Electromagnetic scattering

Graduate Course

Electrical Engineering (Communications)

1st Semester, 1388-1389

Sharif University of Technology

Contents of lecture 3

- **Lecture 3:** Scattering from layered media
  - Introduction
  - Field equations
  - Scattering from the interface between two half-infinite media
  - Scattering matrix of the interface
  - Scattering by a dielectric slab
  - Scattering from a general layered medium
  - Scattering from a continuously changing medium

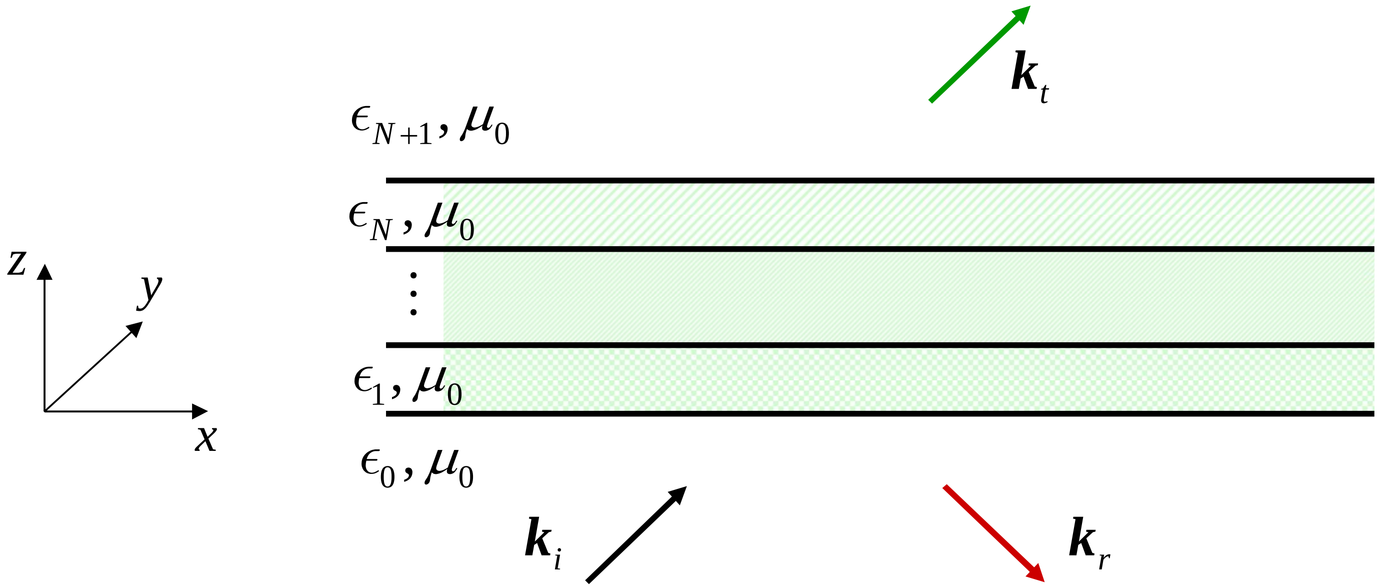
Introduction

- In the formulation of the scattering parameters we did not really explain how the 'internal' field inside an object can be calculated.
- Without this information the scattering problem cannot be solved exactly. Only approximations like Born's can be used which are often not accurate.
- General solution of this problem can be formulated, but usually needs numerical techniques.
- But some cases, even though not always very realistic, lead to exact results. We study some of these cases first.

Introduction

- The easiest case is the scattering of a plane wave by infinitely extended, layered dielectric media.
- Consider an incident wave with

$$\boldsymbol{k}_i = \left(k_{i,x}, k_{i,y}, k_{i,z}\right)$$



Introduction

- Because the system is uniform on the horizontal (x-y) plane, in all layers the horizontal components of the incident wave vector are preserved
- The incident wave is $\boldsymbol{E}_i(\boldsymbol{r}) = \boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right)$
- The reflected and transmitted waves are

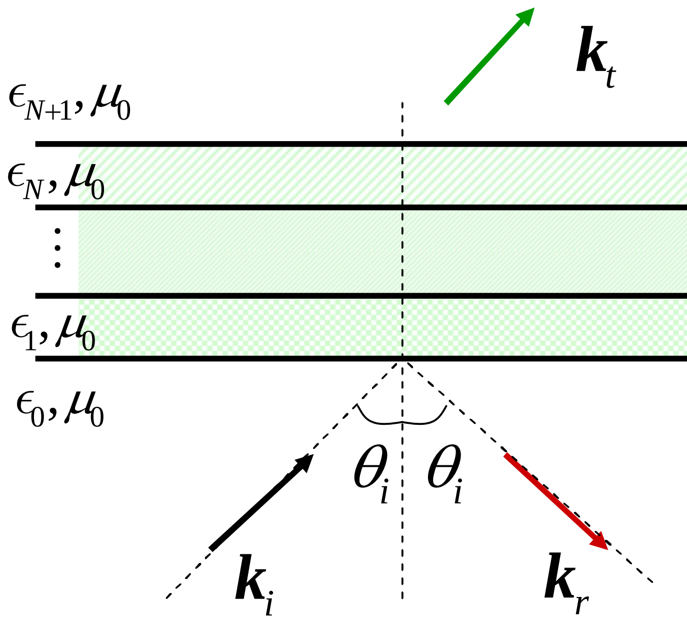
$$\boldsymbol{E}_r(\boldsymbol{r}) = \boldsymbol{E}_r^0 \exp\left(-j\boldsymbol{k}_r \cdot \boldsymbol{r}\right) \qquad \boldsymbol{E}_t(\boldsymbol{r}) = \boldsymbol{E}_t^0 \exp\left(-j\boldsymbol{k}_t \cdot \boldsymbol{r}\right)$$

$$\boldsymbol{k}_t = \boldsymbol{k}_i = \left(k_{i,x}, k_{i,y}, k_{i,z}\right) \qquad \boldsymbol{k}_r = \left(k_{i,x}, k_{i,y}, -k_{i,z}\right)$$

$$\left|\boldsymbol{k}_i\right| = \left|\boldsymbol{k}_r\right| = k_0 = \omega\sqrt{\epsilon_0 \mu_0}$$

Introduction

- Note: incident, reflected and transmitted wave vectors are in the same vertical plane
- Angle of reflection same as angle of incidence because the magnitude of the wave vector is the same for incident and reflected waves



Introduction

- In all layers the horizontal components of the incident wave vector are preserved and we can restrict ourselves to solutions of the type

$$\left\{\boldsymbol{E}(\boldsymbol{r}),\boldsymbol{H}(\boldsymbol{r})\right\}=\left\{\tilde{\boldsymbol{E}}(z),\tilde{\boldsymbol{H}}(z)\right\}\exp\left(-jk_{i,x}x-jk_{i,y}y\right)$$

- It suffices to restrict ourselves to the case where the incident wave vector has no y-component, i.e., $k_{i,y}=0$. If this is not the case, we rotate the axis of coordinates so that this condition is satisfied.

Field equations

- Maxwell equations (in each layer) decouple into

$$-\frac{d\tilde{E}_y}{dz} = -j\omega\mu_0\tilde{H}_x \qquad -\frac{d\tilde{H}_y}{dz} = j\omega\epsilon_\alpha\tilde{E}_x$$

$$-jk_{i,x}\tilde{E}_y = -j\omega\mu_0\tilde{H}_z \qquad -jk_{i,x}\tilde{H}_y = j\omega\epsilon_\alpha\tilde{E}_z$$

$$\frac{d\tilde{H}_x}{dz} + jk_{i,x}\tilde{H}_z = j\omega\epsilon_\alpha\tilde{E}_y \qquad \frac{d\tilde{E}_x}{dz} + jk_{i,x}\tilde{E}_z = -j\omega\mu_0\tilde{H}_y$$

E_y, H_x, H_z — TE scattering

H_y, E_x, E_z — TM scattering

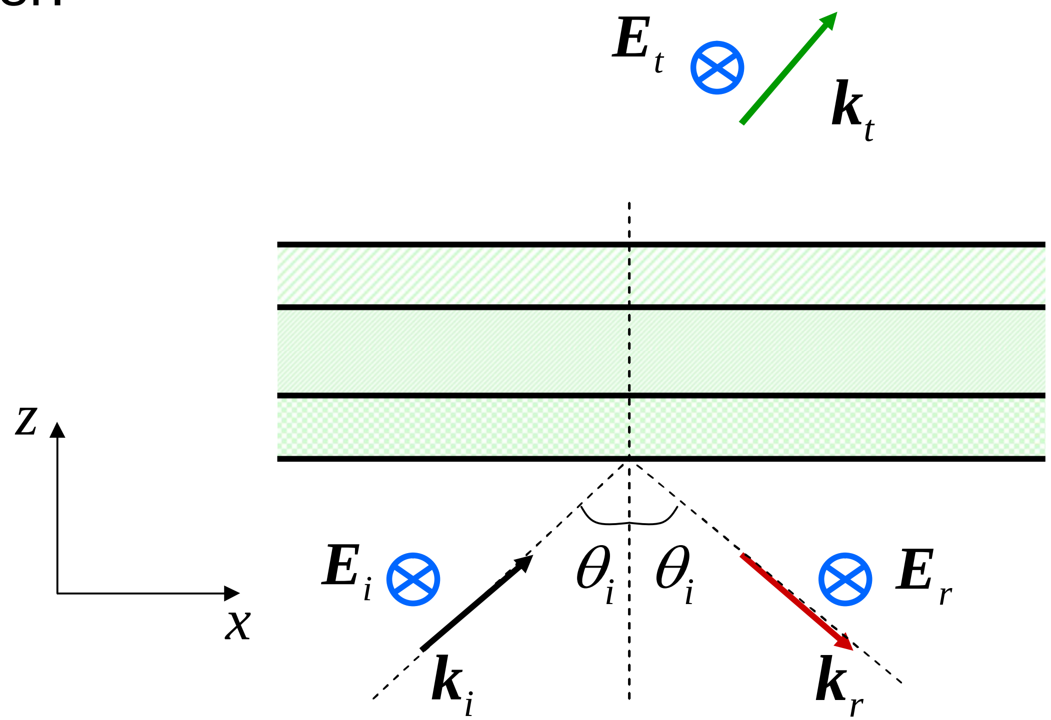
Field equations

- TE^z scattering: electric field in horizontal plane in both incident and scattered waves and in all layers
- Equation inside each layer:

$$\frac{d^2\tilde{E}_y}{dz^2} + k_{\alpha,z}^2 \tilde{E}_y = 0$$

$$k_{\alpha,z}^2 = k_\alpha^2 - k_{i,x}^2$$

$$k_\alpha^2 = \omega^2 \epsilon_\alpha \mu_0$$

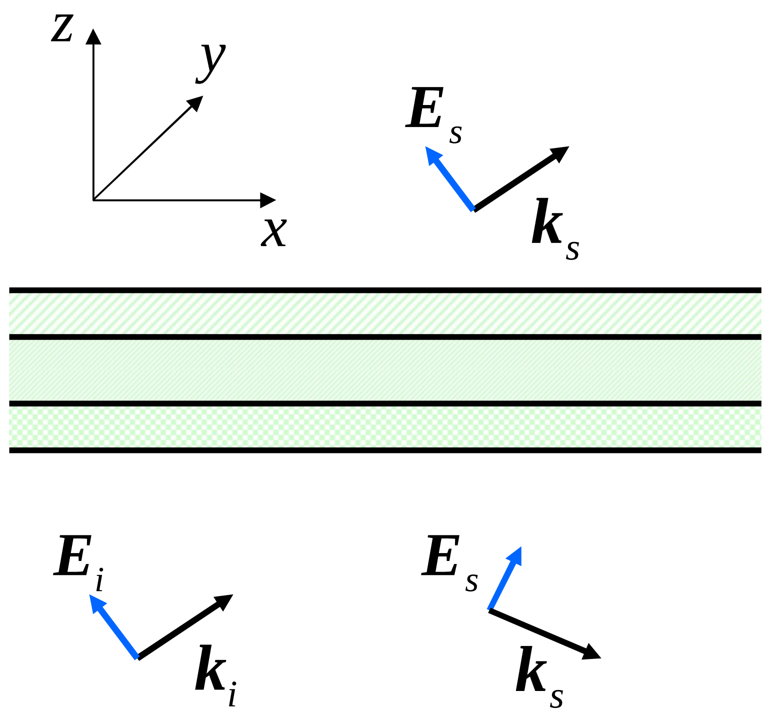


Field equations

- TM^z scattering: electric field in the plane of reflection (normal to horizontal), magnetic field in the horizontal plane
- Equation inside each layer:

$$\frac{d^2 \tilde{H}_y}{dz^2} + k_{\alpha,z}^2 \tilde{H}_y = 0$$

- Note: electric field still perpendicular to wave vector

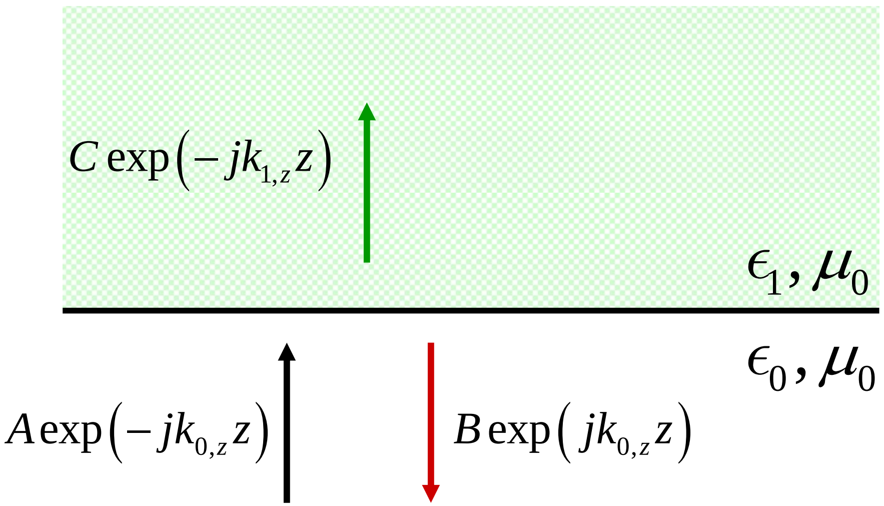


TE^z scattering at the interface between two infinite media

- Consider the TE^z scattering from the interface of an infinitely thick dielectric medium. Solution for the reduced field:

$$z<0 \rightarrow \tilde{E}_y = A\exp\left(-jk_{0,z}z\right)+B\exp\left(jk_{0,z}z\right)$$

$$z>0 \rightarrow \tilde{E}_y = C\exp\left(-jk_{1,z}z\right)$$



$$k_{0,z} = \sqrt{k_0^2 - k_{i,x}^2} = k_{i,z}$$

$$k_{1,z} = \sqrt{k_1^2 - k_{i,x}^2}$$

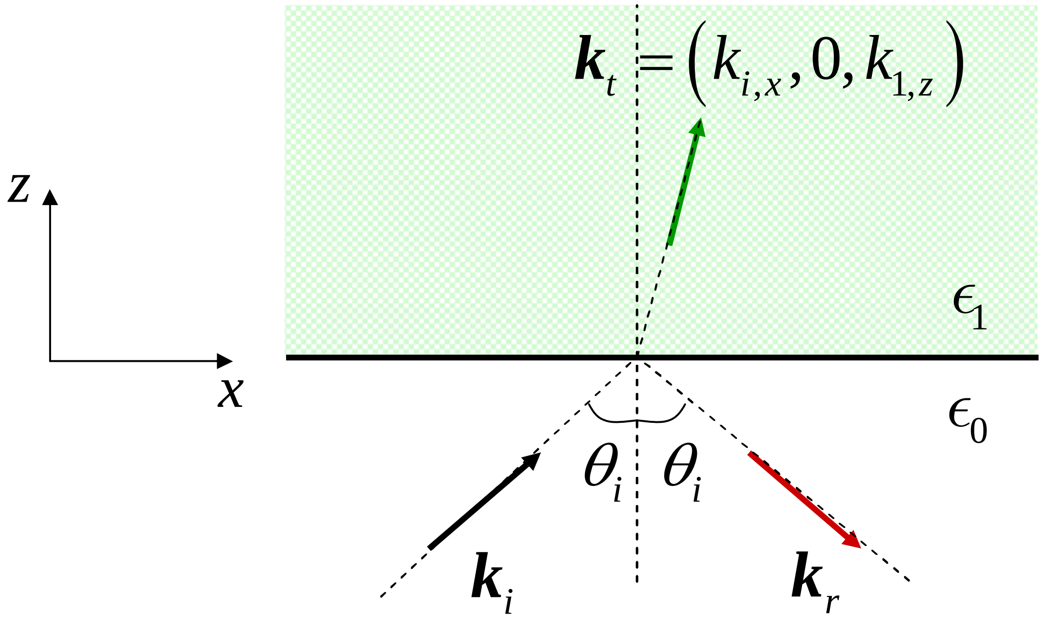
$$k_0^2 = \omega^2\epsilon_0\mu_0$$

$$k_1^2 = \omega^2\epsilon_1\mu_0$$

TE^z scattering at the interface between two infinite media

- Overall solution:

$$C\exp\left(-jk_{i,x}x - jk_{1,z}z\right)$$



$$A\exp\left(-jk_{i,x}x - jk_{0,z}z\right) \qquad B\exp\left(-jk_{i,x}x + jk_{0,z}z\right)$$

$$\boldsymbol{k}_i = \left(k_{i,x}, 0, k_{0,z}\right) \qquad \boldsymbol{k}_r = \left(k_{i,x}, 0, -k_{0,z}\right)$$

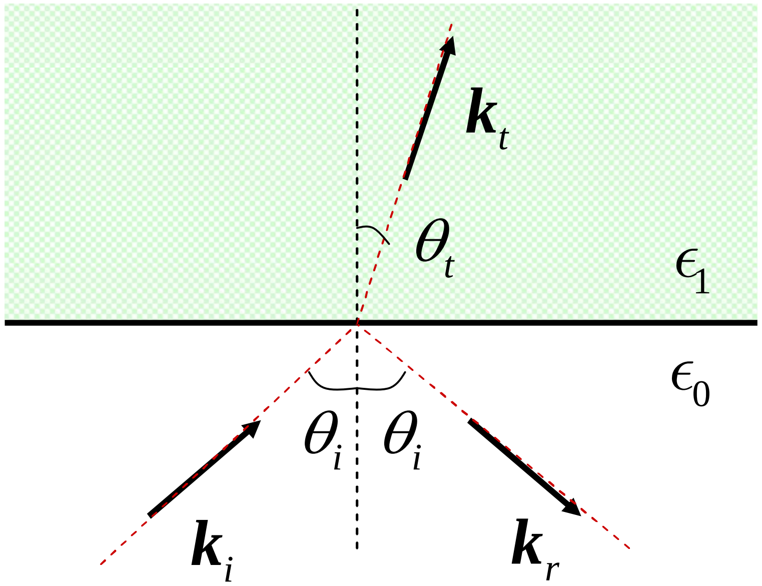
TE^z scattering at the interface between two infinite media

- Before solving the problem note that in all cases

$$k_{1,z} = \sqrt{k_1^2 - k_{i,x}^2} = \sqrt{k_1^2 - k_0^2 \sin^2\theta_i} = \sqrt{\omega^2\mu_0\left(\epsilon_1 - \epsilon_0 \sin^2\theta_i\right)}$$

- If $\epsilon_1 > \epsilon_0$ we always have propagation in top medium

- But if $\epsilon_1 < \epsilon_0$ then in for some angles of incidence no waves will propagate in top medium



TE^z scattering at the interface between two infinite media

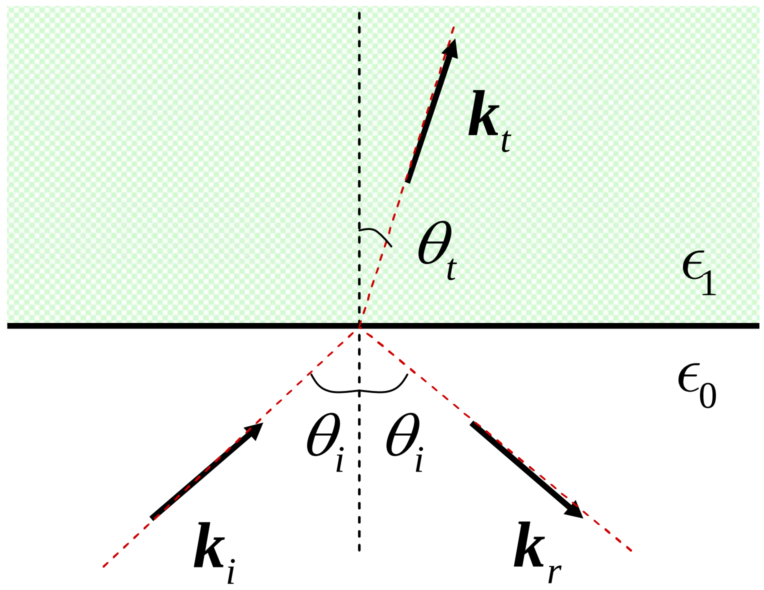
$$\theta_i > \theta_c = \arcsin\left(\sqrt{\frac{\epsilon_1}{\epsilon_0}}\right) \rightarrow k_{1,z}: \text{imaginary}$$

Critical angle

- Fields exponentially drop in top medium for angles larger than critical angle (total reflection).
- But if there is propagation, we always have

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_0}{\epsilon_1}}$$

Snell's law



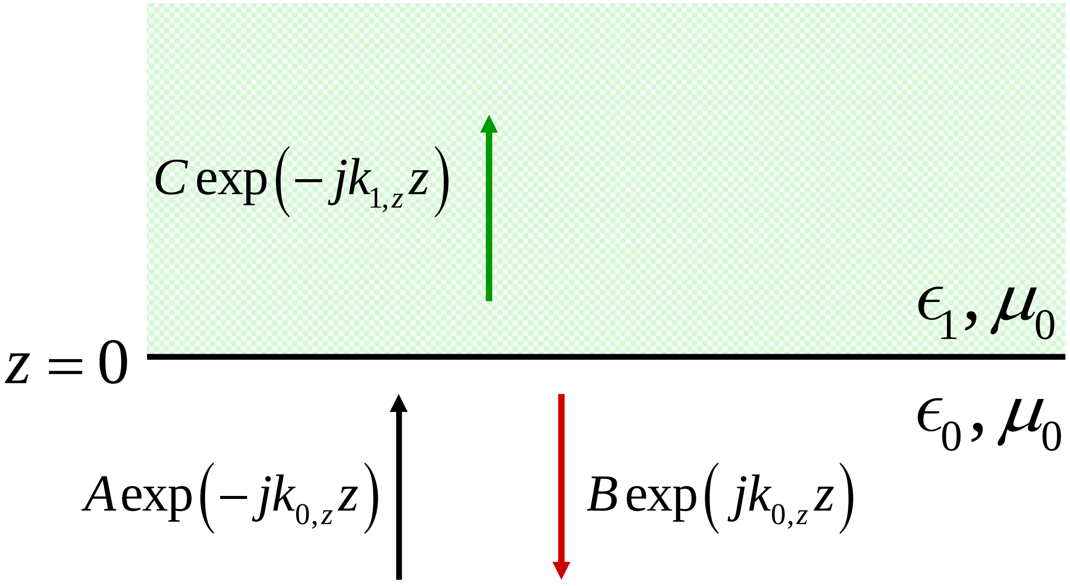
TE^z scattering at the interface between two infinite media

- Solution for electric field by matching the tangential fields at the interface leads to

$$R_{TE} = \frac{B}{A} = \frac{k_{0,z} - k_{1,z}}{k_{0,z} + k_{1,z}}$$

$$T_{TE} = \frac{C}{A} = \frac{2k_{0,z}}{k_{0,z} + k_{1,z}}$$

$$T_{TE} = 1 + R_{TE}$$



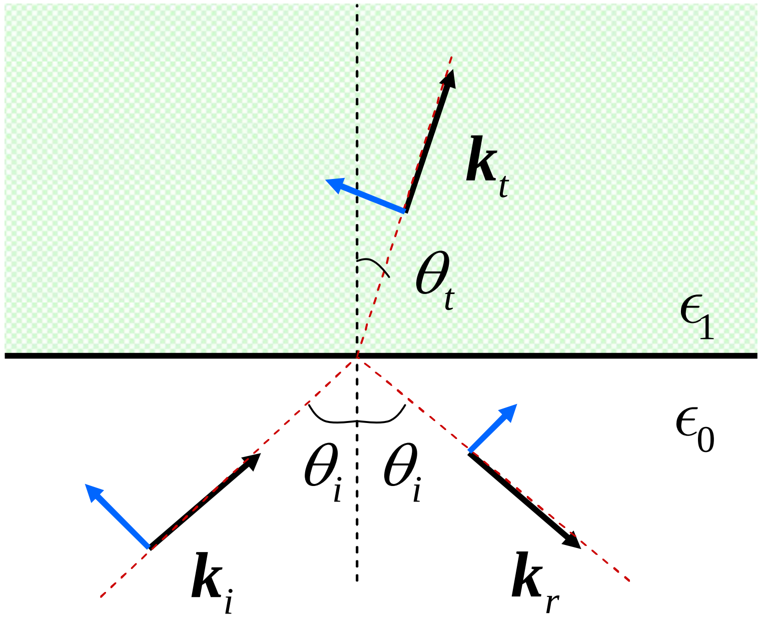
TM^z scattering at the interface between two infinite media

- Now consider the case of TM scattering

$$z<0 \rightarrow \tilde{H}_y = A\exp\left(-jk_{0,z}z\right)+B\exp\left(jk_{0,z}z\right)$$

$$z>0 \rightarrow \tilde{H}_y = C\exp\left(-jk_{1,z}z\right)$$

- Snell's low holds again, there is again a critical angle if $\epsilon_1 < \epsilon_0$



$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_0}{\epsilon_1}} \qquad \theta_c = \arcsin\left(\sqrt{\frac{\epsilon_1}{\epsilon_0}}\right)$$

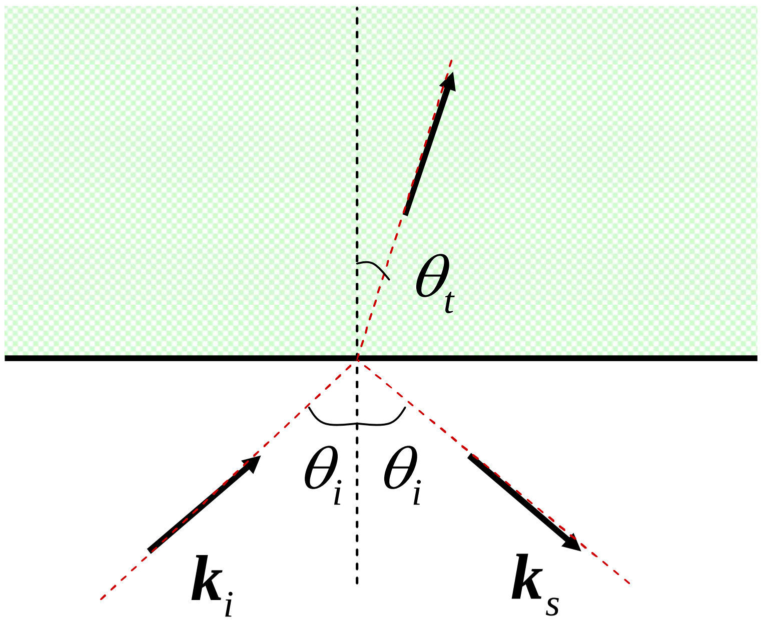
TM^z scattering at the interface between two infinite media

- Matching the fields now yields

$$R_{TM}=\frac{\epsilon_1 k_{0,z}-\epsilon_0 k_{1,z}}{\epsilon_1 k_{0,z}+\epsilon_0 k_{1,z}} \qquad T_{TM}=\frac{2\epsilon_1 k_{0,z}}{\epsilon_1 k_{0,z}+\epsilon_0 k_{1,z}} \qquad T_{TM}=1+R_{TM}$$

- Unlike TE scattering, however, there is an angle where there is no reflection (all power is transmitted)

$$\epsilon_1 k_{0,z}-\epsilon_0 k_{1,z}=0\rightarrow R_{TM}=0$$

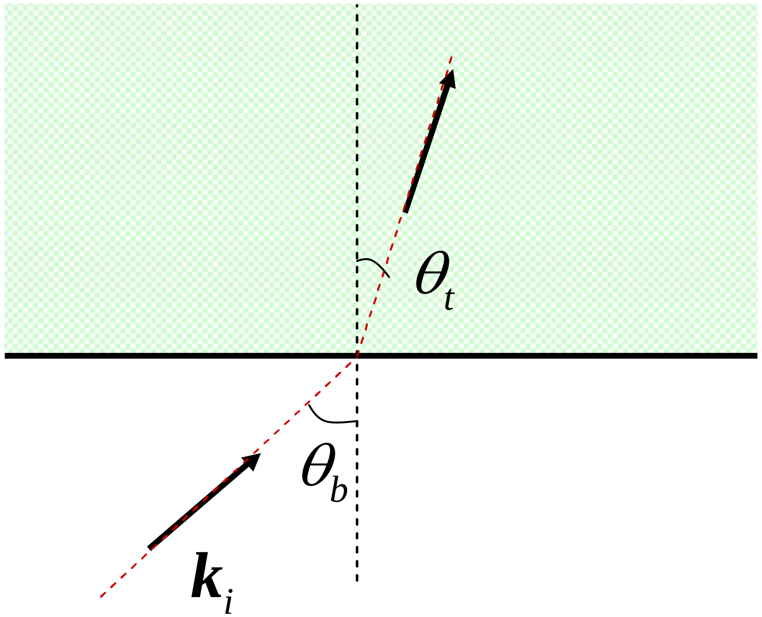


TM^z scattering at the interface between two infinite media

$$\epsilon_1 k_{0,z} - \epsilon_0 k_{1,z} = 0 \rightarrow k_{i,x}^2 = \omega^2 \mu_0 \frac{\epsilon_1 \epsilon_0}{\epsilon_1 + \epsilon_0}$$

$$\sin^2 \theta_b = \frac{\epsilon_2}{\epsilon_1 + \epsilon_2} \qquad \sin^2 \theta_t = \frac{\epsilon_1}{\epsilon_1 + \epsilon_2}$$

- This angle is called the Brewster angle. It always exist for TM scattering, even if the second medium has a lower dielectric constant because $\theta_b < \theta_c$

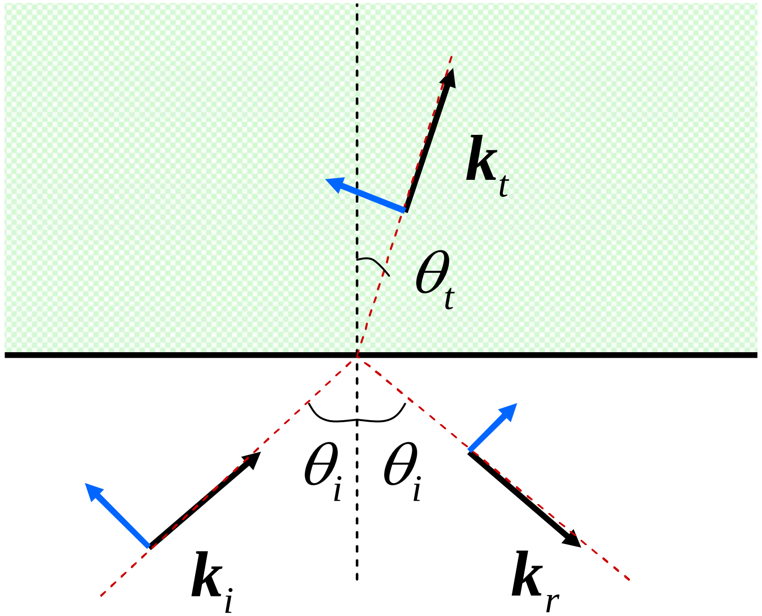


TM^z scattering at the interface between two infinite media

- Note that the TM case is formulated in terms of the in-plane component of the *magnetic field*
- The electric field follows from

$$\tilde{E}_x = -\frac{1}{j\omega\epsilon_\alpha}\frac{d\tilde{H}_y}{dz}$$

$$\tilde{E}_z = -\frac{k_x}{\omega\epsilon_\alpha}\tilde{H}_y$$

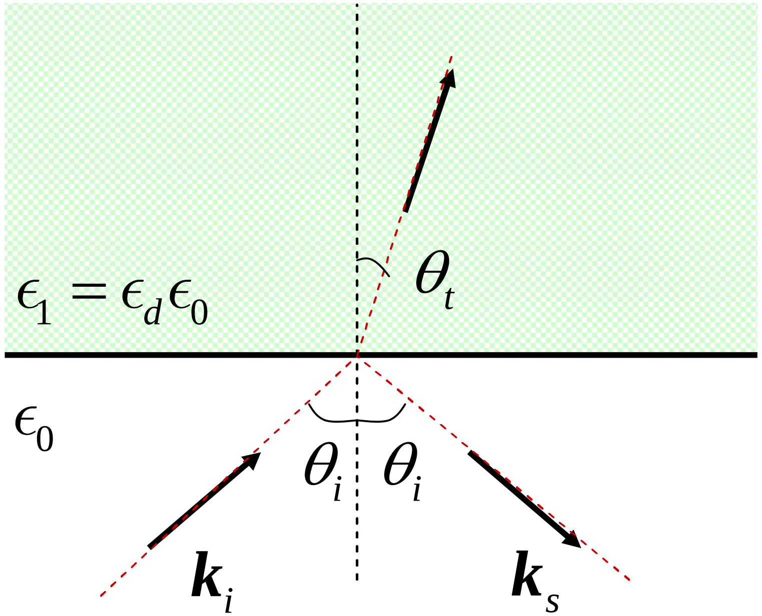


Reflection from a half space medium

- Consider a more practical situation, take bottom medium to be air and top medium to be a dielectric
- Let us study the reflection coefficient for the TE and TM cases as function of the angle of incidence

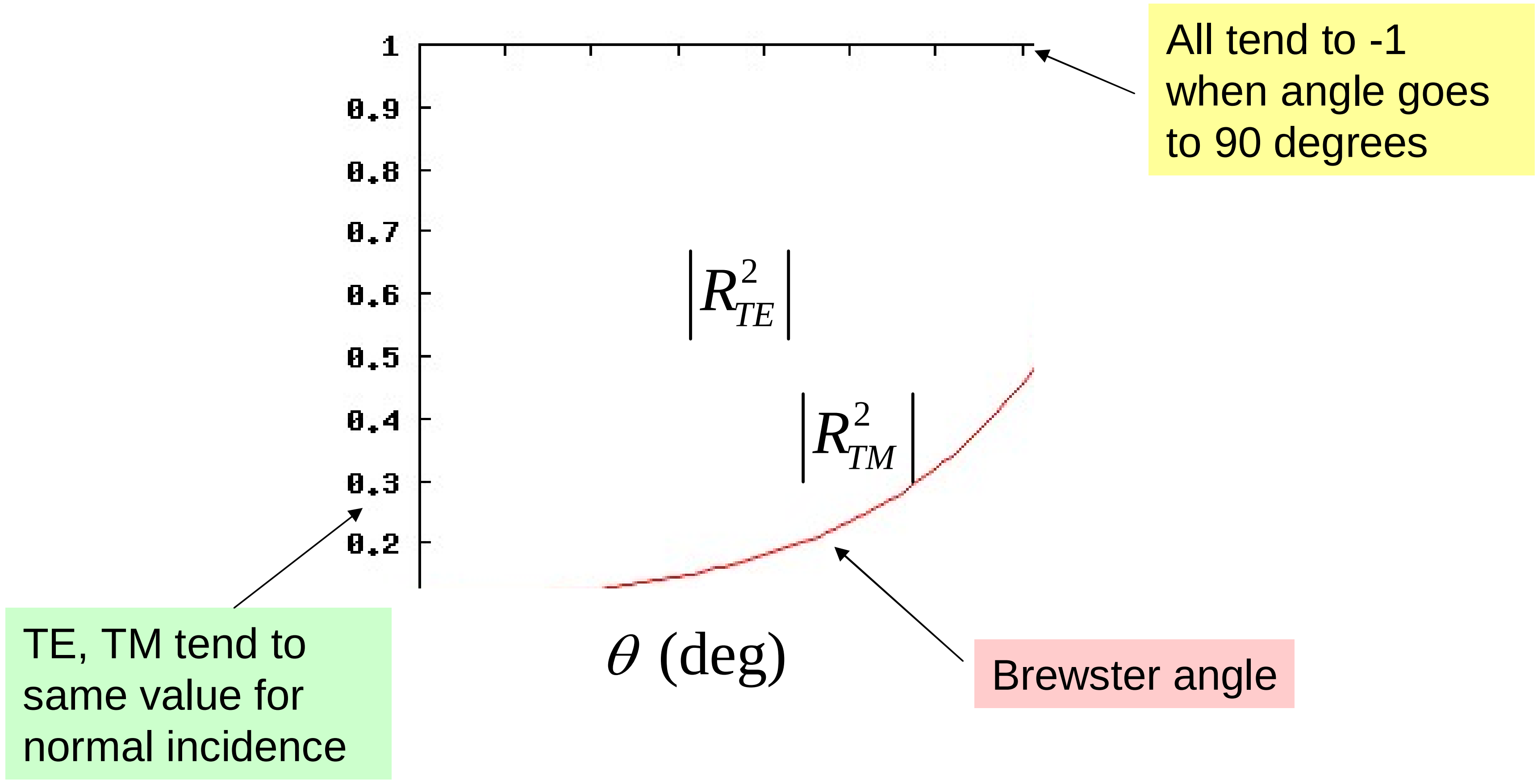
$$R_{TE} = \frac{\cos\theta_i - \sqrt{\epsilon_d - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\epsilon_d - \sin^2\theta_i}}$$

$$R_{TM} = \frac{\epsilon_d \cos\theta_i - \sqrt{\epsilon_d - \sin^2\theta_i}}{\epsilon_d \cos\theta_i + \sqrt{\epsilon_d - \sin^2\theta_i}}$$



Reflection from a half space medium

- Plots for a relative dielectric constant of 4



Reflection from a half space medium

- So far we considered the media to be ideal dielectrics. What if we have polarization loss and/or conductivity?
- Not much changes in the equations, only we make the permittivity complex. In the example shown above

$$\epsilon_d \rightarrow \epsilon_d' - j\epsilon_d'' = \epsilon_d'\left(1 - j\tan\delta\right)$$

Loss tangent