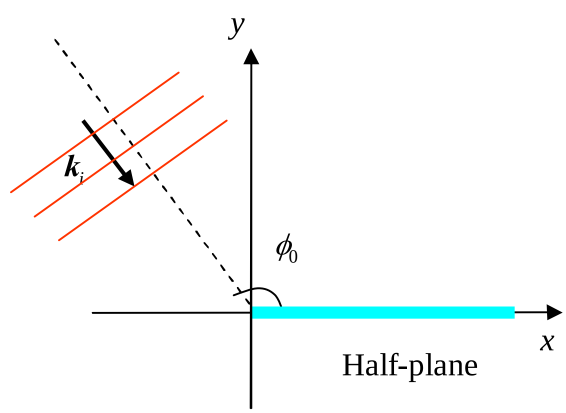
EM Scattering

Homework assignment 4

Problem 1:

A uniform TM[z] plane wave (electric field along the z-direction) is normally incident on a perfectly conducting half-plane at an angle ϕ_0. The amplitude of the incident wave is E_0. Calculate the current density on the top and bottom surfaces of the half plane. The dielectric constant and permeability of the surrounding medium are ε_0, μ_0.



Solution

The surface current density equals $\hat{\boldsymbol{n}} \times \boldsymbol{H}$. Note that $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{\phi}}$ on the top and bottom surfaces, respectively ($\phi = 0, 2\pi$). In the TM[z] case the magnetic field has no z-component. The only component contributing to the surface current is H_ρ. Thus:

$$\phi = 0 \rightarrow \hat{\boldsymbol{n}} \times \boldsymbol{H} = -H_\rho \hat{\boldsymbol{z}} = \frac{\hat{\boldsymbol{z}}}{\rho} \frac{A_0}{j\omega\mu} \frac{4\pi}{2\pi - \beta} \sum_{m=1}^{\infty} (j)^{\nu_m} \nu_m \sin(\nu_m \phi_0) J_{\nu_m}(k\rho) \qquad (1.1)$$

$$\begin{aligned} \phi = 2\pi \rightarrow \hat{\boldsymbol{n}} \times \boldsymbol{H} &= H_\rho \hat{\boldsymbol{z}} = -\hat{\boldsymbol{z}} \frac{A_0}{j\omega\mu} \frac{4\pi}{2\pi - \beta} \sum_{m=1}^{\infty} (j)^{\nu_m} \frac{\nu_m}{\rho} \sin(\nu_m \phi_0) J_{\nu_m}(k\rho) \cos(\nu_m 2\pi) \\ &= -\frac{\hat{\boldsymbol{z}}}{\rho} \frac{A_0}{j\omega\mu} \frac{4\pi}{2\pi - \beta} \sum_{m=1}^{\infty} (-1)^m (j)^{\nu_m} \nu_m \sin(\nu_m \phi_0) J_{\nu_m}(k\rho) \end{aligned} \qquad (1.2)$$

Problem 2:

Repeat the above problem now for a uniform TE[z] plane wave (magnetic field along the z-direction). Take the amplitude of the incident plane magnetic field wave to be H_0.

Solution

Now the magnetic field has only a z-component. Hence,

$$\phi = 0 \rightarrow \hat{\boldsymbol{n}} \times \boldsymbol{H} = H_z \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\rho}} A_0 \frac{2\pi}{2\pi - \beta} \sum_{m=0}^{\infty} \tau_m \left(j \right)^{\nu_m} \cos\left(\nu_m \phi_0 \right) J_{\nu_m}(k\rho) \qquad (1.3)$$

$$\phi = 2\pi \rightarrow \hat{\boldsymbol{n}} \times \boldsymbol{H} = -H_z \hat{\boldsymbol{\rho}} = -\hat{\boldsymbol{\rho}} A_0 \frac{2\pi}{2\pi - \beta} \sum_{m=0}^{\infty} \tau_m (-1)^m \left(j \right)^{\nu_m} \cos\left(\nu_m \phi_0 \right) J_{\nu_m}(k\rho) \qquad (1.4)$$

Problem 3:

Consider the structure shown below which consists of a perfectly conducting half-plane terminated by an infinitely long, perfectly conducting cylinder of radius a whose axis coincides with the z-axis. An incident TM wave with the electric field vector in the z-direction and a wave vector parallel to the x-y plane propagates along a line which makes an angle ϕ_0 with the x-axis (see figure) and is scattered by structure. Find the total electric field in this 2D scattering problem by using the line-source method. The background medium is vacuum.

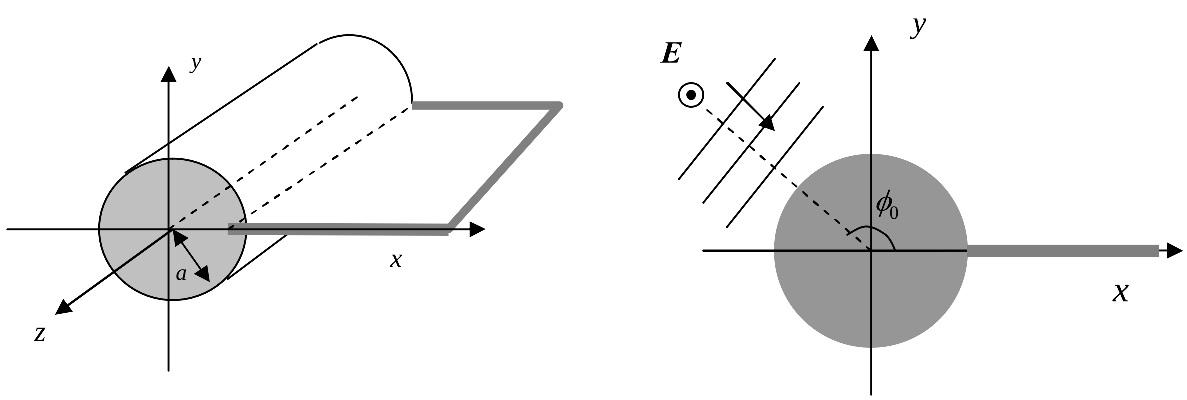
Solution:

Line source method

Consider a line of electric current I at a point $\left(\rho_0, \phi_0\right)$. The field equation for the z-component of the electric field is

$$\frac{\partial}{\rho \partial \rho}\left(\rho \frac{\partial E_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + k^2 E_z = j\omega\mu_0 J_z^i \qquad (1.5)$$

Its solution should become zero when $\phi = 0, 2\pi$ due to the boundary condition on the ground plane. The should be true on the surface of the cylinder, i.e., when $\rho = a$. Now, in the region $\rho < \rho_0$, we write the solution as



$$E_z(\rho,\phi)=\sum_{m=1}^{\infty}\left[c_m J_{\nu_m}(k\rho)+d_m H_{\nu_m}^{(2)}(k\rho)\right]\sin(\nu_m\phi) \qquad \rho<\rho_0 \tag{1.6}$$

Where $\nu_m = m\pi/(2\pi-\beta)=m/2$. For $\rho>\rho_0$

$$E_z(\rho,\phi)=\sum_{m=1}^{\infty} b_m H_{\nu_m}^{(2)}(k\rho)\sin(\nu_m\phi) \qquad \rho>\rho_0 \tag{1.7}$$

On the circle with the radius a we have

$$E_z(\rho,\phi)\Big|_{\rho\downarrow\rho_0}-E_z(\rho,\phi)\Big|_{\rho\uparrow\rho_0}=0 \tag{1.8}$$

$$\frac{\partial E_z(\rho,\phi)}{\partial\rho}\bigg|_{\rho\downarrow\rho_0}-\frac{\partial E_z(\rho,\phi)}{\partial\rho}\bigg|_{\rho\uparrow\rho_0}=\frac{j\omega\mu_0 I}{\rho_0}\delta(\phi-\phi_0) \tag{1.9}$$

Furthermore,

$$E_z(a,\phi)=0 \tag{1.10}$$

Upon substitution of (1.6) and (1.7) into (1.8)-(1.10), multiplication by $\sin(n\phi)$ and integration over the angle from 0 to π, one obtains the following equations

$$c_n J_{\nu_m}(ka)+d_n H_{\nu_n}^{(2)}(ka)=0 \tag{1.11}$$

$$b_n H_{\nu_n}^{(2)}(k\rho_0)=c_n J_{\nu_n}(k\rho_0)+d_n H_{\nu_n}^{(2)}(k\rho_0) \tag{1.12}$$

$$b_n H_{\nu_n}^{(2)\prime}(k\rho_0)-c_n J_{\nu_n}'(k\rho_0)-d_n H_{\nu_n}^{(2)\prime}(k\rho_0)=\frac{j\omega\mu_0 I}{\pi k\rho_0}\sin(n\phi_0) \tag{1.13}$$

Obviously:

$$d_n=-\frac{J_{\nu_n}(ka)}{H_{\nu_n}^{(2)}(ka)}c_n \tag{1.14}$$

$$b_n=\left[\frac{J_{\nu_n}(k\rho_0)}{H_{\nu_n}^{(2)}(k\rho_0)}-\frac{J_{\nu_n}(ka)}{H_{\nu_n}^{(2)}(ka)}\right]c_n \tag{1.15}$$

$$\begin{aligned}&\left[\frac{J_{\nu_n}(k\rho_0)}{H_{\nu_n}^{(2)}(k\rho_0)}-\frac{J_{\nu_n}(ka)}{H_{\nu_n}^{(2)}(ka)}\right]H_{\nu_n}^{(2)\prime}(k\rho_0)c_n\\&\quad-\left[\frac{J_{\nu_n}'(k\rho_0)H_{\nu_n}^{(2)}(ka)-J_{\nu_n}(ka)H_{\nu_n}^{(2)\prime}(k\rho_0)}{H_{\nu_n}^{(2)}(ka)}\right]c_n=\frac{j\omega\mu_0 I}{\pi k\rho_0}\sin(n\phi_0)\end{aligned} \tag{1.16}$$

The last equation leads to

$$\left[\frac{J_{\nu_n}(k\rho_0)H_{\nu_n}^{(2)\prime}(k\rho_0)-J_{\nu_n}'(k\rho_0)H_{\nu_n}^{(2)}(k\rho_0)}{H_{\nu_n}^{(2)}(k\rho_0)}\right]c_n=\frac{j\omega\mu_0 I}{\pi k\rho_0}\sin(n\phi_0) \qquad (1.17)$$

Next, we use the relation

$$\begin{aligned}J_n(k\rho_0)H_n^{(2)\prime}(k\rho_0)-J_n'(k\rho_0)H_n^{(2)}(k\rho_0)&=-jJ_n(k\rho_0)Y_n'(k\rho_0)+jJ_n'(k\rho_0)Y_n(k\rho_0)\\&=-j\frac{2}{\pi k\rho_0}\end{aligned} \qquad (1.18)$$

So that

$$c_n=-\left(\frac{\omega\mu_0 I}{2}\right)H_{\nu_n}^{(2)}(k\rho_0)\sin(n\phi_0) \qquad (1.19)$$

The solution for $\rho<\rho_0$, is, thus

$$E_z(\rho,\phi)=-\left(\frac{\omega\mu_0 I}{2}\right)\sum_{m=1}^{\infty}\left[J_{\nu_m}(k\rho)-\frac{J_{\nu_m}(ka)}{H_{\nu_m}^{(2)}(ka)}H_{\nu_m}^{(2)}(k\rho)\right]H_{\nu_m}^{(2)}(k\rho_0)\sin(m\phi_0)\sin(m\phi) \qquad (1.20)$$

Next, let $\rho_0\to\infty$, while keeping

$$E_0=-\frac{\omega\mu_0 I}{4}\sqrt{\frac{2}{\pi k\rho_0}}\exp\left(-jk\rho_0+j\frac{\pi}{4}\right) \qquad (1.21)$$

a constant. This is the amplitude of an incident plane wave traveling towards the object along the line with the angle ϕ_0. In our solution this leads to

$$E_z(\rho,\phi)=2E_0\sum_{m=1}^{\infty}j^{\nu_m}\left[J_{\nu_m}(k\rho)-\frac{J_{\nu_m}(ka)}{H_{\nu_m}^{(2)}(ka)}H_{\nu_m}^{(2)}(k\rho)\right]\sin(m\phi_0)\sin(m\phi) \qquad (1.22)$$