Electromagnetic scattering

Graduate Course

Electrical Engineering (Communications)

1st Semester, 1390-1391

Sharif University of Technology

Contents of lecture 2

- **Lecture 2: Basic scattering parameters**
  - Formulation of the problem
  - Scattering cross section
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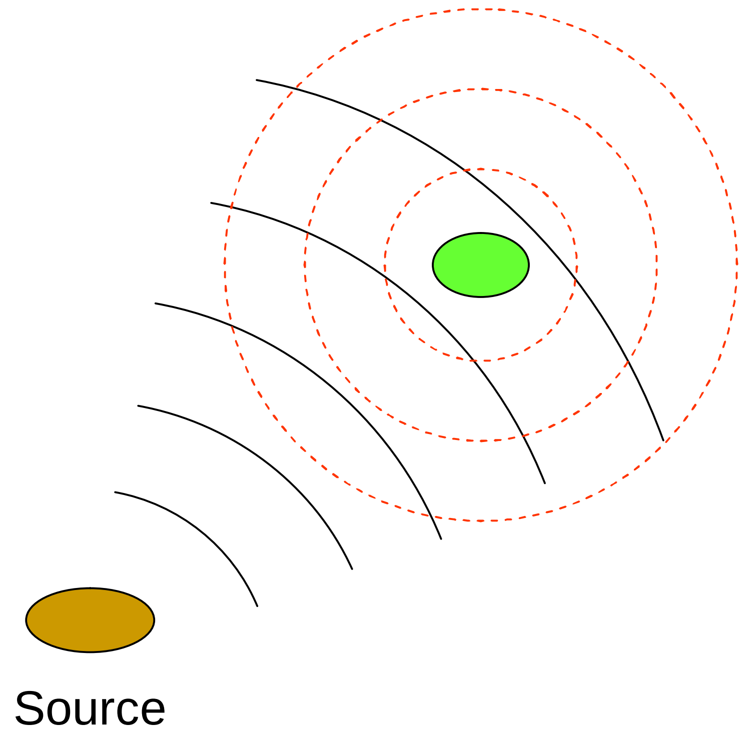
  - Scattering amplitude matrix
  - Unit vector systems
  - Expressions for the scattered field:
    - Born approximation
    - Relation with Fourier transform
    - Optical theorem

Contents of lecture 2

- **Complementary material:**
  - Ishimaru – Chapter 10, pp. 278-288
  - Tsang, Kong, Ding – Chapter 1, pp. 1-9

Introduction

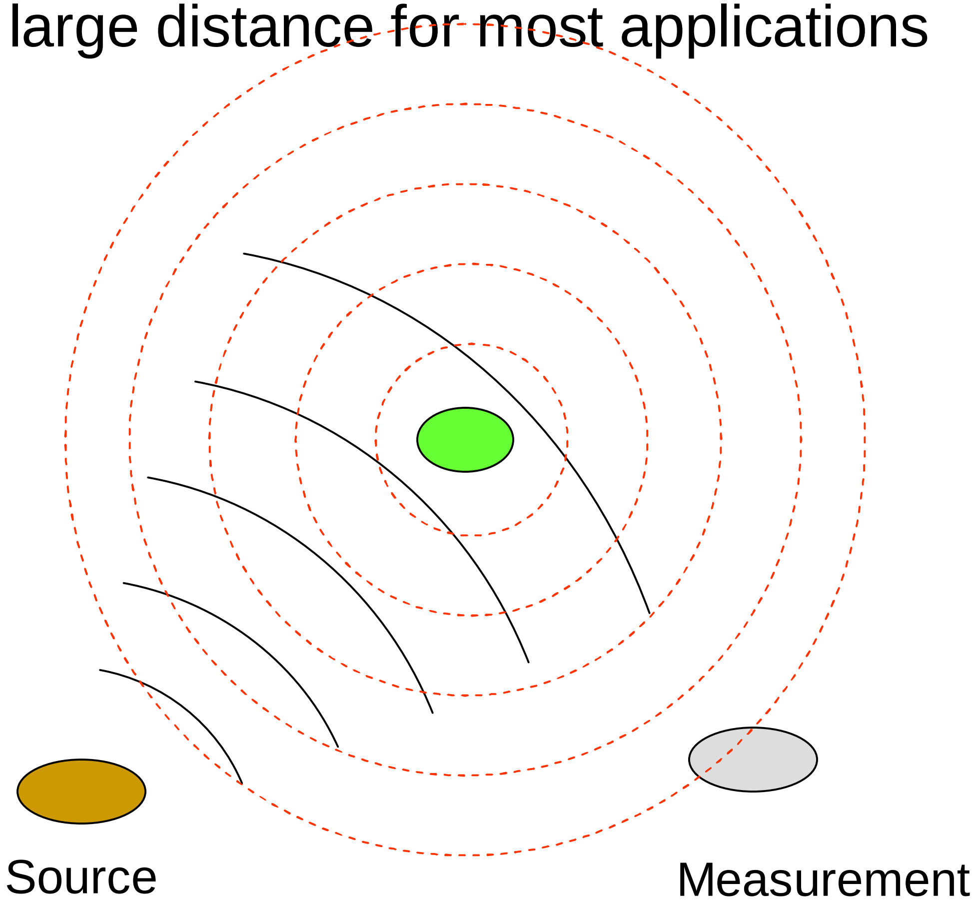
- Often, one's aim is to gain information on an 'object' far away by sending out a wave and analyzing the scattered wave
- At a sufficiently large distance such far field waves are seen as plane TEM waves
- So, we have to study the scattering of plane TEM waves by objects
- Scattering: polarization and conduction charges are set into motion by the incoming field
- Induced currents radiate an extra field (scattered field)



Introduction

- We have to calculate these induced currents and use them to compute the resulting far field, this is because the scattered field is also measured at a large distance for most applications

- But, first, we discuss a number of general parameters used in scattering theory

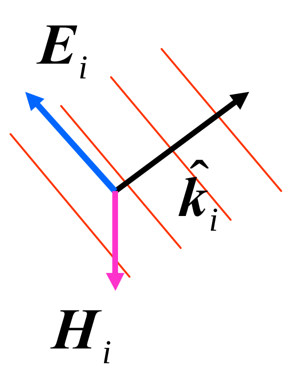


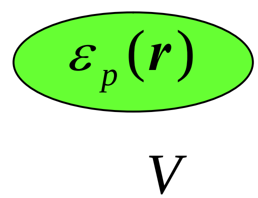
Basic scattering parameters

- Consider an object with a permittivity different from the dielectric constant of the background. A plane wave with a known polarization propagates along $\hat{\boldsymbol{k}}_i$ and hits the objects.
- Far away from the object, the induced scattered field also behaves as a plane wave in each direction $\hat{\boldsymbol{k}}_s$

$$\boldsymbol{E}_i = \boldsymbol{E}_{i,0} \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right)$$

Incident field





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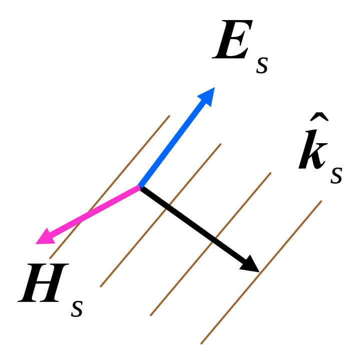
$$\boldsymbol{E}_s = \boldsymbol{E}_{s,0} \exp\left(-j\boldsymbol{k}_s \cdot \boldsymbol{r}\right)$$

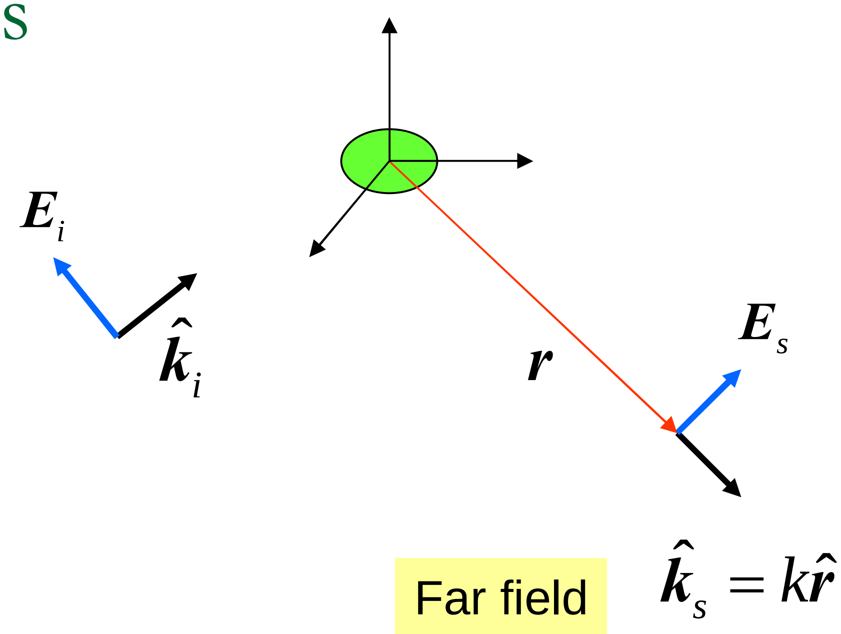


Scattered field, locally behaving as a plane wave in the far-zone

Basic scattering parameters

- The incident electric field

$$\boldsymbol{E}_i = \boldsymbol{E}_{i,0} \exp\left(-jk\hat{\boldsymbol{k}}_i \cdot \boldsymbol{r}\right)$$



- Far-zone scattered field

$$\boldsymbol{E}_s = \tilde{\boldsymbol{E}}_{s,0}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)\frac{\exp(-jkr)}{r}$$

Amplitude vector of the scattered field up to a factor 1/r, also depends on the incoming electric field vector

- Scattering amplitude:

$$\left| f\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)\right| = r\frac{|\boldsymbol{E}_s|}{|\boldsymbol{E}_i|} = \frac{\left|\tilde{\boldsymbol{E}}_{s,0}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)\right|}{|\boldsymbol{E}_{i,0}|}$$

Basic scattering parameters

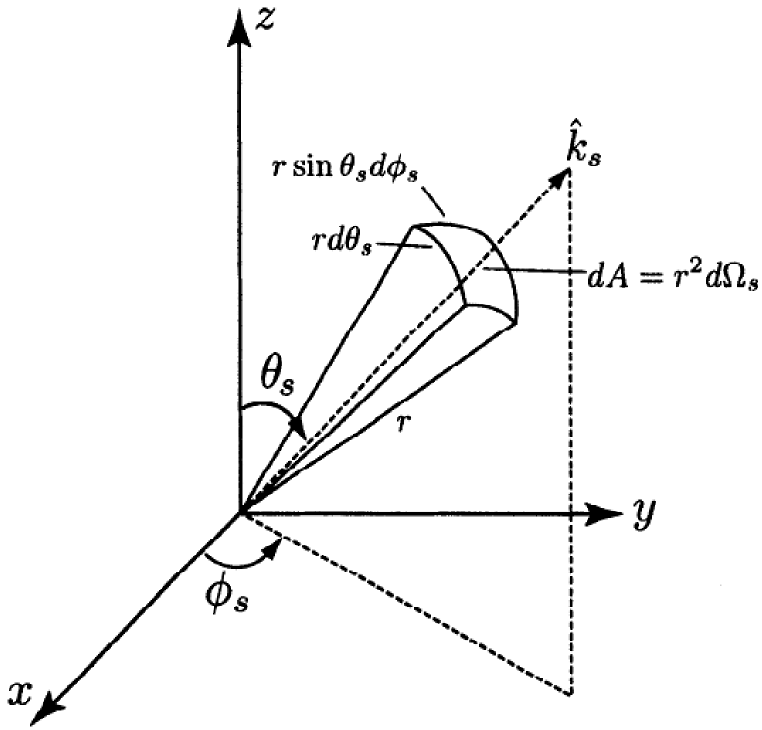
- Incident Poynting vector

$$\boldsymbol{S}_i = \frac{\left|E_{i,0}\right|^2}{2\eta}\hat{\boldsymbol{k}}_i$$

- Poynting vector of the scattered wave

$$\boldsymbol{S}_s = \frac{\left|E_s\right|^2}{2\eta}\hat{\boldsymbol{k}}_s = \frac{\left|f\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)\right|^2}{r^2}\frac{\left|E_{i,0}\right|^2}{2\eta}\hat{\boldsymbol{k}}_s$$

$$\hat{\boldsymbol{k}}_s = \left(\sin\theta_s\cos\phi_s, \sin\theta_s\sin\phi_s, \cos\theta_s\right)$$



Scattering cross section

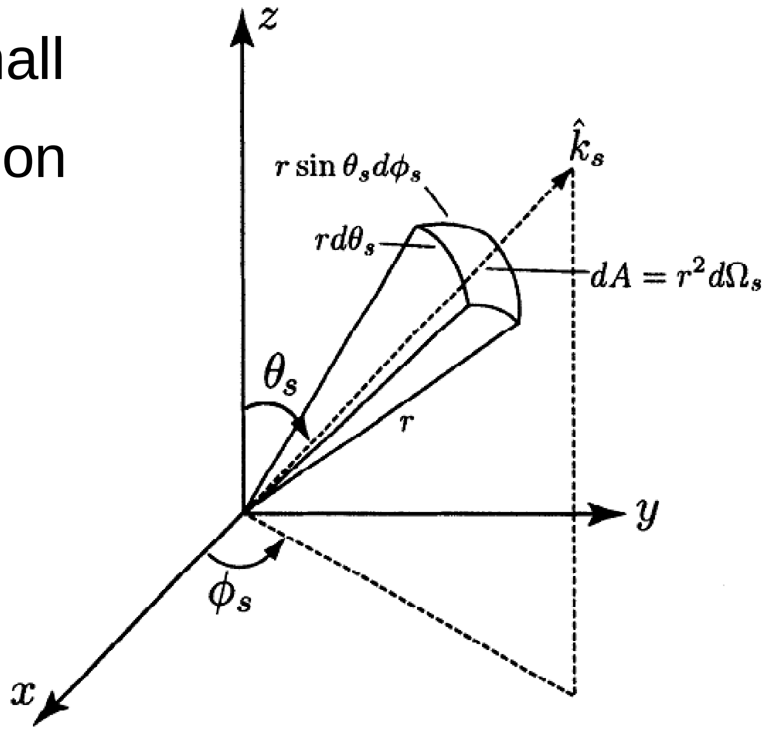
- Scattered power flowing through a small surface normal to the observed direction

$$dP_s = \boldsymbol{S}_s \cdot \hat{\boldsymbol{n}} dA = |S_i| \left| f\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \right|^2 d\Omega_s$$

$$\frac{dP_s}{|S_i|} = \underbrace{\left| f\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \right|^2}_{\text{Differential scattering cross section}} d\Omega_s$$

Differential scattering cross section

$$\sigma_d\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) = \left| f\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \right|^2$$



$$d\Omega_s = \sin\theta_s d\theta_s d\phi_s$$

Scattering cross section

- Differential scattering cross section has the dimension of *area:*

$$dP_s = |S_i| \sigma_d \left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i \right) d\Omega_s$$

- Power scattered into a small normal surface with a solid angle $d\Omega_s$ equals incident power passing through a surface normal to the incident wave with the area

$$\sigma_d \left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i \right) d\Omega_s$$

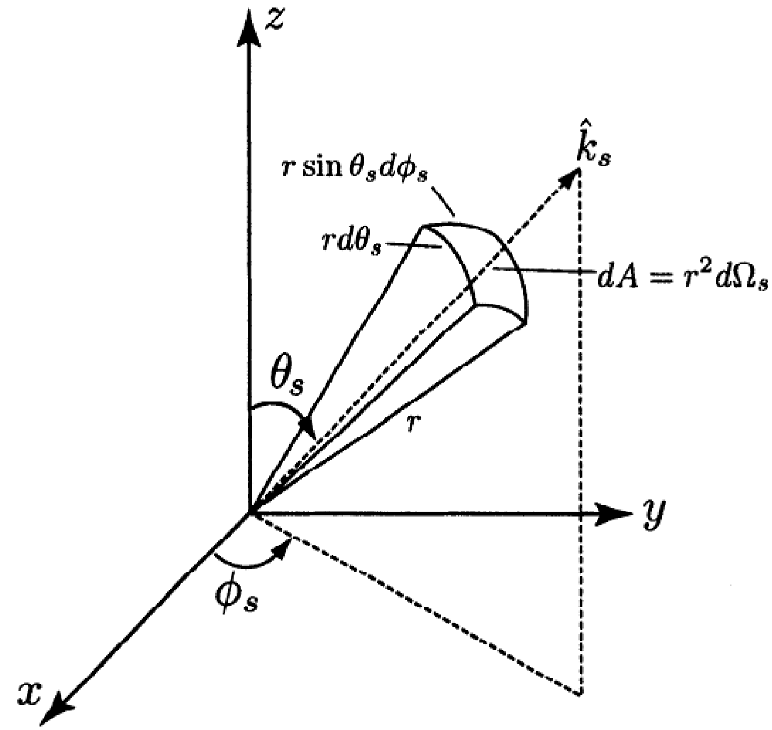
- It is a measurable quantity if we know the intensity of the incident wave and the distance from the observation point to the scatterer

Scattering cross section

- Total scattering cross section:

$$P_s = |S_i| \int_{4\pi} \left| f\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \right|^2 d\Omega_s$$

$$\frac{P_s}{|S_i|} = \underbrace{\int_{4\pi} \left| f\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \right|^2 d\Omega_s}_{\text{Scattering cross section}}$$

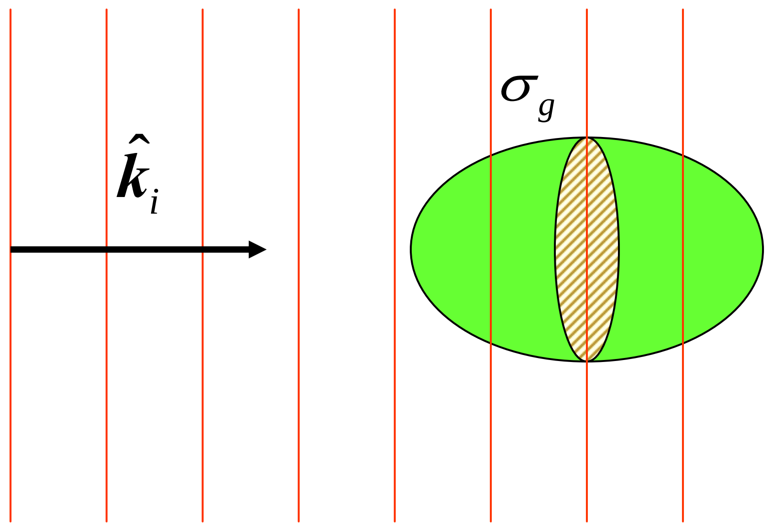


$$d\Omega_s = \sin\theta_s d\theta_s d\phi_s$$

$$\sigma_s = \frac{P_s}{|S_i|} = \int_{4\pi} \sigma_d\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) d\Omega_s$$

Scattering cross section

- Scattering cross section: area of a surface normal to the incident wave which captures an incident power equal to the total power scattered
- Not equal to the geometrical cross section σ_g: area projected onto a plane perpendicular to the incident wave vector



$$\sigma_s \neq \sigma_g$$

Absorption cross section

- Part of the power captured by the object may be dissipated due to polarization loss or conductivity

- Total power absorbed by the object (magnetic loss neglected):

$$P_a = \frac{\omega\varepsilon''}{2}\int_V \left|\boldsymbol{E}_{\text{int}}(\boldsymbol{r})\right|^2 dV$$

Field inside the object

- Absorption cross section:

$$\sigma_a = \frac{P_a}{|S_i|}$$

Total cross section

- Total cross section

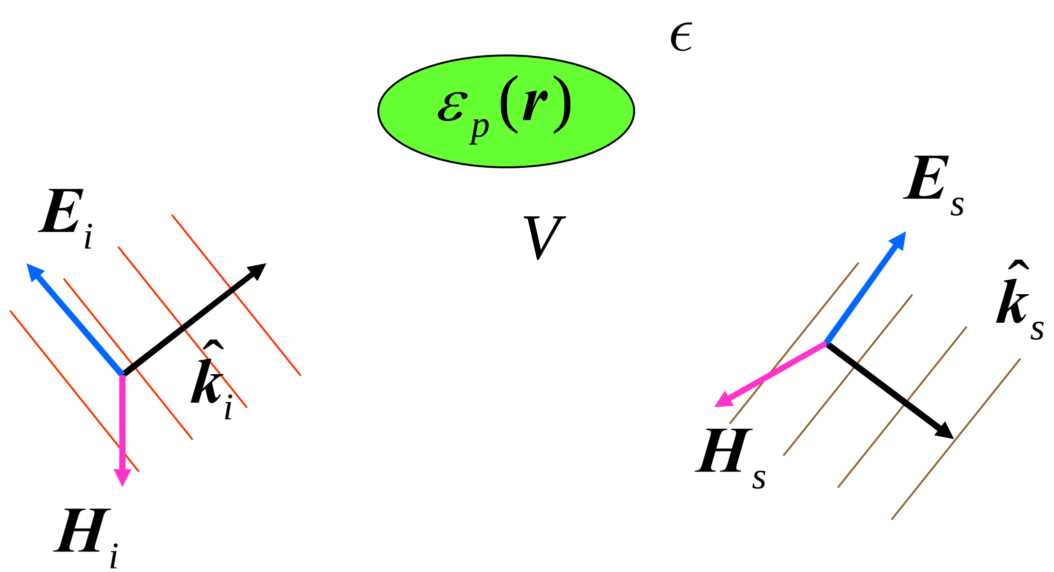
$$\sigma_t = \frac{P_s}{|S_i|} + \frac{P_a}{|S_i|} = \sigma_s + \sigma_a$$

- Albedo of the object

$$\varpi = \frac{\sigma_s}{\sigma_t} = \frac{\sigma_s}{\sigma_s + \sigma_a} \leq 1$$

Scattering amplitude matrix

- So far, we only discussed the relations between the incident and scattered energy flow (power)
- But what about the fields themselves? What about their amplitudes and directions?

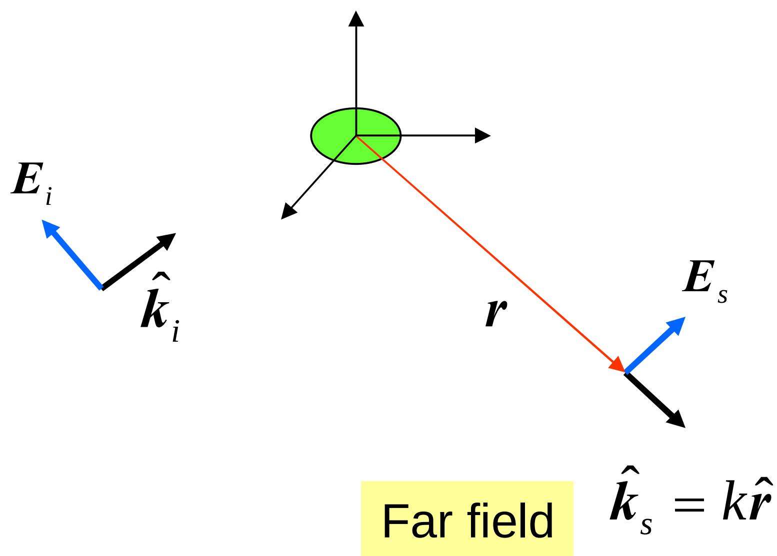


Scattering amplitude matrix

- The scattering amplitude $f\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)$ was introduced without any reference to the polarization of the incident and reflected waves

$$\boldsymbol{E}_i = \boldsymbol{E}_{i,0} \exp\left(-jk\hat{\boldsymbol{k}}_i \cdot \boldsymbol{r}\right)$$

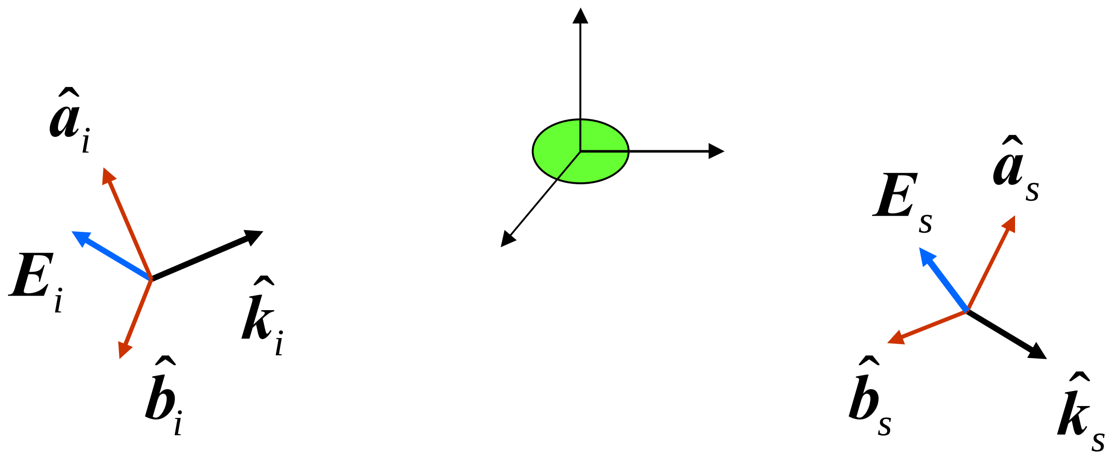
$$\boldsymbol{E}_s = \tilde{\boldsymbol{E}}_{s,0}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)\frac{\exp\left(-jkr\right)}{r}$$



- But, in reality, this function depends on these directions

Scattering amplitude matrix

- General formulation: choose a system of coordinates for the electric field of the incident wave and for the electric field of the scattered wave for each scattering direction
- The unit vectors of the incident wave are perpendicular to $\hat{\boldsymbol{k}}_i$ and the unit vectors of the scattered wave are perpendicular to $\hat{\boldsymbol{k}}_s$



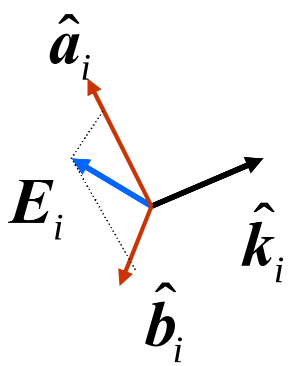
Scattering amplitude matrix

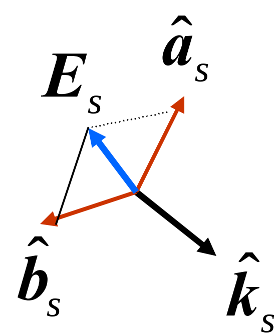
$$\boldsymbol{E}_i = \left(E_i^a \hat{\boldsymbol{a}}_i + E_i^b \hat{\boldsymbol{b}}_i\right) \exp\left(-jk\hat{\boldsymbol{k}}_i \cdot \boldsymbol{r}\right)$$

$$\boldsymbol{E}_s = \left(E_s^a \hat{\boldsymbol{a}}_s + E_s^b \hat{\boldsymbol{b}}_s\right) \frac{\exp(-jkr)}{r}$$

$$\begin{pmatrix} E_s^a \\ E_s^b \end{pmatrix} = \begin{bmatrix} f_{aa}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) & f_{ab}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \\ f_{ba}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) & f_{bb}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \end{bmatrix} \begin{pmatrix} E_i^a \\ E_i^b \end{pmatrix}$$

Scattering amplitude matrix





Scattering amplitude matrix

- Note that the waves are not necessarily linearly polarized.
- Depending on their coefficients, they may have linear, circular or elliptic polarization in the plane perpendicular to the propagation direction
- Also note that the system of unit vectors for the scattered wave may depend on the scattering direction considered
- Finally, there are two main system of unit vectors which are usually used as we see next

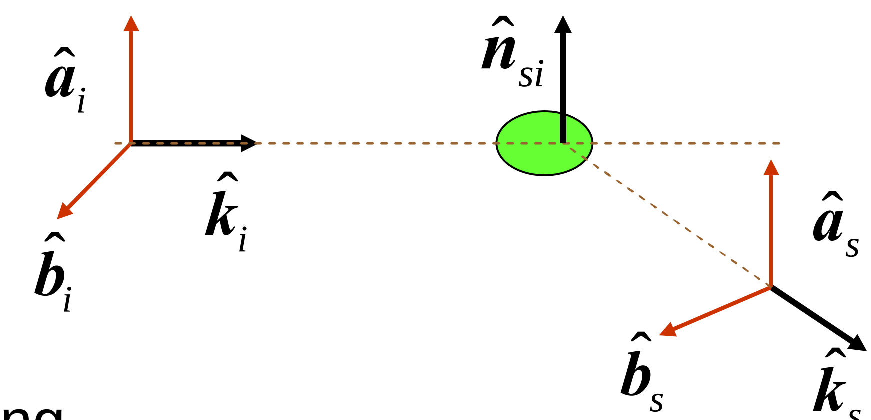
Unit vector systems

- (1) System based on *scattering plane*: for every scattering direction, $\hat{\boldsymbol{k}}_i$ and $\hat{\boldsymbol{k}}_s$ lie on a plane with the normal vector

$$\hat{\boldsymbol{n}}_{si} = \frac{\hat{\boldsymbol{k}}_s \times \hat{\boldsymbol{k}}_i}{\left|\hat{\boldsymbol{k}}_s \times \hat{\boldsymbol{k}}_i\right|} \quad \Rightarrow \quad \text{choose} \quad \hat{\boldsymbol{a}}_i = \hat{\boldsymbol{a}}_s = \hat{\boldsymbol{n}}_{si}$$

$$\hat{\boldsymbol{b}}_i = \hat{\boldsymbol{k}}_i \times \hat{\boldsymbol{a}}_i$$

$$\hat{\boldsymbol{b}}_s = \hat{\boldsymbol{k}}_s \times \hat{\boldsymbol{a}}_s$$



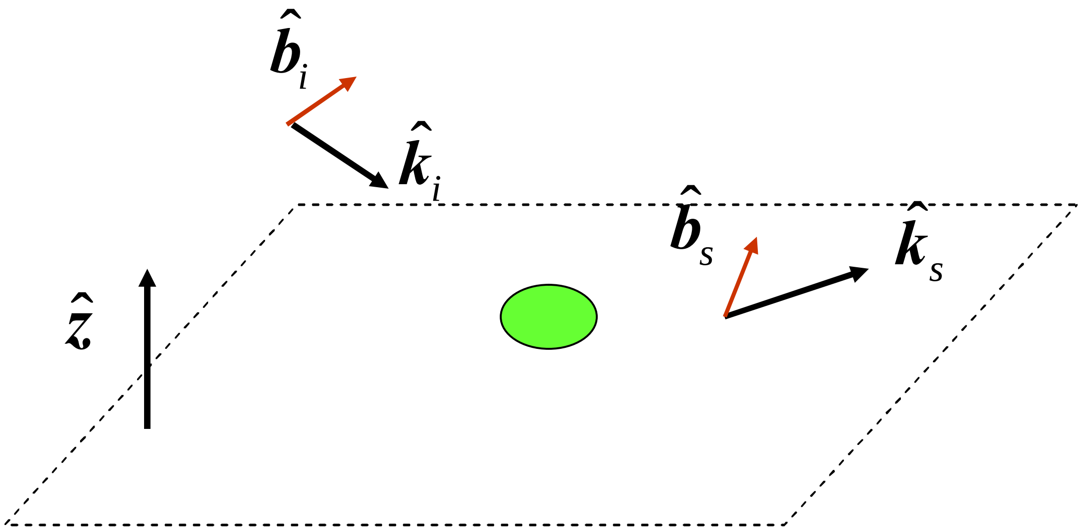
- Vectors depend on scattering direction considered

Unit vector systems

- (2) Vertical and horizontal polarization: often there is a preferred plane like the earth surface (in geographical systems). Let normal to this plane be the axis z
- Then: 2nd unit vector chosen to lie in horizontal plane

$$\hat{\boldsymbol{b}}_i = \frac{\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_i}{\left|\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_i\right|}$$

$$\hat{\boldsymbol{b}}_s = \frac{\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_s}{\left|\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_s\right|}$$

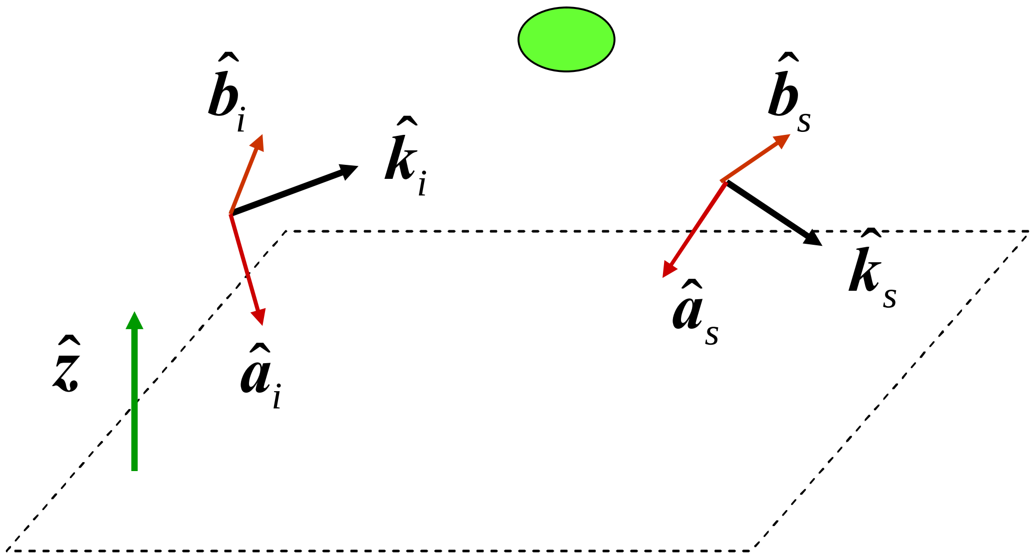


Unit vector systems

- The other unit vector follows:

$$\hat{\boldsymbol{a}}_i = \hat{\boldsymbol{b}}_i \times \hat{\boldsymbol{k}}_i \qquad \hat{\boldsymbol{a}}_s = \hat{\boldsymbol{b}}_s \times \hat{\boldsymbol{k}}_i$$

- Terminology: horizontal and vertical polarization
- Later we may use the notions TE and TM which should not be confused with waveguide modes



Unit vector systems

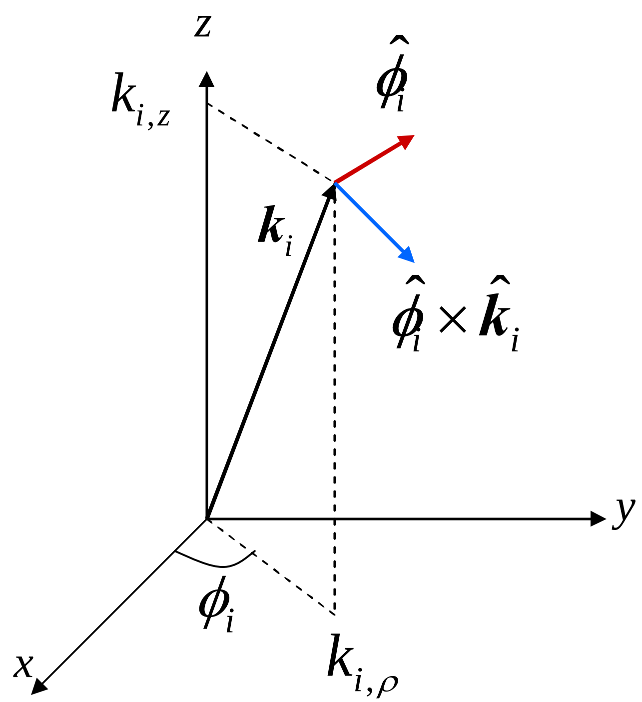
- In terms of cylindrical coordinate system:

$$\hat{\boldsymbol{k}}_i = \frac{1}{k}\left(k_{i,\rho}\cos\phi_i, k_{i,\rho}\sin\phi_i, k_{i,z}\right)$$

$$\hat{\boldsymbol{b}}_i = \frac{\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_i}{\left|\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_i\right|} = \left(-\sin\phi_i, \cos\phi_i, 0\right) = \hat{\boldsymbol{\phi}}_i$$

$$\hat{\boldsymbol{a}}_i = \hat{\boldsymbol{b}}_i \times \hat{\boldsymbol{k}}_i$$

$$= \frac{1}{k}\left(k_{i,z}\cos\phi_i, k_{i,z}\sin\phi_i, -k_{i,\rho}\right)$$



Unit vector systems

- In terms of spherical coordinate system:

$$\hat{\boldsymbol{k}}_i = (\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i)$$

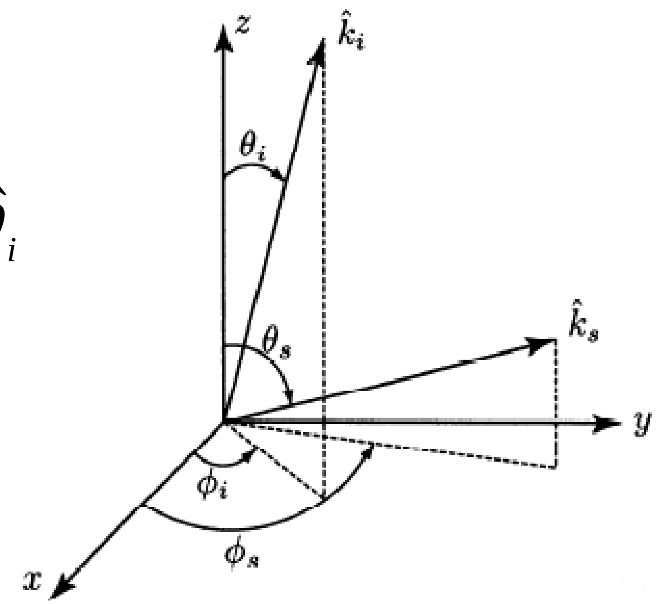
$$\hat{\boldsymbol{b}}_i = (-\sin\phi_i, \cos\phi_i, 0) = \hat{\boldsymbol{\phi}}_i$$

$$\hat{\boldsymbol{a}}_i = (\cos\theta_i \cos\phi_i, \cos\theta_i \sin\phi_i, -\sin\theta_i) = \hat{\boldsymbol{\theta}}_i$$

$$\hat{\boldsymbol{k}}_s = (\sin\theta_s \cos\phi_s, \sin\theta_s \sin\phi_s, \cos\theta_s)$$

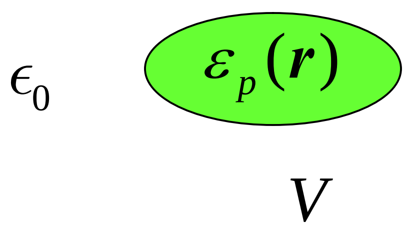
$$\hat{\boldsymbol{b}}_s = (-\sin\phi_s, \cos\phi_s, 0) = \hat{\boldsymbol{\phi}}_s$$

$$\hat{\boldsymbol{a}}_s = (\cos\theta_s \cos\phi_s, \cos\theta_s \sin\phi_s, -\sin\theta_s) = \hat{\boldsymbol{\theta}}_s$$



Expressions for the scattered field

- So far we just discussed a number of parameters which are used in scattering problems
- But how can one express the scattered field in terms of what is happening inside the object?
- Let us focus on an object which may be a dielectric and/or conductive in *free space*. This object will be characterized by a complex permittivity



$$\varepsilon_p(\boldsymbol{r}) = \epsilon_p(\boldsymbol{r}) - \frac{j}{\omega}\sigma_p(\boldsymbol{r})$$

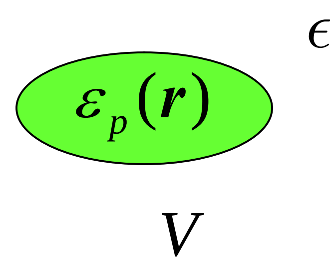
Expressions for the scattered field

- When the incident field radiates the object, currents are induced. The sum of induced polarization and conduction currents is

$$\boldsymbol{J}_p(\boldsymbol{r}) = j\omega\left[\epsilon_p(\boldsymbol{r}) - \epsilon_0\right]\boldsymbol{E}(\boldsymbol{r}) \qquad \boldsymbol{J}_c(\boldsymbol{r}) = \sigma_p(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r})$$

$$\rightarrow \boldsymbol{J}(\boldsymbol{r}) = \boldsymbol{J}_p(\boldsymbol{r}) + \boldsymbol{J}_c(\boldsymbol{r}) = j\omega\left[\varepsilon_p(\boldsymbol{r}) - \epsilon_0\right]\boldsymbol{E}(\boldsymbol{r})$$

- What is the far field generated by this current? (This is the scattering field)



Expressions for the scattered field

- Far field expressions for the scattered field:

$$\boldsymbol{E}_s^f(\boldsymbol{r}) = jk\eta \frac{\exp(-jkr)}{4\pi r}\hat{\boldsymbol{r}} \times \left[\hat{\boldsymbol{r}} \times \boldsymbol{F}(\hat{\boldsymbol{r}})\right]$$

$$\boldsymbol{H}_s^f(\boldsymbol{r}) = -\frac{jk\exp(-jkr)}{4\pi r}\hat{\boldsymbol{r}} \times \boldsymbol{F}(\hat{\boldsymbol{r}})$$

$$\boldsymbol{F}(\hat{\boldsymbol{r}}) = \int_V \exp\left(jk\hat{\boldsymbol{r}} \cdot \boldsymbol{r}'\right)\boldsymbol{J}(\boldsymbol{r}')dV'$$

- Note that in all directions the scattered wave is along the position vector

$$\boxed{\hat{\boldsymbol{k}}_s = \hat{\boldsymbol{r}}}$$

Expressions for the scattered field

- (Far-zone) scattered electric field: for simplicity let us define

$$\boldsymbol{Q}\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)=-\frac{jk\eta}{4\pi}\boldsymbol{F}\left(\hat{\boldsymbol{k}}_s\right)$$

$$\boldsymbol{E}_s^f(\boldsymbol{r})=-\frac{\exp(-jkr)}{r}\hat{\boldsymbol{k}}_s\times\left[\hat{\boldsymbol{k}}_s\times\boldsymbol{Q}\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)\right]$$

- Scattering from equivalent currents in a dielectric:

$$\boldsymbol{Q}\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)=\frac{k^2}{4\pi\epsilon_0}\int_V \exp\left(j\boldsymbol{k}_s\cdot\boldsymbol{r}'\right)\delta\varepsilon_p(\boldsymbol{r}')\boldsymbol{E}(\boldsymbol{r}')dV'$$

$$\delta\varepsilon_p(\boldsymbol{r}')=\varepsilon_p(\boldsymbol{r}')-\epsilon_0$$

This is the total field inside the object: incident plus the field generated by the object itself

Expressions for the scattered field

- Equivalently:

$$\boldsymbol{E}_s^f(\boldsymbol{r}) = \frac{\exp(-jkr)}{r}\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)$$

$$\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) = \boldsymbol{Q}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) - \left[\hat{\boldsymbol{k}}_s \cdot \boldsymbol{Q}\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)\right]\hat{\boldsymbol{k}}_s$$

- Compare with the definition of scattering amplitude:

$$\left|f\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)\right| = r\frac{|\boldsymbol{E}_s|}{|\boldsymbol{E}_i|} \rightarrow \left|f\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)\right| = \frac{1}{|\boldsymbol{E}_{i,0}|}\left|\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)\right|$$

Born approximation

- In principle, we should know the total field inside the scattering object in order to be able to compute the scattered field
- But, if $\varepsilon_p(\boldsymbol{r}') \approx \epsilon_0$ an approximation can be made
- In that case the field generated by the object is small and the total field is close to the incident field

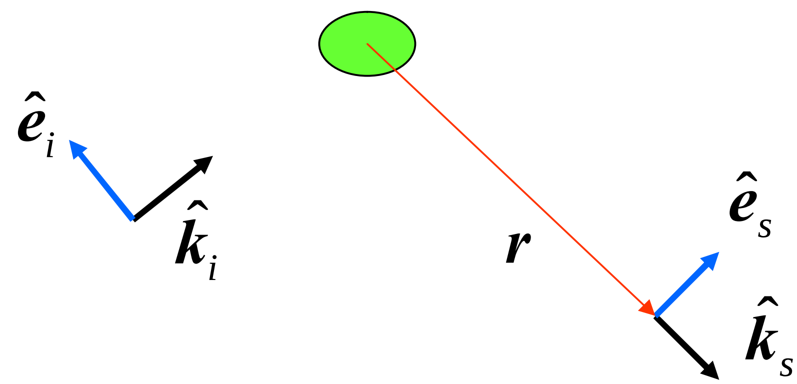
$$\boldsymbol{E}(\boldsymbol{r}) \approx \boldsymbol{E}_i(\boldsymbol{r}) = \boldsymbol{E}_{i,0} \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right)$$

Born approximation

- Result:

$$\boldsymbol{Q}\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)=\frac{k^2}{4\pi\epsilon_0}\left\{\int_V \exp\left[j\left(\boldsymbol{k}_s-\boldsymbol{k}_i\right)\cdot\boldsymbol{r}'\right]\delta\varepsilon_p(\boldsymbol{r}')dV'\right\}\boldsymbol{E}_{i,0}$$

$$\boldsymbol{E}_s^f(\boldsymbol{r})=-\frac{\exp(-jkr)}{r}\hat{\boldsymbol{k}}_s\times\left[\hat{\boldsymbol{k}}_s\times\boldsymbol{Q}\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)\right]$$



Born approximation

- Equivalently

$$\boldsymbol{E}_s^f(\boldsymbol{r}) = \frac{\exp(-jkr)}{r}\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)$$

$$\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) = \frac{k^2}{4\pi\epsilon_0}\left\{\int_V \exp\left[j\left(\boldsymbol{k}_s - \boldsymbol{k}_i\right)\cdot\boldsymbol{r}'\right]\delta\varepsilon_p(\boldsymbol{r}')dV'\right\}$$

$$\left[\boldsymbol{E}_{i,0} - \left(\boldsymbol{E}_{i,0}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right]$$

Born approximation

- So that

$$\boldsymbol{E}_s^f(\boldsymbol{r}) = \frac{\exp(-jkr)}{r}\left[\boldsymbol{E}_{i,0} - \left(\boldsymbol{E}_{i,0}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right] S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)$$

$$S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right) = \frac{k^2}{4\pi\epsilon_0}\int_V \exp\left[j\left(\boldsymbol{k}_s-\boldsymbol{k}_i\right)\cdot\boldsymbol{r}'\right]\delta\varepsilon_p(\boldsymbol{r}')dV'$$

- The polarization of the scattered field in the direction $\hat{\boldsymbol{k}}_s$ is given by the direction of the vector

$$\boldsymbol{E}_{i,0} - \left(\boldsymbol{E}_{i,0}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s$$

Born approximation

- From the definition of the scattering amplitude it follows that

$$\left| f\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \right| = \frac{1}{\left| \boldsymbol{E}_{i,0} \right|} \left| \boldsymbol{E}_{i,0} - \left(\boldsymbol{E}_{i,0} \cdot \hat{\boldsymbol{k}}_s \right) \hat{\boldsymbol{k}}_s \right| \left| S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \right|$$

- Note that the polarization of the scattered wave in this approximation does not depend on material properties
- Also the dependence of f on polarization is the same for all objects because S is independent of polarization.
- Finally note that S is the Fourier transform of $\delta\varepsilon_p$ for

$$\boldsymbol{k}_d = \boldsymbol{k}_s - \boldsymbol{k}_i = k\left(\hat{\boldsymbol{k}}_s - \hat{\boldsymbol{k}}_i \right)$$

Born approximation

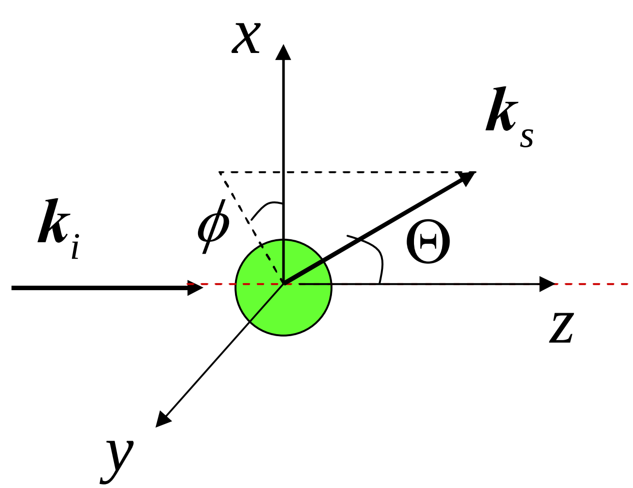
- Example: sphere of radius R with constant ϵ

- Using spherical coordinates, and choosing a linearly polarized incident wave propagating along z:

$$S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) = S(\Theta) = \frac{k^2\left(\epsilon_r - 1\right)}{k_d^3}\left[\sin\left(k_d R\right) - k_d R \cos\left(k_d R\right)\right]$$

$$k_d = \left|\boldsymbol{k}_d\right| = 2k \sin\frac{\Theta}{2}$$

- Because of the symmetry of the structure we do not need to consider other incident directions



Born approximation

- Details of calculation: $S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)=\dfrac{k^2\delta\varepsilon_p}{4\pi\epsilon_0}\displaystyle\int_{sphere}\exp\left(j\boldsymbol{k}_d\cdot\boldsymbol{r}'\right)dV'$

- Change coordinate system to one where z-is along $\boldsymbol{k_d}$

$$S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)=\frac{k^2\delta\varepsilon_p}{4\pi\epsilon_0}\int_0^R\int_0^\pi\int_0^{2\pi}\exp\left(jk_d r\cos\theta\right)r^2\sin\theta\,dr\,d\theta\,d\varphi$$

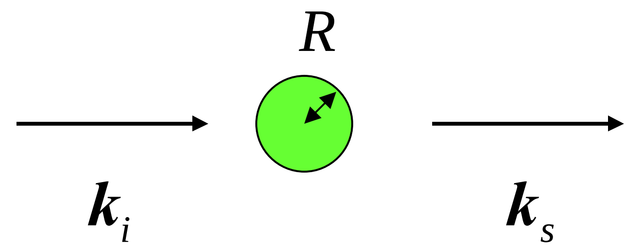
$$=\frac{k^2\delta\varepsilon_p}{2\epsilon_0}\int_0^R\int_0^\pi\exp\left(jk_d r\cos\theta\right)r^2\sin\theta\,dr\,d\theta$$

$$=\frac{k^2\delta\varepsilon_p}{2\epsilon_0}\int_0^R\int_{-1}^1\exp\left(jk_d ru\right)r^2\,dr\,du=\frac{k^2\delta\varepsilon_p}{\epsilon_0}\int_0^R\frac{\sin\left(k_d r\right)}{k_d}r\,dr$$

$$=\frac{k^2\delta\varepsilon_p}{\epsilon_0 k_d^3}\int_0^{k_d R}v\sin v\,dv=\frac{k^2\delta\varepsilon_p}{\epsilon_0 k_d^3}\left[\sin\left(k_d R\right)-k_d R\cos\left(k_d R\right)\right]$$

Born approximation

- Forward scattering limit: $\Theta \to 0$, $k_d \to 0$

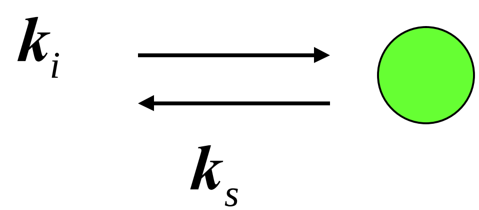


$$S\left(\hat{\boldsymbol{k}}_i, \hat{\boldsymbol{k}}_i\right) = S_{\mathrm{f}} = \frac{1}{3} R^3 k^2 \left(\epsilon_r - 1\right) = \frac{V_s}{4\pi} k^2 \left(\epsilon_r - 1\right)$$

$$V_s = \frac{4\pi R^3}{3}$$

$$\left| f\left(\hat{\boldsymbol{k}}_i, \hat{\boldsymbol{k}}_i\right)\right| = \left| S\left(\hat{\boldsymbol{k}}_i, \hat{\boldsymbol{k}}_i\right)\right|$$

- Backward scattering limit: $\Theta \to \pi$, $k_d \to 2k$

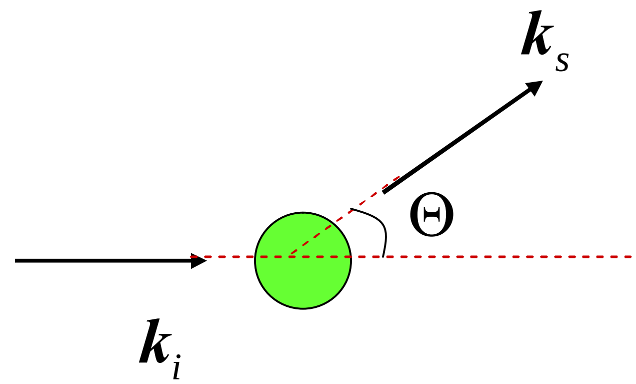


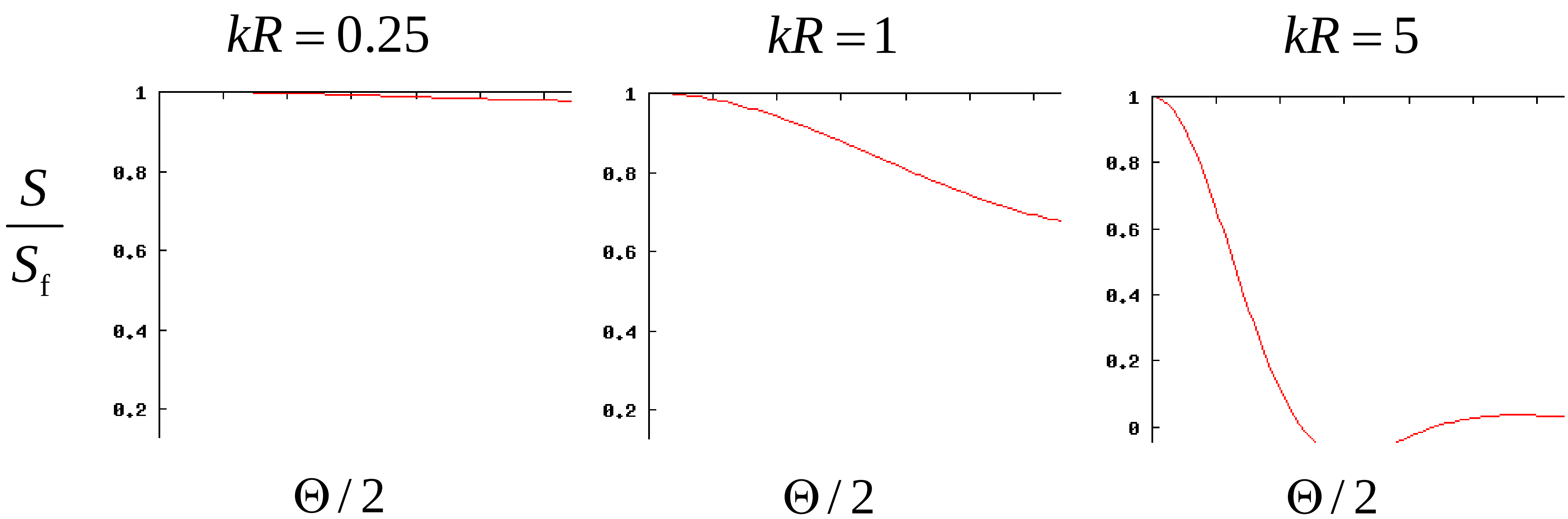
$$S\left(-\hat{\boldsymbol{k}}_i, \hat{\boldsymbol{k}}_i\right) = S_{\mathrm{b}} = \frac{\left(\epsilon_r - 1\right)}{8k}\left[\sin\left(2kR\right) - 2kR\cos\left(2kR\right)\right]$$

$$\left| f\left(-\hat{\boldsymbol{k}}_i, \hat{\boldsymbol{k}}_i\right)\right| = \left| S\left(-\hat{\boldsymbol{k}}_i, \hat{\boldsymbol{k}}_i\right)\right|$$

Born approximation

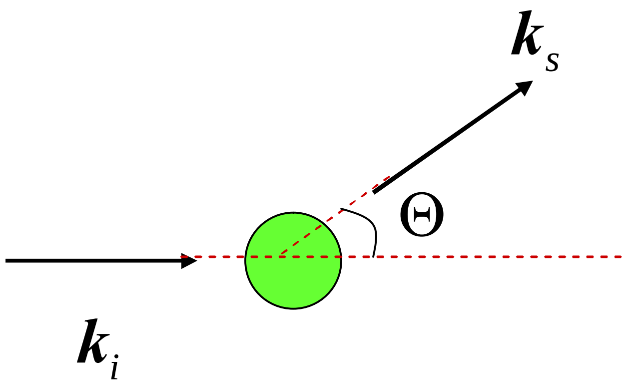
- As function of scattering angle

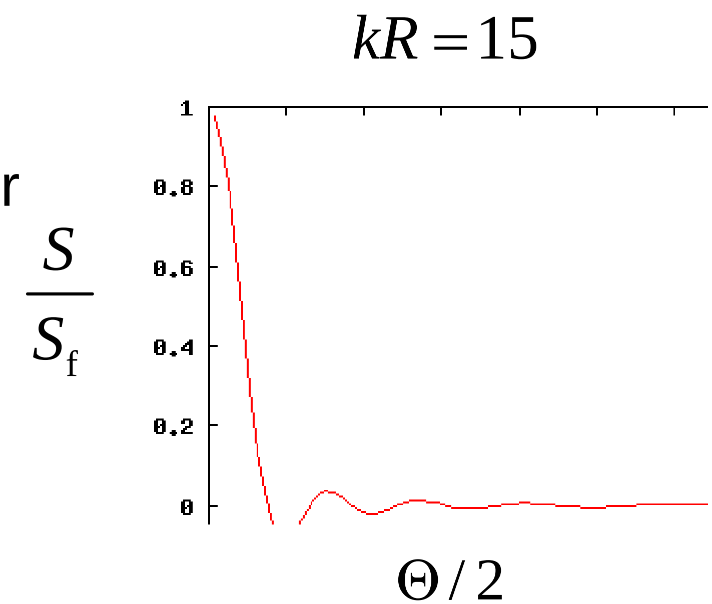




Born approximation

- Spheres small compared to wavelength → small change with angle
- Large spheres show oscillatory behavior with decreasing envelop
- In all cases the maximum S is in forward direction $\Theta \to 0$ (different for different radii)



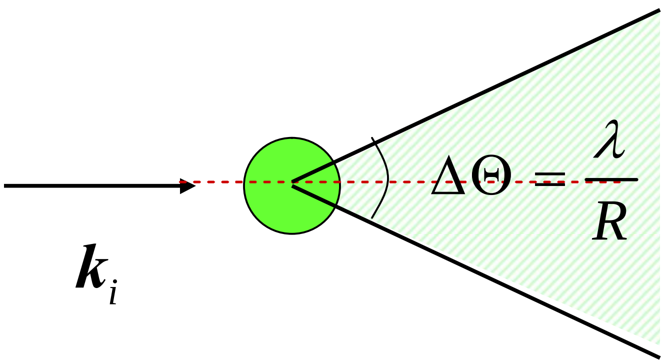


$$S_f = \frac{V_s}{4\pi} k^2 \left(\epsilon_r - 1\right)$$

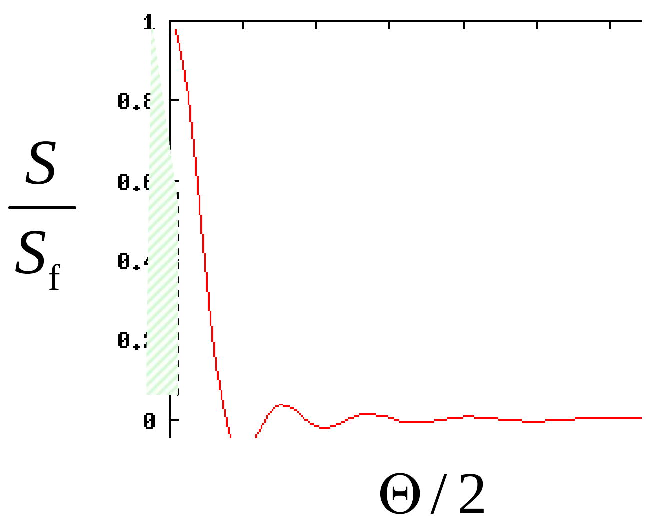
Born approximation

- If the sphere is large compared to wavelength (kR>>1) then S is largest in the forward direction within the range given by

$$-\pi < k_d R < \pi \rightarrow |\Theta| < \frac{\pi}{kR} = \frac{\lambda}{2R}$$



$kR = 15$



Born approximation

- Note: in all these discussions, the scattering amplitude f follows from S by including

$$\frac{1}{\left|\boldsymbol{E}_{i,0}\right|}\left|\boldsymbol{E}_{i,0}-\left(\boldsymbol{E}_{i,0}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right|=\left|\sin\vartheta\right| \quad \vartheta\text{: angle between } \hat{\boldsymbol{k}}_s \text{ and } \boldsymbol{E}_i$$

- Forward, backward scattering: we look in a direction along incident wave ($\hat{\boldsymbol{k}}_s=\pm\hat{\boldsymbol{k}}_i$) so that the incident polarization is normal to $\hat{\boldsymbol{k}}_s$ and f and S are the same

- In other directions this is not the case, and we have to include the relative angle with respect to incident polarization to find the true scattering strength

Born approximation

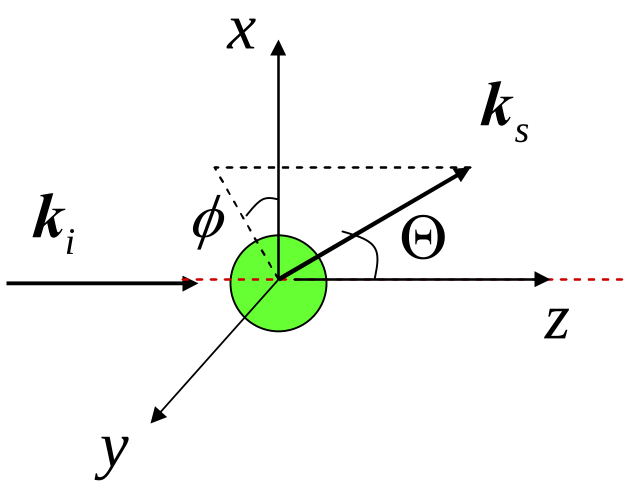
- Let us consider some scattering parameters of a dielectric sphere in Born approximation
- Differential scattering cross section

$$\sigma_d\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)=\left|f\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)\right|^2=\sin^2\vartheta\left|S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)\right|^2$$

- Total scattering cross section

$$\sigma_s=\int_{4\pi}\sigma_d\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)d\Omega_s$$

$$=\int_0^{2\pi}\int_0^{\pi}\sigma_d\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)\sin\Theta d\Theta d\phi$$



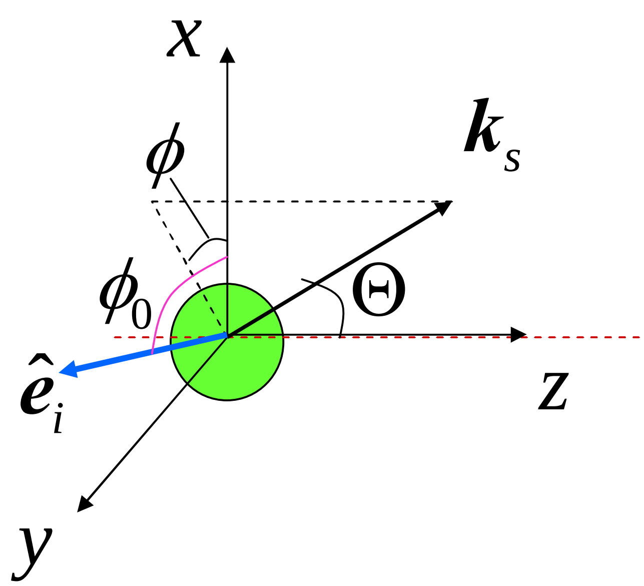
Born approximation

$$\sigma_s = \int_0^{2\pi}\int_0^{\pi} \sin^2\vartheta \left|S(\Theta)\right|^2 \sin\Theta d\Theta d\phi$$

- Since incident wave is along z-direction, take its polarization to be on the x-y plane making an angle ϕ_0 with x-axis
- Then

$$\sin^2\vartheta = 1 - \left(\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{k}}_s\right)^2$$

$$= 1 - \sin^2\Theta\cos^2\left(\phi - \phi_0\right)$$



Born approximation

$$\sigma_s = \pi \int_0^\pi \left| S(\Theta) \right|^2 \left(1+\cos^2\Theta\right) \sin\Theta d\Theta$$

$$S(\Theta) = \frac{k^2 \delta\varepsilon_p}{\epsilon_0 \left(2k\sin\frac{\Theta}{2}\right)^3} \left[\sin\left(2kR\sin\frac{\Theta}{2}\right) - \left(2kR\sin\frac{\Theta}{2}\right)\cos\left(2kR\sin\frac{\Theta}{2}\right) \right]$$

- For a small sphere

$$kR \ll 1 \rightarrow k_d R \ll 1 \rightarrow S(\Theta) \sim \frac{k^2 \delta\varepsilon_p R^3}{3\epsilon_0}$$

Born approximation

$$\sigma_s = \pi \left(\frac{k^2 \delta\varepsilon_p R^3}{3\epsilon_0} \right)^2 \int_0^{\pi} \left(1+\cos^2\Theta\right) \sin\Theta d\Theta$$

$$= \pi \left(\frac{k^2 \delta\varepsilon_p R^3}{3\epsilon_0} \right)^2 \frac{8}{3} = \frac{2}{9} k^4 R^3 \left(\frac{\delta\varepsilon_p}{\epsilon_0} \right)^2 V_s$$

Born approximation

- Why is the sky blue and the sunset red?
- Because the scattering cross section is proportional to the 4th order of frequency (*k*). Blue light is scattered more than red by small particles in the atmosphere (dilute gas)
- Light received from direction other than that of the incident beam reaches us by scattering. Thus is shifted to blue.
- Direct light tends to be red as it is the light which did not undergo scattering processes in the atmosphere.

Born approximation*

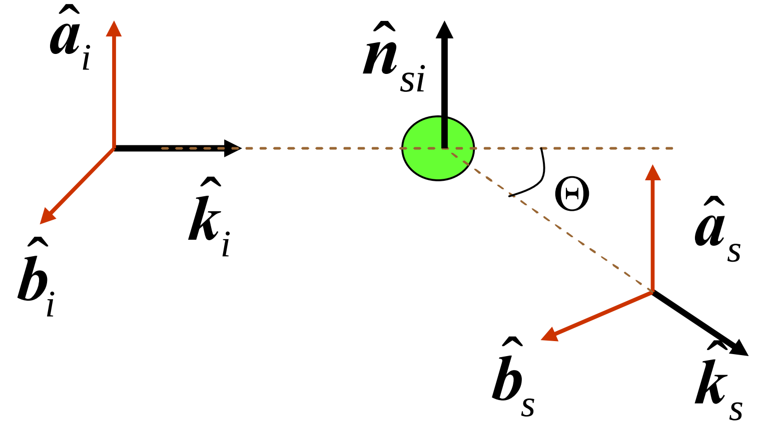
- Scattering amplitude matrix (in the scattering plane)

$$\boldsymbol{E}_s^f(\boldsymbol{r}) = \frac{\exp(-jkr)}{r}\left[\boldsymbol{E}_{i,0} - \left(\boldsymbol{E}_{i,0}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right] S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)$$

$$\boldsymbol{E}_i = \left(E_i^a \hat{\boldsymbol{a}}_i + E_i^b \hat{\boldsymbol{b}}_i\right)\exp\left(-jk\hat{\boldsymbol{k}}_i\cdot\boldsymbol{r}\right)$$

$$\boldsymbol{E}_s^f = \left(E_s^{f,a}\hat{\boldsymbol{a}}_s + E_s^{f,b}\hat{\boldsymbol{b}}_s\right)\frac{\exp(-jkr)}{r}$$

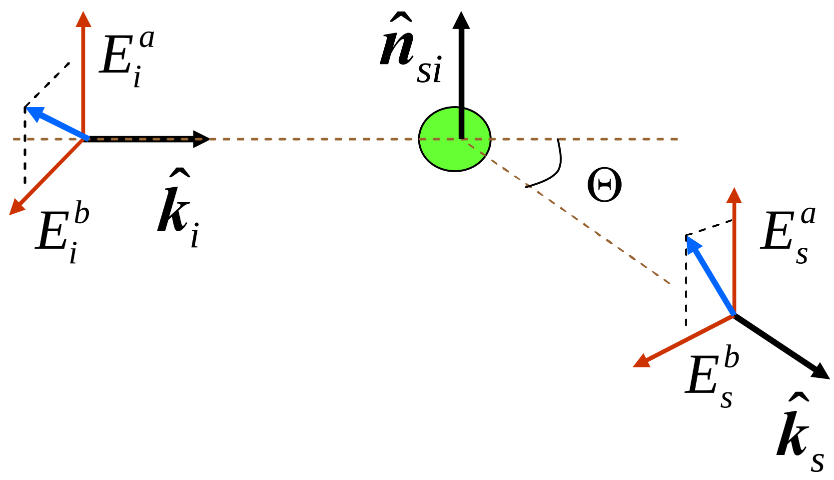
$$\begin{bmatrix} f_{aa} & f_{ab} \\ f_{ba} & f_{bb} \end{bmatrix} = S\begin{bmatrix} 1 & 0 \\ 0 & \cos\Theta \end{bmatrix}$$



Born approximation*

- Note that
$$\begin{pmatrix} E_s^a \\ E_s^b \end{pmatrix} = S \begin{bmatrix} 1 & 0 \\ 0 & \cos\Theta \end{bmatrix} \begin{pmatrix} E_i^a \\ E_i^b \end{pmatrix}$$

- Component normal to scattering plane (depends on observation direction) gets multiplied by *S.* The other component first projected on the unit axis of the incident wave.



Born approximation

- So, if Born approximation applies, it seems that if we can measure the scattering amplitude, we can reconstruct the permittivity profile by an inverse Fourier transform:

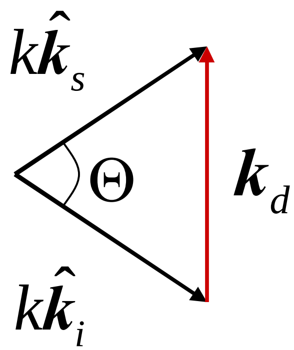
$$\delta\varepsilon_p(\boldsymbol{r}) = \frac{4\pi\epsilon_0}{k^2}\int S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)\exp\left(-j\boldsymbol{k}_d\cdot\boldsymbol{r}\right)\frac{d\boldsymbol{k}_d}{(2\pi)^3}$$

- But the range of observation in k-space is limited:

$$\boldsymbol{k}_d = k\left(\hat{\boldsymbol{k}}_s - \hat{\boldsymbol{k}}_i\right)$$

$$\left|\boldsymbol{k}_d\right| = 2k\sin\frac{\Theta}{2}$$

$$0 < \left|\boldsymbol{k}_d\right| < 2k$$



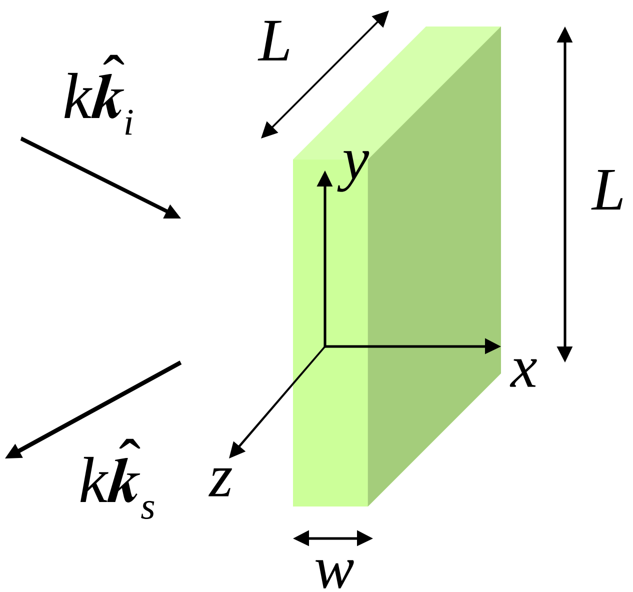
Born approximation

- Let us be more specific by considering an example. We had

$$S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) = \frac{k^2}{4\pi\epsilon_0}\int_V \exp\left(j\boldsymbol{k}_d \cdot \boldsymbol{r}\right)\delta\varepsilon_p(\boldsymbol{r})dV$$

- Let us consider a dielectric slab with an internal dielectric constant only depending on x,

$$\delta\varepsilon_p(\boldsymbol{r}) = \delta\varepsilon_p(x)$$



Born approximation

- Then

$$S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)=\frac{k^2}{4\pi\epsilon_0}\int_{-w/2}^{w/2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\exp\left(j\boldsymbol{k}_d\cdot\boldsymbol{r}\right)\delta\varepsilon_p(x)dx\,dy\,dz$$

$$=\frac{k^2}{\pi\epsilon_0}\frac{\sin\left(k_{d,y}L/2\right)\sin\left(k_{d,z}L/2\right)}{k_{d,y}k_{d,z}}\int_{-w/2}^{w/2}\exp\left(jk_{d,x}x\right)\delta\varepsilon_p(x)dx$$

- Hence, we can extract the following function from measurement

$$I\left(k_{d,x}\right)=\int_{-w/2}^{w/2}\exp\left(jk_{d,x}x\right)\delta\varepsilon_p(x)dx$$

$$k_{d,x}=k\left(\hat{\boldsymbol{k}}_s-\hat{\boldsymbol{k}}_i\right)\cdot\hat{\boldsymbol{x}}\rightarrow -2k<k_{d,x}<2k$$

Born approximation

- We apply the inverse Fourier transform to this function, neglecting the values outside the range of observation:

$$e(x) = \frac{1}{2\pi} \int_{-2k}^{2k} I\left(k_{d,x}\right) \exp\left(-jk_{d,x} x\right) dk_{d,x}$$

$$= \int_{-w/2}^{w/2} \Lambda\left(x - x'\right) \delta\varepsilon_p (x') dx'$$

$$\Lambda\left(x - x'\right) = \frac{1}{2\pi} \int_{-2k}^{2k} \exp\left[-jk_{d,x}\left(x - x'\right)\right] dk_{d,x}$$

$$= \frac{\sin\left[2k\left(x - x'\right)\right]}{\pi\left(x - x'\right)}$$

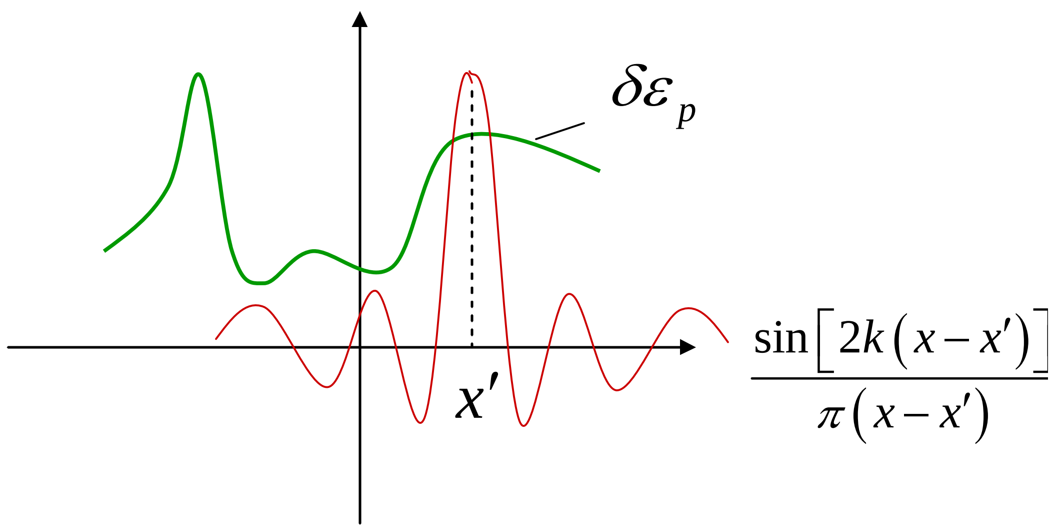
Born approximation

- The reconstructed function e(x) is not the same as δε. It is a smoothened version averaged over the effective width of the sinc function

$$e(x) = \int_{-w/2}^{w/2} \frac{\sin\left[2k\left(x - x'\right)\right]}{\pi\left(x - x'\right)} \delta\varepsilon_p(x')dx'$$

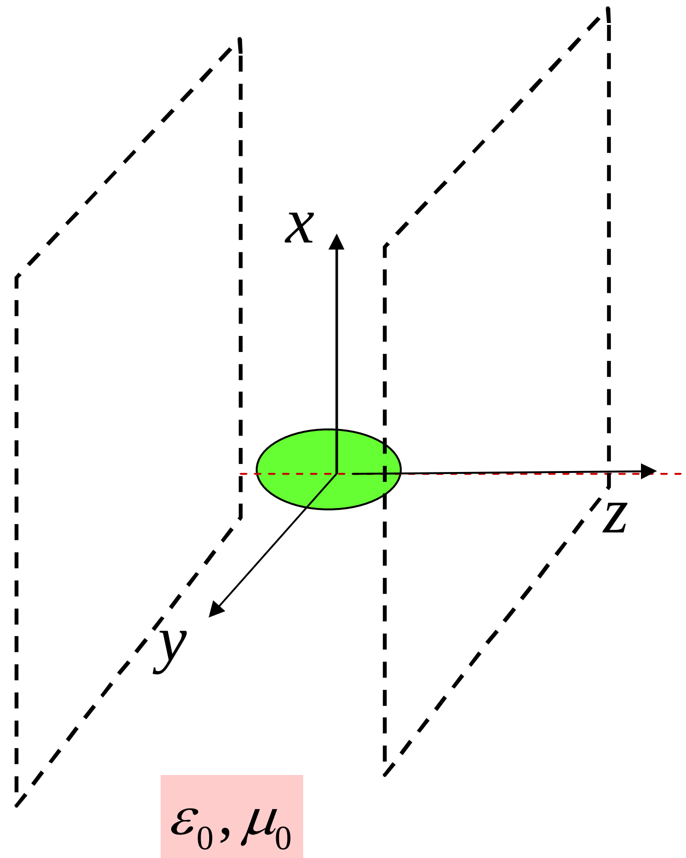
- Width of the sinc function

$$\Delta x \sim \frac{\pi}{2k} = \frac{\lambda}{4}$$



Relation with Fourier transform

- Remember that $$\boldsymbol{A}(\boldsymbol{r}) = \mu_0 \int_V G_0(\boldsymbol{r} - \boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}')dV'$$
- Assume that these currents are inside the object. Consider two infinite planes on the two sides of the object



Relation with Fourier transform

- Let us apply a Fourier transform on a plane

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{A}(\boldsymbol{r})\exp\left(jk_x x + jk_y y\right)dxdy =$$

$$\mu_0 \int_V \tilde{G}_0\left(z - z' \mid k_x, k_y\right)\exp\left(jk_x x' + jk_y y'\right)\boldsymbol{J}\left(\boldsymbol{r}'\right)dV'$$

$$\tilde{G}_0\left(z \mid k_x, k_y\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_0(\boldsymbol{r})\exp\left(jk_x x + jk_y y\right)dxdy$$

$$= \frac{\exp\left(-jk_z |z|\right)}{2jk_z}, \qquad k_z = \sqrt{k^2 - k_x^2 - k_y^2} \longleftarrow$$

Not necessarily real!
But should satisfy

$$\mathrm{Re}\left[k_z\right] > 0, \mathrm{Im}\left[k_z\right] < 0$$

Relation with Fourier transform

- For the plane on the right: $z > z'$

$$\tilde{\boldsymbol{A}}(z\,|\,k_x,k_y)=\frac{\mu_0\exp\left(-jk_z z\right)}{2\,jk_z}\int_V \exp\left(j\boldsymbol{k}^R\cdot\boldsymbol{r}'\right)\boldsymbol{J}\left(\boldsymbol{r}'\right)dV'$$

$$\boldsymbol{k}^R=(k_x,k_y,k_z)$$

- On the left: $z < z'$

$$\tilde{\boldsymbol{A}}(z\,|\,k_x,k_y)=\frac{\mu_0\exp\left(jk_z z\right)}{2\,jk_z}\int_V \exp\left(j\boldsymbol{k}^L\cdot\boldsymbol{r}'\right)\boldsymbol{J}\left(\boldsymbol{r}'\right)dV'$$

$$\boldsymbol{k}^L=(k_x,k_y,-k_z)$$

Relation with Fourier transform

- Inverse Fourier transform

$$\boldsymbol{A}^{R,L}(\boldsymbol{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\boldsymbol{A}}(z^{R,L} \mid k_x, k_y)\exp\left(-jk_x x - jk_y y\right)\frac{dk_x dk_y}{\left(2\pi\right)^2}$$

- Magnetic field

$$\boldsymbol{H}^{R,L}(\boldsymbol{r}) = \frac{1}{\mu_0}\nabla \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\boldsymbol{A}}(z^{R,L} \mid k_x, k_y)\exp\left(-jk_x x - jk_y y\right)\frac{dk_x dk_y}{\left(2\pi\right)^2}$$

- Fourier transform of the magnetic field

$$\rightarrow \tilde{\boldsymbol{H}}(z^{R,L} \mid k_x, k_y) = -\frac{j\boldsymbol{k}^{R,L}}{\mu_0} \times \tilde{\boldsymbol{A}}(z^{R,L} \mid k_x, k_y)$$

Relation with Fourier transform

- Similarly, Fourier transform of the electric field

$$\rightarrow \tilde{\boldsymbol{E}}(z^{R,L} \,|\, k_x, k_y) = -\frac{1}{j\omega\varepsilon_0\mu_0}\boldsymbol{k}^{R.L} \times \left[\boldsymbol{k}^{R,L} \times \tilde{\boldsymbol{A}}^{R,L}(z^{R,L} \,|\, k_x, k_y)\right]$$

$$= \frac{\exp\left(\mp jk_z z^{R,L}\right)}{2k_z\omega\varepsilon_0}\boldsymbol{k}^{R.L} \times \left[\boldsymbol{k}^{R,L} \times \int_V \exp\left(j\boldsymbol{k}^{R,L} \cdot \boldsymbol{r}'\right)\boldsymbol{J}\left(\boldsymbol{r}'\right)dV'\right]$$

Relation with Fourier transform

- Consider the case where $k_x^{\,2} + k_y^{\,2} < k^{\,2} \to \boldsymbol{k}^{\,R,L}$: real vectors

$$\boldsymbol{F}\left(\hat{\boldsymbol{k}}^{R,L}\right) = \int_V \exp\left(j\boldsymbol{k}^{R,L} \cdot \boldsymbol{r}'\right)\boldsymbol{J}\left(\boldsymbol{r}'\right) dV'$$

$$\to \tilde{\boldsymbol{E}}(z^{R,L} \,|\, k_x, k_y) = \frac{k\eta \exp\left(\mp jk_z z^{R,L}\right)}{2k_z} \hat{\boldsymbol{k}}^{R.L} \times \left[\hat{\boldsymbol{k}}^{R,L} \times \boldsymbol{F}\left(\hat{\boldsymbol{k}}^{R,L}\right)\right]$$

- Let $\hat{\boldsymbol{k}}^{R.L} = \hat{\boldsymbol{k}}_s$ and use

$$\boldsymbol{E}_s^{\,f}(\boldsymbol{r}) = -jk\eta \frac{\exp\left(-jkr\right)}{4\pi r} \hat{\boldsymbol{k}}_s \times \left[\hat{\boldsymbol{k}}_s \times \boldsymbol{F}\left(\hat{\boldsymbol{k}}_s\right)\right]$$

Relation with Fourier transform

- Hence, with $\hat{\boldsymbol{r}} = \hat{\boldsymbol{k}}_s = \hat{\boldsymbol{k}}^{R,L}$

$$\boldsymbol{E}_s^{f,R,L}(\boldsymbol{r}) = -\frac{\exp(-jkr)}{r} \frac{jk_z \exp\left(\pm jk_z z^{R,L}\right)}{2\pi} \tilde{\boldsymbol{E}}(z^{R,L} \,|\, k_{s,x}, k_{s,y})$$

- The far electric field along any direction in space is proportional to the 2D Fourier transform of the electric field on an arbitrary plane (between the object and the observer) with the Fourier components:

$$k_x = k_{s,x}, \quad k_y = k_{s,y}$$

Relation with Fourier transform

- From the definition of the scattering amplitude it follows that

$$\left| f\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \right| = \frac{|k_z|}{2\pi} \left| \tilde{\boldsymbol{E}}(z \mid k_{s,x}, k_{s,y}) \right|$$

- Also note that

$$\tilde{\boldsymbol{E}}(z^{R,L} \mid k_{s,x}, k_{s,y}) = -\frac{k\eta \exp\left(\mp jk_z z^{R,L}\right)}{2k_z} \boldsymbol{F}_\perp\left(\hat{\boldsymbol{k}}^{R,L}\right)$$

$$\boldsymbol{E}_s^{f,R,L}(\boldsymbol{r}) = jk\eta \frac{\exp\left(-jkr\right)}{4\pi r} \boldsymbol{F}_\perp\left(\hat{\boldsymbol{k}}^{R,L}\right)$$

Optical theorem

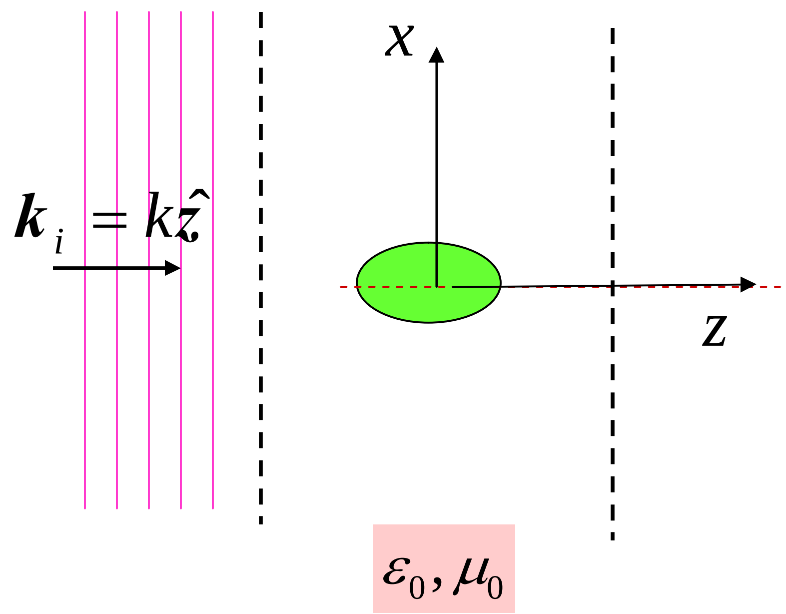
- Now, consider the scattering problem when the incident wave is along z, normal to the fictitious planes

- Total field:

$$\boldsymbol{E} = \boldsymbol{E}_i + \boldsymbol{E}_s$$

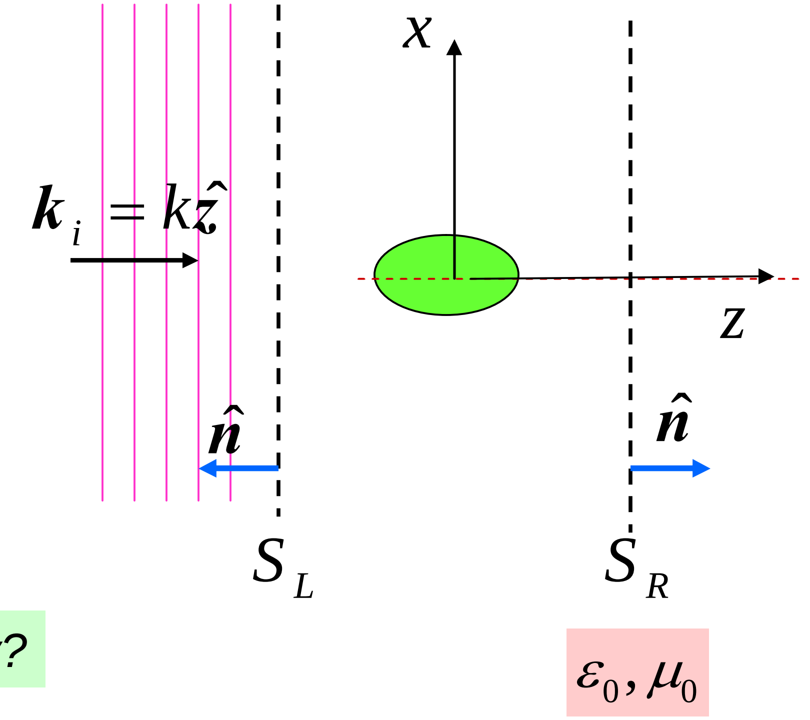
$$\boldsymbol{H} = \boldsymbol{H}_i + \boldsymbol{H}_s$$

Not necessarily far-zone, but the field generated by induced currents at any point



Optical theorem

- Poynting's theorem applied to the surface consisting of two infinite planes normal to the z-axis, and surrounding the object



$$-\frac{1}{2}\mathrm{Re}\left[\oint_{S_L+S_R}\left(\boldsymbol{E}\times\boldsymbol{H}^*\right)\cdot\hat{\boldsymbol{n}}dS\right] = P_l$$

=0 why?

$$= -\frac{1}{2}\mathrm{Re}\left[\oint_{S_L+S_R}\cancel{\left(\boldsymbol{E}_i\times\boldsymbol{H}_i^*\right)\cdot\hat{\boldsymbol{n}}dS}\right] - \frac{1}{2}\mathrm{Re}\left[\oint_{S_L+S_R}\left(\boldsymbol{E}_s\times\boldsymbol{H}_s^*\right)\cdot\hat{\boldsymbol{n}}dS\right]$$

$$-\frac{1}{2}\mathrm{Re}\left[\oint_{S_L+S_R}\left(\boldsymbol{E}_i\times\boldsymbol{H}_s^*\right)\cdot\hat{\boldsymbol{n}}dS\right] - \frac{1}{2}\mathrm{Re}\left[\oint_{S_L+S_R}\left(\boldsymbol{E}_s\times\boldsymbol{H}_i^*\right)\cdot\hat{\boldsymbol{n}}dS\right]$$

Optical theorem

- Result:

Scattered power

Dissipated power

$$\frac{1}{2}\mathrm{Re}\left[\oint_{S_L+S_R}\left(\boldsymbol{E}_s\times\boldsymbol{H}_s^*\right)\cdot\hat{\boldsymbol{n}}dS\right]+P_l=$$

$$-\frac{1}{2}\mathrm{Re}\left[\oint_{S_L+S_R}\left(\boldsymbol{E}_i\times\boldsymbol{H}_s^*\right)\cdot\hat{\boldsymbol{n}}dS\right]-\frac{1}{2}\mathrm{Re}\left[\oint_{S_L+S_R}\left(\boldsymbol{E}_s\times\boldsymbol{H}_i^*\right)\cdot\hat{\boldsymbol{n}}dS\right]$$

Optical theorem

- Using the definition of a Fourier transform:

$$\oint_{S_L+S_R}\left(\boldsymbol{E}_s\times\boldsymbol{H}_i^*\right)\cdot\hat{\boldsymbol{n}}dS=\hat{\boldsymbol{z}}\cdot\oint_{S_R}\exp\left(jkz^R\right)\left(\boldsymbol{E}_s\times\boldsymbol{H}_{i,0}^*\right)dS$$

$$-\hat{\boldsymbol{z}}\cdot\oint_{S_L}\exp\left(jkz^L\right)\left(\boldsymbol{E}_s\times\boldsymbol{H}_{i,0}^*\right)dS$$

$$=\exp\left(jkz^R\right)\hat{\boldsymbol{z}}\cdot\tilde{\boldsymbol{E}}_s\left(z^R;0,0\right)\times\boldsymbol{H}_{i,0}^*-\exp\left(jkz^L\right)\hat{\boldsymbol{z}}\cdot\tilde{\boldsymbol{E}}_s\left(z^L;0,0\right)\times\boldsymbol{H}_{i,0}^*$$

$$=\left(\boldsymbol{H}_{i,0}^*\times\hat{\boldsymbol{z}}\right)\cdot\left[\exp\left(jkz^R\right)\tilde{\boldsymbol{E}}_s\left(z^R;0,0\right)-\exp\left(jkz^L\right)\tilde{\boldsymbol{E}}_s\left(z^L;0,0\right)\right]$$

$$=\frac{1}{\eta}\boldsymbol{E}_{i,0}^*\cdot\left[\exp\left(jkz^R\right)\tilde{\boldsymbol{E}}_s\left(z^R;0,0\right)-\exp\left(jkz^L\right)\tilde{\boldsymbol{E}}_s\left(z^L;0,0\right)\right]$$

$$=\frac{1}{\eta}\boldsymbol{E}_{i,0}^*\cdot\left[-\frac{k\eta}{2k_z}\boldsymbol{F}_\perp\left(\hat{\boldsymbol{z}}\right)-\exp\left(jkz^L\right)\tilde{\boldsymbol{E}}_s\left(z^L;0,0\right)\right]$$

$$\boxed{k_z=0\text{ since }k_x=k_y=0}$$

Optical theorem

- Using the definition of a Fourier transform:

$$\oint_{S_L+S_R}\left(\boldsymbol{E}_i\times\boldsymbol{H}_s^*\right)\cdot\hat{\boldsymbol{n}}dS=\hat{\boldsymbol{z}}\cdot\oint_{S_R}\exp\left(-jkz^R\right)\left(\boldsymbol{E}_{i,0}\times\boldsymbol{H}_s^*\right)dS$$

$$-\hat{\boldsymbol{z}}\cdot\oint_{S_L}\exp\left(-jkz^L\right)\left(\boldsymbol{E}_{i,0}\times\boldsymbol{H}_s^*\right)dS$$

$$=\exp\left(-jkz^R\right)\hat{\boldsymbol{z}}\cdot\boldsymbol{E}_{i,0}\times\tilde{\boldsymbol{H}}_s^*\left(z^R;0,0\right)-\exp\left(-jkz^L\right)\hat{\boldsymbol{z}}\cdot\boldsymbol{E}_{i,0}\times\tilde{\boldsymbol{H}}_s^*\left(z^L;0,0\right)$$

$$=\boldsymbol{E}_{i,0}\cdot\left[\exp\left(-jkz^R\right)\tilde{\boldsymbol{H}}_s^*\left(z^R;0,0\right)\times\hat{\boldsymbol{z}}-\exp\left(-jkz^L\right)\tilde{\boldsymbol{H}}_s^*\left(z^L;0,0\right)\times\hat{\boldsymbol{z}}\right]$$

$$=\frac{1}{\eta}\boldsymbol{E}_{i,0}\cdot\left[\exp\left(-jkz^R\right)\tilde{\boldsymbol{E}}_s^*\left(z^R;0,0\right)+\exp\left(-jkz^L\right)\tilde{\boldsymbol{E}}_s^*\left(z^L;0,0\right)\right]$$

$$=\frac{1}{\eta}\boldsymbol{E}_{i,0}\cdot\left[-\frac{k\eta}{2k_z}\boldsymbol{F}_\perp^*\left(\hat{\boldsymbol{z}}\right)+\exp\left(-jkz^L\right)\tilde{\boldsymbol{E}}_s^*\left(z^L;0,0\right)\right]$$

Plus sign because it is the returning wave

Optical theorem

- Collecting the results:

$$P_s + P_l = -\frac{1}{2}\mathrm{Re}\left[\oint_{S_L+S_R}\left(\boldsymbol{E}_i \times \boldsymbol{H}_s^*\right)\cdot\hat{\boldsymbol{n}}dS\right] - \frac{1}{2}\mathrm{Re}\left[\oint_{S_L+S_R}\left(\boldsymbol{E}_s \times \boldsymbol{H}_i^*\right)\cdot\hat{\boldsymbol{n}}dS\right]$$

$$= -\frac{1}{2\eta}\mathrm{Re}\left\{\boldsymbol{E}_{i,0}^*\cdot\left[-\frac{\eta}{2}\boldsymbol{F}_\perp(\hat{\boldsymbol{z}}) - \exp\left(jkz^L\right)\tilde{\boldsymbol{E}}_s\left(z^L;0,0\right)\right]\right.$$

$$\left. + \boldsymbol{E}_{i,0}\cdot\left[-\frac{\eta}{2}\boldsymbol{F}_\perp^*(\hat{\boldsymbol{z}}) + \exp\left(-jkz^L\right)\tilde{\boldsymbol{E}}_s^*\left(z^L;0,0\right)\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left[\boldsymbol{E}_{i,0}^*\cdot\boldsymbol{F}_\perp\left(\hat{\boldsymbol{z}}\right)\right]$$

Optical theorem

- Conclusion: the total scattered plus absorbed power is related to the forward scattering amplitude because if we look at the scattered field along the incident field

$$\boldsymbol{E}_s^{f,R}(\boldsymbol{r}) = jk\eta \frac{\exp(-jkr)}{4\pi r} \boldsymbol{F}_\perp(\hat{z})$$

- In terms of scattering cross section:

Total cross section

$$\sigma_t = \sigma_s + \frac{P_l}{|S_i|} = \frac{1}{2|S_i|} \mathrm{Re}\left[\boldsymbol{E}_{i,0}^* \cdot \boldsymbol{F}_\perp(\hat{\boldsymbol{z}})\right]$$

Scattering cross section

Absorption cross section σ_a

Optical theorem

- Rewriting this equation:

$$\left|S_i\right| = \frac{\left|\boldsymbol{E}_{i,0}\right|^2}{2\eta} \rightarrow \sigma_t = \frac{\eta}{\left|\boldsymbol{E}_{i,0}\right|^2} \mathrm{Re}\left[\boldsymbol{E}_{i,0}^* \cdot \boldsymbol{F}_\perp\left(\hat{\boldsymbol{z}}\right)\right]$$

- Same result could have been easily obtained using the conservation of energy inside the object!