**Problem 1:**

A uniform TM\(_x\) plane wave (electric field along the z-direction) is normally incident on a perfectly conducting half-plane at an angle \(\phi_0\). The amplitude of the incident wave is \(E_0\). Calculate the current density on the top and bottom surfaces of the half plane. The dielectric constant and permeability of the surrounding medium are \(\varepsilon_0, \mu_0\).

**Solution**

The surface current density equals \(\hat{n} \times \mathbf{H}\). Note that \(\hat{n} = \pm \hat{z}\) on the top and bottom surfaces, respectively (\(\phi = 0, 2\pi\)). In the TM\(_x\) case the magnetic field has no z-component. The only component contributing to the surface current is \(H_\rho\). Thus:

\[
\phi = 0 \rightarrow \hat{n} \times \mathbf{H} = -H_\rho \hat{z} = \hat{z} \frac{A_0}{\rho} \frac{4\pi}{j \omega \mu} \sum_{m=1}^{\infty} (j)^m v_m \sin(v_m \phi_0) J_{\nu_m}(k \rho) \tag{1.1}
\]

\[
\phi = 2\pi \rightarrow \hat{n} \times \mathbf{H} = H_\rho \hat{z} = -\hat{z} \frac{A_0}{\rho} \frac{4\pi}{j \omega \mu} \sum_{m=1}^{\infty} (j)^m v_m \sin(v_m \phi_0) J_{\nu_m}(k \rho) \cos(v_m 2\pi) = -\hat{z} \frac{A_0}{\rho} \frac{4\pi}{j \omega \mu} \sum_{m=1}^{\infty} (-1)^m (j)^m v_m \sin(v_m \phi_0) J_{\nu_m}(k \rho) \tag{1.2}
\]

**Problem 2:**

Repeat the above problem now for a uniform TE\(_x\) plane wave (magnetic field along the z-direction). Take the amplitude of the incident plane magnetic field wave to be \(H_0\).

**Solution**

Now the magnetic field has only a z-component. Hence,
\[
\phi = 0 \rightarrow \hat{n} \times \mathbf{H} = H_z \hat{\rho} = \hat{\rho} A_0 \frac{2\pi}{2\pi - \beta} \sum_{m=0}^{\infty} \tau_m (j)^m \cos (v_m \phi_0) J_{v_m}(k\rho) \quad (1.3)
\]

\[
\phi = 2\pi \rightarrow \hat{n} \times \mathbf{H} = -H_z \hat{\rho} = -\hat{\rho} A_0 \frac{2\pi}{2\pi - \beta} \sum_{m=0}^{\infty} \tau_m (-1)^m (j)^m \cos (v_m \phi_0) J_{v_m}(k\rho) \quad (1.4)
\]

**Problem 3:**

Consider the structure shown below which consists of a perfectly conducting half-plane terminated by an infinitely long, perfectly conducting cylinder of radius \( a \) whose axis coincides with the z-axis. An incident TM wave with the electric field vector in the z-direction and a wave vector parallel to the x-y plane propagates along a line which makes an angle \( \phi_0 \) with the x-axis (see figure) and is scattered by structure. Find the total electric field in this 2D scattering problem by using the line-source method. The background medium is vacuum.

**Solution:**

*Line source method*

Consider a line of electric current \( I \) at a point \((\rho_0, \phi_0)\). The field equation for the z-component of the electric field is

\[
\frac{\partial}{\partial \rho} \left( \rho \frac{\partial E_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + k^2 E_z = j\omega \mu_0 j_z \quad (1.5)
\]

Its solution should become zero when \( \phi = 0, 2\pi \) due to the boundary condition on the ground plane. The should be true on the surface of the cylinder, i.e., when \( \rho = a \). Now, in the region \( \rho < \rho_0 \), we write the solution as
\[ E_z(\rho, \phi) = \sum_{m=1}^{\infty} \left[ c_m J_{v_m}(k \rho) + d_m H_{v_m}^{(2)}(k \rho) \right] \sin(v_m \phi) \quad \rho < \rho_0 \]  \hspace{1cm} (1.6)

Where \( v_m = m \pi / (2 \pi - \beta) = m / 2 \). For \( \rho > \rho_0 \)

\[ E_z(\rho, \phi) = \sum_{m=1}^{\infty} b_m H_{v_m}^{(2)}(k \rho) \sin(v_m \phi) \quad \rho > \rho_0 \]  \hspace{1cm} (1.7)

On the circle with the radius \( a \) we have

\[ E_z(\rho, \phi) \bigg|_{\rho=a} - E_z(\rho, \phi) \bigg|_{\rho=0} = 0 \]  \hspace{1cm} (1.8)

\[ \frac{\partial E_z(\rho, \phi)}{\partial \rho} \bigg|_{\rho=a} - \frac{\partial E_z(\rho, \phi)}{\partial \rho} \bigg|_{\rho=0} = \frac{j \omega \mu_0 I}{\pi \rho_0} \delta(\phi - \phi_0) \]  \hspace{1cm} (1.9)

Furthermore,

\[ E_z(a, \phi) = 0 \]  \hspace{1cm} (1.10)

Upon substitution of (1.6) and (1.7) into (1.8)-(1.10), multiplication by \( \sin(n \phi) \) and integration over the angle from 0 to \( \pi \), one obtains the following equations

\[ c_n J_{v_n}(ka) + d_n H_{v_n}^{(2)}(ka) = 0 \]  \hspace{1cm} (1.11)

\[ b_n H_{v_n}^{(2)}(k \rho_0) = c_n J_{v_n}(k \rho_0) + d_n H_{v_n}^{(2)}(k \rho_0) \]  \hspace{1cm} (1.12)

\[ b_n H_{v_n}^{(2)}(k \rho_0) - c_n J_{v_n}'(k \rho_0) - d_n H_{v_n}^{(2)}(k \rho_0) = \frac{j \omega \mu_0 I}{\pi k \rho_0} \sin(n \phi_0) \]  \hspace{1cm} (1.13)

Obviously:

\[ d_n = -\frac{J_{v_n}(ka)}{H_{v_n}^{(2)}(ka)} c_n \]  \hspace{1cm} (1.14)

\[ b_n = \left[ \frac{J_{v_n}(k \rho_0)}{H_{v_n}^{(2)}(k \rho_0)} - \frac{J_{v_n}(ka)}{H_{v_n}^{(2)}(ka)} \right] c_n \]  \hspace{1cm} (1.15)

\[ \left[ \frac{J_{v_n}(k \rho_0)}{H_{v_n}^{(2)}(k \rho_0)} - \frac{J_{v_n}(ka)}{H_{v_n}^{(2)}(ka)} \right] H_{v_n}^{(2)}(k \rho_0) c_n \]

\[ - \left[ \frac{J_{v_n}'(k \rho_0) H_{v_n}^{(2)}(ka) - J_{v_n}(ka) H_{v_n}^{(2)}(k \rho_0)}{H_{v_n}^{(2)}(ka)} \right] c_n = \frac{j \omega \mu_0 I}{\pi k \rho_0} \sin(n \phi_0) \]  \hspace{1cm} (1.16)
The last equation leads to
\[
J_n(k \rho_0) H_{\nu n}^{(2)\prime}(k \rho_0) - J'_{\nu n}(k \rho_0) H_{\nu n}^{(2)}(k \rho_0) = \frac{j \omega \mu_0 I}{\pi k \rho_0} \sin(n \phi) \tag{1.17}
\]

Next, we use the relation
\[
J_n(k \rho_0) H_{\nu n}^{(2)\prime}(k \rho_0) - J'_{\nu n}(k \rho_0) H_{\nu n}^{(2)}(k \rho_0) = -j \frac{2}{\pi k \rho_0} \tag{1.18}
\]

So that
\[
c_n = -\left(\frac{\omega \mu_0 I}{2}\right) H_{\nu n}^{(2)}(k \rho_0) \sin(n \phi) \tag{1.19}
\]

The solution for \( \rho < \rho_0 \), is, thus
\[
E_z(\rho, \phi) = -\left(\frac{\omega \mu_0 I}{2}\right) \sum_{m=1}^{\infty} \left[ J_{\nu n}(k \rho) - \frac{J_{\nu n}(ka)}{H_{\nu n}^{(2)}(ka)} H_{\nu n}^{(2)}(k \rho) \right] H_{\nu n}^{(2)}(k \rho_0) \sin(m \phi) \sin(m \phi) \tag{1.20}
\]

Next, let \( \rho_0 \to \infty \), while keeping
\[
E_0 = -\frac{\omega \mu_0 I}{4} \sqrt{\frac{2}{\pi k \rho_0}} \exp\left(-j k \rho_0 + j \frac{\pi}{4}\right) \tag{1.21}
\]
a constant. This is the amplitude of an incident plane wave traveling towards the object along the line with the angle \( \phi_0 \). In our solution this leads to
\[
E_z(\rho, \phi) = 2E_0 \sum_{m=1}^{\infty} j^{\nu n} \left[ J_{\nu n}(k \rho) - \frac{J_{\nu n}(ka)}{H_{\nu n}^{(2)}(ka)} H_{\nu n}^{(2)}(k \rho) \right] \sin(m \phi_0) \sin(m \phi) \tag{1.22}
\]