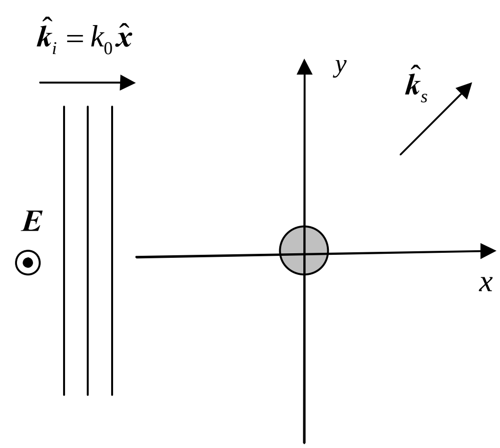
EM Scattering

Homework assignment 6

Problem 1:

An incident plane wave is propagating in vacuum along the +x-axis and with a polarization along the z-axis (in the electric sense). The wave is scattered by a small dielectric sphere with the radius a and relative dielectric constant ε_r. The center of the sphere coincides with the origin of the coordinate system. Find the far-zone scattered field along an arbitrary direction $\hat{\boldsymbol{k}}_s$ by using the Rayleigh approximation (small-particle, electrostatic approximation where the electric field is assumed constant inside the object and depolarization factors are used)



Solution

The incident wave may be approximated by a constant field ($\boldsymbol{r}_0 = 0$):

$$\boldsymbol{E}_i(\boldsymbol{r}) = \boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right) \approx \boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}_0\right) = \boldsymbol{E}_i^0 = E_0\hat{\boldsymbol{z}} \quad (1.1)$$

The total field inside the small sphere in Rayleigh approximation

$$\boldsymbol{E} = \left[\bar{\bar{\boldsymbol{I}}} + \delta\varepsilon_r \bar{\bar{\boldsymbol{N}}}\right]^{-1} \cdot \boldsymbol{E}_i^0 \quad (1.2)$$

where for a sphere

$$\bar{\bar{\boldsymbol{N}}} = \begin{bmatrix} N_x & 0 & 0 \\ 0 & N_y & 0 \\ 0 & 0 & N_z \end{bmatrix}, \; N_x = N_y = N_z = 1/3 \quad (1.3)$$

As a result:

$$\boldsymbol{E} = \frac{3E_0}{2+\varepsilon_r}\hat{\boldsymbol{z}} \quad (1.4)$$

Far zone scattered field

$$\begin{aligned}
\boldsymbol{E}_s(\boldsymbol{r}) &= -\frac{k^2\exp(-jkr)}{4\pi r}\hat{\boldsymbol{k}}_s\times\left[\hat{\boldsymbol{k}}_s\times\delta\varepsilon_r\int_V\exp\left(j\boldsymbol{k}_s\cdot\boldsymbol{r}'\right)\boldsymbol{E}(\boldsymbol{r}')dV'\right] \\
&\approx -\frac{k^2\exp(-jkr)}{4\pi r}\hat{\boldsymbol{k}}_s\times\left[\hat{\boldsymbol{k}}_s\times\left(\delta\varepsilon_r V_{sphere}\right)\boldsymbol{E}\right] \\
&= -\frac{k^2\exp(-jkr)V_{sphere}E_0}{4\pi r}\frac{3(\varepsilon_r-1)}{2+\varepsilon_r}\hat{\boldsymbol{k}}_s\times\left(\hat{\boldsymbol{k}}_s\times\hat{\boldsymbol{z}}\right) \\
&= -\frac{\exp(-jkr)k^2a^3E_0}{r}\frac{(\varepsilon_r-1)}{2+\varepsilon_r}\hat{\boldsymbol{k}}_s\times\left(\hat{\boldsymbol{k}}_s\times\hat{\boldsymbol{z}}\right)
\end{aligned} \tag{1.5}$$

Problem 2:

A plane wave is normally incident on a perfectly conducting, infinitely long cylinder of radius a whose axis lies on the z-axis. The incoming wave is propagating along the x-axis.

1. For the TM_z case (electric field along the z-axis) find the current density on the cylinder as function of the angle ϕ. Use the exact series in terms of Bessel and Hankel functions found in previous chapters.
2. Compare the result found with the physical optic approximation by plotting the magnitude of the current density as function of ϕ for $ka = 1$ and $ka = 20$. Here k is the wave number in the background medium.

Solution

(1) The first part was solved in homework 3. The current density equals

$$\boldsymbol{J}_s^{TM} = \hat{\boldsymbol{\rho}}\times\boldsymbol{H}^{TM} = \hat{\boldsymbol{z}}H_\phi^{TM}(a,\phi) \tag{1.6}$$

Where

$$H_\phi^{TM}(a,\phi) = \frac{jE_0}{\eta}\left(\frac{2}{\pi ka}\right)\sum_{m=-\infty}^{\infty}\frac{(-j)^m\exp\left[-jm\left(\phi-\phi_i\right)\right]}{H_m^{(2)}(ka)} \tag{1.7}$$

Here E_0 is the amplitude of the incident wave and $\phi_i = 0$. (There is a sign difference with the solution of problem 1 of homework set 3 since, there, the incident electric field was assumed along the $-z$ direction.) Thus

$$\boldsymbol{J}_s^{TM} = \hat{\boldsymbol{z}} \frac{jE_0}{\eta} \left(\frac{2}{\pi ka} \right) \sum_{m=-\infty}^{\infty} \frac{(-j)^m \exp(-jm\phi)}{H_m^{(2)}(ka)} \tag{1.8}$$

(2) In physics optics approximation one has

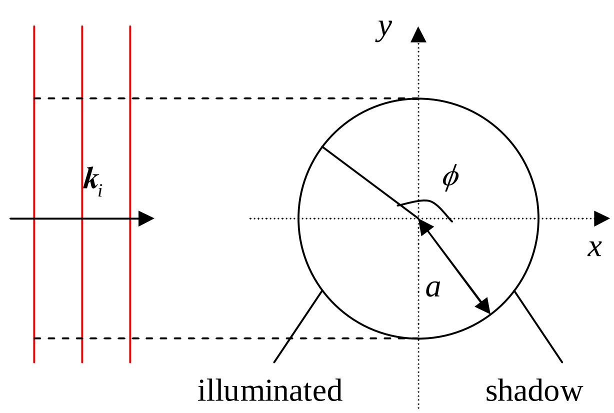
$$\boldsymbol{J}_s^{PO} = \begin{cases} 2\hat{\boldsymbol{n}} \times \boldsymbol{H}_i & \text{in the illuminated region} \\ 0 & \text{in the shadow region} \end{cases} \tag{1.9}$$

$$\boldsymbol{H}_i = \boldsymbol{H}_i^0 \exp(-j\boldsymbol{k}_i \cdot \boldsymbol{r}), \quad \boldsymbol{H}_i^0 = \frac{1}{\eta} \hat{\boldsymbol{k}}_i \times \boldsymbol{E}_i^0 \tag{1.10}$$

Therefore,

$$\boldsymbol{H}_i = -\hat{\boldsymbol{y}} \frac{E_0}{\eta} \exp(-jkx) \tag{1.11}$$

$$\boldsymbol{J}_s^{PO} = \begin{cases} -2\hat{\boldsymbol{z}} \frac{E_0}{\eta} \cos\phi \exp(-jka\cos\phi) & \text{for } \frac{\pi}{2} < \phi < \frac{3\pi}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.12}$$



Magnitude of the current density in the two cases:

Exact:

$$\left| \boldsymbol{J}_s^{TM} \right| = 2 \frac{|E_0|}{\eta} \left| \sum_{m=-\infty}^{\infty} \frac{(-j)^m \exp(-jm\phi)}{\pi ka H_m^{(2)}(ka)} \right| \tag{1.13}$$

Physics optics:

$$\left|\boldsymbol{J}_s^{PO}\right| = \begin{cases} 2\dfrac{\left|E_0\right|}{\eta}\left|\cos\phi\right| & \text{for } \dfrac{\pi}{2} < \phi < \dfrac{3\pi}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.14}$$