_0)\right]$$

The sum can be written as

$$\sum_{m=-\infty}^{\infty}(-j)^m J_m(k\rho)\exp\left[-jm(\phi-\phi_0-\pi)\right] = \exp\left[-jk\rho\cos(\phi-\phi_0-\pi)\right]$$
$$= \exp(-j\boldsymbol{k}_i\cdot\boldsymbol{r})$$

Where $\boldsymbol{k}_i = \left[k\cos(\phi_0+\pi), k\sin(\phi_0+\pi), 0\right]$. This is a plane wave moving towards the origin whose wave vector makes an angle $\phi_i = \phi_0 + \pi$ with the x-axis. Hence, if we change the current as we move away to ensure that

$$E_z^0 = -\frac{\omega\mu I}{4}\sqrt{\frac{2}{\pi k\rho_0}}\exp\left(-jk\rho_0 + j\frac{\pi}{4}\right)$$

Is a constant, then the field generated by the source far away is the incident plane wave $E_z^f = E_z^0\exp(-j\boldsymbol{k}_i\cdot\boldsymbol{r})$.

Now, with this result in mind, let us return to the solution in presence of the cylinder which can be written as

$$E_z(\rho,\phi) = E_z^0\exp(-j\boldsymbol{k}_i\cdot\boldsymbol{r}) - E_z^0\sum_{m=-\infty}^{\infty} j^m\frac{J_m(ka)}{H_m^{(2)}(ka)}H_m^{(2)}(k\rho)\exp\left[-jm(\phi-\phi_0)\right]$$

The 2nd term on the right hand side is just the scattered field.

5. Compare the solution to the one presented during the lectures.

Solution

Using the M and N vector functions, we found the scattered field in the TM case, for $k_{i,z} = 0, k_{i,\rho} = k$ to be given by

$$E_{s,z}^{TM}(\boldsymbol{r}) = \left(\boldsymbol{E}_i^0 \cdot \hat{\boldsymbol{v}}_i\right) \sum_{m=-\infty}^{\infty} \frac{(-j)^m J_m(ka)}{H_m^{(2)}(ka)} H_m^{(2)}\left(k_{i,\rho}\rho\right) \exp\left[-jm\left(\phi - \phi_i\right)\right]$$

Here $\hat{\boldsymbol{v}}_i = \hat{\boldsymbol{\phi}}_i \times \hat{\boldsymbol{k}}_i$. In our case (see figure below) we have $\hat{\boldsymbol{v}}_i = -\hat{\boldsymbol{z}} \rightarrow \boldsymbol{E}_i^0 \cdot \hat{\boldsymbol{v}}_i = -E_z^0$. Also note that the replacement $\phi_i = \phi_0 + \pi$ will turn the – sign in $(-j)^m$ into a plus sign so that the expression becomes identical to the one derived above.

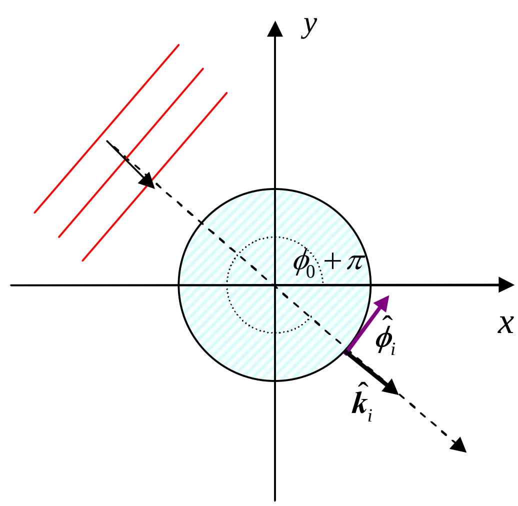
6. How can we solve the TE scattering case by using this technique?

Solution

One should then into a *magnetic* line current. The details will not be given here.

Problem 3:

Consider an infinitely long, dielectric circular cylinder whose axis coincides with the *z*-axis. A plane wave with the wave vector $\boldsymbol{k}_i = (k_{i,x}, k_{i,y}, 0)$ is normally incident on the cylinder where $|\boldsymbol{k}_i| = k_0 = \omega\sqrt{\varepsilon_0 \mu_0}$ with ε_0, μ_0 respectively denoting the permittivity and permeability of the background medium. The electric field of the incident wave is polarized along the z-direction. The dielectric constant and radius of the cylinder are ε and a, respectively.



1. Write down the equations governing the electric field inside and outside the cylinder and give their general solution

2. Compute the scattered field by using the expansion of a (scalar) plane wave and matching the solutions at the boundary of the cylinder.

Solution

Since the fields are independent of the z-coordinate, we have the equation

$$\frac{\partial}{\rho\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2}+k^2E_z=0 \tag{1.15}$$

where $k^2=\omega^2\varepsilon\mu_0$ (see previous problem, but in the absence of a source). Furthermore:

$$\frac{\partial E_z}{\rho\partial\phi}=-j\omega\mu_0H_\rho,\ \frac{\partial E_z}{\partial\rho}=j\omega\mu_0H_\phi \tag{1.16}$$

Inside the cylinder the total field may be expanded as

$$E_z(\rho,\phi)=\sum_{m=-\infty}^{\infty}c_mJ_m\left(k_d\rho\right)\exp\left(-jm\phi\right),\ \rho<a \tag{1.17}$$

Where $k_d^2=\omega^2\varepsilon\mu_0$. Outside the cylinder we decompose the total field into an incident and a scattered field. The scattered electric field is expanded as

$$E_{s,z}(\rho,\phi)=\sum_{m=-\infty}^{\infty}b_mH_m^{(2)}\left(k_0\rho\right)\exp\left(-jm\phi\right),\ \rho>a \tag{1.18}$$

where $k_0^2=\omega^2\varepsilon_0\mu_0$. The incident plane wave is

$$E_{i,z}=E_0\exp\left(-j\boldsymbol{k}_i\cdot\boldsymbol{r}\right)=E_0\sum_{m=-\infty}^{\infty}(-j)^mJ_m\left(k_0\rho\right)\exp\left[-jm\left(\phi-\phi_i\right)\right],\ \rho>a \tag{1.19}$$

The total tangential electric field must be continuous across the surface of the cylinder so that

$$\begin{aligned}\sum_{m=-\infty}^{\infty}c_mJ_m\left(k_da\right)\exp\left(-jm\phi\right)=E_0\sum_{m=-\infty}^{\infty}(-j)^mJ_m\left(k_0a\right)\exp\left[-jm\left(\phi-\phi_i\right)\right]+\\ \sum_{m=-\infty}^{\infty}b_mH_m^{(2)}\left(k_0a\right)\exp\left(-jm\phi\right)\end{aligned} \tag{1.20}$$

From the continuity of the tangential magnetic field (ϕ component) it follows that

$$k_d \sum_{m=-\infty}^{\infty} c_m J'_m(k_d a)\exp(-jm\phi) = E_0 k_0 \sum_{m=-\infty}^{\infty} (-j)^m J'_m(k_0 a)\exp\left[-jm(\phi-\phi_i)\right] + k_0 \sum_{m=-\infty}^{\infty} b_m H_m^{(2)\prime}(k_0 a)\exp(-jm\phi) \tag{1.21}$$

By applying a discrete Fourier transform, we arrive at the equations

$$J_m(k_d a)c_m - H_m^{(2)}(k_0 a)b_m = (-j)^m J_m(k_0 a)\exp(jm\phi_i)E_0 \tag{1.22}$$

$$\frac{k_d}{k_0} J'_m(k_d a)c_m - H_m^{(2)\prime}(k_0 a)b_m = (-j)^m J'_m(k_0 a)\exp(jm\phi_i)E_0 \tag{1.23}$$

The solution is

$$\begin{aligned} c_m &= \frac{1}{\Delta}\left[H_m^{(2)}(k_0 a)J'_m(k_0 a) - H_m^{(2)\prime}(k_0 a)J_m(k_0 a)\right](-j)^m \exp(jm\phi_i)E_0 \\ &= -\frac{1}{\Delta}\frac{2}{\pi k_0 a}(-j)^m \exp(jm\phi_i)E_0 \end{aligned} \tag{1.24}$$

$$b_m = \frac{1}{\Delta}\left[J_m(k_d a)J'_m(k_0 a) - \frac{k_d}{k_0}J'_m(k_d a)J_m(k_0 a)\right](-j)^m \exp(jm\phi_i)E_0 \tag{1.25}$$

$$\Delta = \frac{k_d}{k_0}H_m^{(2)}(k_0 a)J'_m(k_d a) - H_m^{(2)\prime}(k_0 a)J_m(k_d a) \tag{1.26}$$

The scattered field is thus

$$E_{s,z}(\rho,\phi) = -E_0 \sum_{m=-\infty}^{\infty}\left[\frac{J_m(k_d a)J'_m(k_0 a) - \frac{k_d}{k_0}J'_m(k_d a)J_m(k_0 a)}{J_m(k_d a)H_m^{(2)\prime}(k_0 a) - \frac{k_d}{k_0}J'_m(k_d a)H_m^{(2)}(k_0 a)}\right] H_m^{(2)}(k_0\rho)(-j)^m \exp\left[-jm(\phi-\phi_i)\right] \tag{1.27}$$