Electromagnetic scattering

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Introduction

- Scattering from flat, layered media is an example of an exactly solvable problem.
- Another class of exactly solvable systems is the scattering by cylinders of infinite length.
- In case of dielectric cylinders they are solvable for a constant dielectric constant.
Introduction

- The system has rotational symmetry but, of course, the incident plane wave is not rotationally symmetric.
- Nonetheless, we can still analyze the problem using the ‘natural’ solutions of the wave equation in cylindrical coordinates.
- First, consider *scalar* waves in a *homogeneous* medium.

$$\left( \nabla^2 + k^2 \right) \psi = 0$$

$$k^2 = \omega^2 \mu \epsilon$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0$$

In cylindrical coordinates.
Scalar waves in cylindrical coordinates

- Try solutions of the type

\[
\psi(\rho, \phi, z) = f(\rho) \exp(-jm\phi - jk_z z)
\]

\[
\rho^2 \frac{d^2 f(\rho)}{d \rho^2} + \rho \frac{df(\rho)}{d \rho} + \left(k^2 \rho^2 - m^2\right) f(\rho) = 0
\]

\[
k_\rho = \sqrt{k^2 - k_z^2}
\]

- General solution:

\[
f_{m,k_z}(\rho) = AJ_m\left(k_\rho \rho\right) + BY_m\left(k_\rho \rho\right) \Rightarrow
\]

\[
\psi_{m,k_z}(\rho, \phi, z) = \left[ AJ_m\left(k_\rho \rho\right) + BY_m\left(k_\rho \rho\right) \right] \exp(-jm\phi - jk_z z)
\]
Scalar waves in cylindrical coordinates

- Alternatively, one can represent the solution in terms of Hankel functions of the 1\textsuperscript{st} and 2\textsuperscript{nd} kind

\[ H_{m}^{(1,2)}(k_{\rho}\rho) = J_{m}(k_{\rho}\rho) \pm jY_{m}(k_{\rho}\rho) \]

\[ \psi_{m,k_{z}}(\rho, \phi, z) = \left[ A H_{m}^{(1)}(k_{r}\rho) + B H_{m}^{(2)}(k_{r}\rho) \right] \exp(-jm\phi - jk_{z}z) \]

- Hankel functions preserve the ‘wave’ picture. For large radial distances they represent inward and outward moving waves

\[ \rho \to \infty : H_{m}^{(1,2)}(k_{\rho}\rho) \sim (\mp j)^{m} \sqrt{\frac{2}{\pi k_{\rho}\rho}} \exp\left[ \pm\left( jk_{\rho}\rho - j\pi / 4 \right) \right] \]
Vector wave equation

- These were the solutions in a homogeneous medium
- At the first sight the scattering problem for a cylinder can now be easily solved: just solve the Bessel and Hankel waves outside and inside the cylinder (if dielectric) and match them at the interface
- But there are two problems:
  - We have to deal with a vector problem and a vector wave equation: EM fields are vector fields and have a polarization.
  - Incident wave is not a ‘cylindrical’ wave, but a normal plane wave
- These facts make the analysis ‘a bit’ more complicated!
Vector wave equation

- Remember vector wave equation in a homogeneous medium

\[ \nabla \times (\nabla \times E) - k^2 E = -\nabla^2 E + \nabla (\nabla \cdot E) - k^2 E = 0 \]

- Instead of trying to move to cylindrical coordinates and directly solve the equation, consider the following general theorem

- Assume that \( \psi(r) \) is a solution of the ‘scalar’ wave equation solved in any coordinate system

- Also consider a known vector field \( a(r) \), such that

\[ \nabla \times a(r) = 0 \]
Vector wave equation

- Now, let us introduce the new vector fields

\[
M(r) = \frac{1}{k} \nabla \times [\psi(r)a(r)] \\
N(r) = \frac{1}{k} \nabla \times M(r)
\]

- It can then be shown that the 1st vector field is a solution of the vector wave equation if

\[
\nabla \times \left[ \nabla \psi \nabla \cdot a - 2 (\nabla \psi \cdot \nabla) a \right] = 0
\]

- For the derivation use

\[
\nabla \times \nabla \times (\psi a) = \nabla \times (\nabla \psi \times a) = \nabla \psi \nabla \cdot a - a \nabla^2 \psi + (a \cdot \nabla) \nabla \psi - (\nabla \psi \cdot \nabla) a
\]

\[
= \nabla \psi \nabla \cdot a - a \nabla^2 \psi + \nabla (a \cdot \nabla \psi) - 2 (\nabla \psi \cdot \nabla) a
\]
Vector wave equation

- The 2\textsuperscript{nd} vector field is then automatically a solution as well.
- These two vector fields are linearly independent from each other.
- They both have zero divergence.
- Every solution of the vector wave equation in a homogeneous medium can be written as a combination of these vectors, generated by different solutions of the scalar wave equation.
- One cannot generate more functions by taking more curls. Note that

\[ \nabla \times N(r) = \frac{1}{k} \nabla \times \nabla \times M(r) = kM(r) \]
Vector wave equation

- **Solution 1:**

\[
E(r) = M(r) = \frac{1}{k} \nabla \times [\psi(r)a(r)]
\]

\[
H(r) = -\frac{1}{j\omega \mu_0} \nabla \times E(r) = -\frac{1}{j\omega \mu_0} \nabla \times M(r) = \frac{j}{\eta} N(r)
\]

- **Solution 2:**

\[
E(r) = N(r) = \frac{1}{k} \nabla \times M(r)
\]

\[
H(r) = -\frac{1}{j\omega \mu_0} \nabla \times N(r) = \frac{j}{\eta} M(r)
\]
Vector wave equation

- Example: choose the scalar function to be a solution of the wave equation (Helmholtz equation) in the Cartesian system

- This would be a simple scalar plane wave

\[(\nabla^2 + k^2)\psi = 0 \rightarrow \psi = \exp(-jk \cdot r)\]

- Take, as an example, a constant \(a(r)\), i.e., \(a(r) = \hat{x}\)

\[M(r) = -j\hat{k} \times \hat{x} \exp(-jk \cdot r)\]

\[N(r) = -\hat{k} \times (\hat{k} \times \hat{x}) \exp(-jk \cdot r)\]

- These correspond to two independent polarizations
Vector waves in cylindrical coordinates

- For cylindrical objects, a proper choice is to use the constant vector field
  \[ a(r) = \hat{z} \]

- This field satisfies all the requirements and results in
  \[
  M = \frac{1}{k} \nabla \times \left[ \psi \hat{z} \right] = \frac{1}{k} \nabla \psi \times \hat{z}
  \]
  \[
  N = \frac{1}{k^2} \nabla \times \left[ \nabla \psi \times \hat{z} \right] = \frac{1}{k^2} \left[ -\nabla^2 \psi \hat{z} + \frac{\partial}{\partial z} \nabla \psi \right]
  \]
  \[
  = \frac{1}{k^2} \left[ k^2 \psi \hat{z} + \frac{\partial}{\partial z} \nabla \psi \right]
  \]
Vector waves in cylindrical coordinates

- We next use cylindrical coordinates

\[
\nabla \psi_{m,k_z} = \hat{\rho} \frac{\partial \psi_{m,k_z}}{\partial \rho} + \hat{\phi} \frac{\partial \psi_{m,k_z}}{\rho \partial \phi} + \hat{z} \frac{\partial \psi_{m,k_z}}{\partial z}
\]

\[
M_{m,k_z} = \frac{1}{k} \nabla \psi_{m,k_z} \times \hat{z} = -\hat{\phi} \frac{\partial \psi_{m,k_z}}{k \partial \rho} + \hat{\rho} \frac{\partial \psi_{m,k_z}}{k \rho \partial \phi}
\]

\[
= -\hat{\rho} \frac{jm}{k \rho} \psi_{m,k_z} - \hat{\phi} \frac{\partial \psi_{m,k_z}}{k \partial \rho}
\]

\[
N_{m,k_z} = \frac{1}{k^2} \left[ -\hat{\rho} \frac{jk_z}{\rho} \frac{\partial \psi_{m,k_z}}{\partial \rho} - \hat{\phi} \frac{mk_z}{\rho} \psi_{m,k_z} + \hat{z} k^2 \psi_{m,k_z} \right]
\]
Vector waves in cylindrical coordinates

- Let us see what kind of fields they represent. 1st solution:

\[ E = M_{m,k_z} = \exp(-jm\phi - jk_zz) \left[ -\hat{\rho} \frac{j m}{k \rho} f_{m,k_z}(\rho) - \hat{\phi} \frac{df_{m,k_z}(\rho)}{kd\rho} \right] \]

- This field has no vertical component (TEz wave). It has an elliptic polarization in the x-y plane.

- Its corresponding magnetic field is

\[ H = -\frac{1}{j \omega \mu} \nabla \times E = j \frac{j}{\eta} N_{m,k_z} = \frac{j}{\eta} \exp(-jm\phi - jk_zz) \]

\[ \left[ -\hat{\rho} \frac{j k_z}{k^2} \frac{df_{m,k_z}(\rho)}{d\rho} - \hat{\phi} \frac{k_z m}{k^2} f_{m,k_z}(\rho) + \hat{z} \frac{k^2}{k_z} f_{m,k_z}(\rho) \right] \]
Vector waves in cylindrical coordinates

- The 2\textsuperscript{nd} solution for the electric field is:

\[
E = N_{m,k_z} = \exp(-jm\phi - jk_z z)
\]

\[
\left[-\hat{\rho} \frac{jk_z}{k^2} \frac{df_{m,k_z}(\rho)}{d\rho} - \hat{\phi} \frac{k_z m}{k^2} f_{m,k_z}(\rho) + \hat{z} \frac{k^2}{k^2} f_{m,k_z}(\rho)\right]
\]

- Here, the electric field does have a vertical component.

- It again has elliptic polarization, but in a plane rotated around the radial unit vector.
Vector waves in cylindrical coordinates

- The corresponding magnetic field is

\[ H = -\frac{1}{j\omega \mu} \nabla \times E = -\frac{1}{j\omega \mu k} \nabla \times (\nabla \times M_{m,k_z}) = \frac{j}{\eta} M_{m,k_z} \]

\[ = \frac{j}{\eta} \exp(-jm\phi - jk_z z) \left[ -\hat{\rho} \frac{jm}{k\rho} f_{m,k_z}(\rho) - \hat{\phi} \frac{df_{m,k_z}(\rho)}{kd\rho} \right] \]

- Now the magnetic field lies in the horizontal plane. It has no vertical component (TM^2 wave)

- Note also that in both cases

\[ M_{m,k_z} \cdot N_{m,k_z} = 0 \rightarrow E \cdot H = 0 \]
Vector waves in cylindrical coordinates

- Example: $m = 0$ corresponds to the cylindrically symmetric fields

\[
M_{0,k_z} = -\hat{\phi} \exp(-jk_z z) \frac{df_{m,k_z}(\rho)}{kd\rho}
\]

\[
N_{0,k_z} = \exp(-jk_z z) \left[ -\hat{\rho} \frac{jk_z}{k^2} \frac{df_{m,k_z}(\rho)}{d\rho} + \hat{z} \frac{k^2}{k^2} f_{m,k_z}(\rho) \right]
\]

- Example: $k_z = 0$ corresponds to solutions uniform along $z$

\[
M_{m,0} = \exp(-jm\phi) \left[ -\hat{\rho} \frac{jm}{k\rho} f_{m,0}(\rho) - \hat{\phi} \frac{df_{m,0}(\rho)}{kd\rho} \right]
\]

\[
N_{m,0} = \hat{z} \exp(-jm\phi) f_{m,0}(\rho)
\]

$k_\rho = k$
Vector waves in cylindrical coordinates

- For later use, let us consider the vector functions when
  \[ f_{m,k_z}(r) = H_{m}^{(2)}(k_\rho \rho) \]

- This choice is useful for representing scattered wave since it satisfies the radiation condition.

- Consider the functions at a large distance from the center. Asymptotic relation for the Hankel function
  \[ k_\rho \rho \gg 1 \rightarrow H_{m}^{(2)}(k_\rho \rho) \sim j^m \sqrt{\frac{2}{\pi k_\rho \rho}} \exp\left(-j k_\rho \rho + \frac{j\pi}{4}\right) \]
Vector waves in cylindrical coordinates

- The TE solution has the asymptotic behavior: $k_{\rho} \rho \gg 1 \rightarrow$

\[
E = M_{m,k_{z}} \sim \phi \frac{j^{m} k_{\rho}}{k} C_{0} \frac{\exp(-jk_{\rho}\rho - jm\phi - jk_{z}z)}{\sqrt{k_{\rho}\rho}}
\]

\[
H = \frac{j}{\eta} N_{0,k_{z}} \sim \left[-\frac{k_{z}}{k} \hat{\rho} + \frac{k_{\rho}}{k} \hat{z}\right]\left(\frac{j^{m} k_{\rho}}{\eta k} C_{0}\right) \frac{\exp(-jk_{\rho}\rho - jm\phi - jk_{z}z)}{\sqrt{k_{\rho}\rho}}
\]

\[
C_{0} = j\sqrt{\frac{2}{\pi}} \exp\left(\frac{j\pi}{4}\right)
\]
Vector waves in cylindrical coordinates

- At large distance these are conical waves propagating with the wave vector $k_\rho \hat{\rho} + k_z \hat{z}$ for every angle $\phi$. They all make an angle $\sin \theta = k_z / k$ with the $z$-axis.

- They behave as TEM waves: fields are perpendicular to each other and to the wave vector in all directions.
Vector waves in cylindrical coordinates

- Magnetic field has no $\phi$ component, electric field has only a $\phi$ component
- Again the electric and magnetic fields are orthogonal
- At a large distance these wave behave as TEM waves

\[ k_\rho \hat{\rho} + k_z \hat{z} \]

Remember: the above analysis of the asymptotic behavior is only valid for solutions based on Hankel function of the 2nd kind.
The behavior of the TM solution at large distance is similar to TE but now the electric field has no $\phi$ component and magnetic field only has a $\phi$ component.

\[
E = N_{0,k_z} \sim (-j) \left[ -\frac{k_z}{k} \hat{\rho} + \frac{k_\rho}{k} \hat{z} \right] \left( \frac{j^m k_\rho}{k} C_0 \right) \frac{\exp(-jk_\rho \rho - jm\phi - jk_z z)}{\sqrt{k_\rho \rho}}
\]

\[
H = \frac{j}{\eta} M_{m,k_z} \sim \frac{j}{\eta} \hat{\phi} \left( \frac{j^m k_\rho}{k} C_0 \right) \frac{\exp(-jk_\rho \rho - jm\phi - jk_z z)}{\sqrt{k_\rho \rho}}
\]
Expansion of a plane wave

- So far we have analyzed the ‘natural’ solutions to the vector wave equation in cylindrical coordinates. We expect them to be useful when treating scattering by cylindrically symmetric objects.

- We have seen that asymptotically, these solutions behave as TEM waves at large distance.

- But the actual problem we are interested in is not the scattering of these waves, but the scattering of a simple plane wave such as

\[ E_i(r) = E_i^0 \exp(-j k_i \cdot r) \]

\[ |k_i| = k \]
Expansion of a plane wave

- To be able to use the cylindrical vector solutions, we first have to expand the plane wave into these functions.

- Let us write

\[ \mathbf{k}_i = (k_{i,x}, k_{i,y}, k_{i,z}) = (k_{i,\rho} \cos \phi_i, k_{i,\rho} \sin \phi_i, k_{i,z}) \]

\[ k_{i,\rho}^2 + k_{i,z}^2 = k^2 \]
Expansion of a plane wave

- Remember that
  \[ r = (x, y, z) = (\rho \cos \phi, \rho \sin \phi, z) \]
  \[ k_i \cdot r = k_{i,\rho} \rho \cos (\phi - \phi_i) + k_{i,z} z \]

- We next use the following relationship from the theory of Bessel functions
  \[ \exp(-jz \cos \theta) = \sum_{m=-\infty}^{\infty} (-j)^m J_m(z) \exp(-jm\theta) \]
  \[ \exp(-jk_i \cdot r) = \exp(-jk_{i,z} z) \sum_{m=-\infty}^{\infty} (-j)^m J_m(k_{i,\rho} \rho) \exp[-jm(\phi - \phi_i)] \]
Expansion of a plane wave

- Remember that

\[ \psi_{m,k_{i,z}}^J (\rho, \phi, z) = J_m (k_{i,\rho} \rho) \exp \left( -jm\phi - jk_{i,z} z \right) \]

- is one solution of the scalar wave equation with the Bessel function of the first kind

\[ \exp \left( -jk_{i} \cdot r \right) = \sum_{m=-\infty}^{\infty} (-j)^m \psi_{m,k_{i,z}}^J (\rho, \phi, z) \exp \left( jm\phi_i \right) \]

- This is the expansion of a scalar plane wave for an incident wave with \( k_{i,z}, \phi_i \). What about the vector plane wave?
Expansion of a plane wave

1st result:

\[
\frac{1}{k} \nabla \times \left[ \hat{z} \exp(-jk_i \cdot r) \right] = \sum_{m=-\infty}^{\infty} (-j)^m M_{m,k_i,z}^J (\rho, \phi, z) \exp(jm\phi_i)
\]

\[
M_{m,k_i,z}^J = \frac{1}{k} \nabla \times \left( \psi_{m,k_i,z}^J \hat{z} \right) = -\hat{\rho} \frac{jm}{k \rho} \psi_{m,k_i,z}^J - \hat{\phi} \frac{\partial \psi_{m,k_i,z}^J}{k \partial \rho}
\]

\[
= \left[ -\hat{\rho} \frac{jm}{k \rho} J_m(k_i,\rho \rho) - \hat{\phi} \frac{k_i,\rho}{k} J_m'(k_i,\rho \rho) \right] \exp(-jm\phi - jk_{i,z}z)
\]

\[
J_m'(z) \equiv \frac{dJ_m(z)}{dz}
\]
Expansion of a plane wave

- The first vector relation now follows:

\[
\left( \hat{z} \times \hat{k}_i \right) \exp(-j k_i \cdot r) = -j \sum_{m=-\infty}^{\infty} (-j)^m M_{m,k_i,z}^J (\rho, \phi, z) \exp(j m \phi_i)
\]

- For the 2nd vector relation take the curl of the above equation

\[
\frac{1}{k} \nabla \times \left[ \left( \hat{z} \times \hat{k}_i \right) \exp(-j k_i \cdot r) \right] = -j \sum_{m=-\infty}^{\infty} (-j)^m N_{m,k_i,z}^J (\rho, \phi, z) \exp(j m \phi_i)
\]

\[
N_{m,k_i,z}^J = \frac{1}{k} \nabla \times M_{m,k_i,z}^J
\]
Expansion of a plane wave

- Explicit form of $N$:

$$N_{m,k_i,z}^J = \exp\left(-jm\phi - jk_{i,z}z\right)$$

$$\left[-\hat{\rho} \frac{jk_{i,z}k_{i,\rho}}{k^2} J'_m\left(k_{i,\rho}\rho\right) - \hat{\phi} \frac{mk_{i,z}}{k^2 \rho} J_m\left(k_{i,\rho}\rho\right) + \hat{z} \frac{k_{i,\rho}^2}{k^2} J_m\left(k_{i,\rho}\rho\right)\right]$$

- The 2\textsuperscript{nd} vector relation now becomes

$$\left[(\hat{z} \times \hat{k}_i) \times \hat{k}_i \right] \exp(-jk_i \cdot r) = -\sum_{m=-\infty}^{\infty} (-j)^m N_{m,k_i,z}^J(\rho, \phi, z) \exp(jm\phi_i)$$
Expansion of a plane wave

- Now, we define the unit vector system

\[ \hat{h}_i = \frac{\hat{z} \times \hat{k}_i}{|\hat{z} \times \hat{k}_i|} \]

\[ \hat{v}_i = \hat{h}_i \times \hat{k}_i = \frac{\hat{z} \times \hat{k}_i}{|\hat{z} \times \hat{k}_i|} \times \hat{k}_i \]
Expansion of a plane wave

- Let us see what these vectors are:

\[ \mathbf{k}_i = (k_{i,x}, k_{i,y}, k_{i,z}) = (k_{i,\rho} \cos \phi_i, k_{i,\rho} \sin \phi_i, k_{i,z}) \]

\[ \hat{k}_i = \frac{1}{k} (k_{i,\rho} \cos \phi_i, k_{i,\rho} \sin \phi_i, k_{i,z}) \]

\[ \hat{h}_i = \frac{\hat{z} \times \hat{k}_i}{|\hat{z} \times \hat{k}_i|} = (-\sin \phi_i, \cos \phi_i, 0) = \hat{\phi}_i \]

\[ \hat{v}_i = \hat{h}_i \times \hat{k}_i = \hat{\phi}_i \times \hat{k}_i = \frac{k_{i,z}}{k} \hat{\rho}_i - \frac{k_{i,\rho}}{k} \hat{z} \]

\[ \hat{\rho}_i = (\cos \phi_i, \sin \phi_i, 0) \]

These cylindrical unit vectors are defined with respect to \( k_i \). They have a subscript \( i \). Do not confuse them with the cylindrical unit vectors for position!
Expansion of a plane wave

We thus have the relations

\[ \hat{h}_i \exp(-jk_i \cdot r) = \hat{\phi}_i \exp(-jk_i \cdot r) = \]

\[ -j \frac{k}{k_{i,\rho}} \sum_{m=-\infty}^{\infty} (-j)^m M^J_{m,k_{i,z}} (\rho, \phi, z) \exp(jm\phi_i) \]

\[ \hat{v}_i \exp(-jk_i \cdot r) = \left( \frac{k_{i,z}}{k} \hat{\rho}_i - \frac{k_{i,\rho}}{k} \hat{z} \right) \exp(-jk_i \cdot r) = \]

\[- \frac{k}{k_{i,\rho}} \sum_{m=-\infty}^{\infty} (-j)^m N^J_{m,k_{i,z}} (\rho, \phi, z) \exp(jm\phi_i) \]
Expansion of a plane wave

- For any polarization of the incident plane wave:

$$ E_i(r) = E_i^0 \exp(-jk_i \cdot r) = \left[ (E_i^0 \cdot \hat{h}_i) \hat{h}_i + (E_i^0 \cdot \hat{v}_i) \hat{v}_i \right] \exp(-jk_i \cdot r) $$

$$ E_i(r) = \sum_{m=-\infty}^{\infty} u_m M^J_{m,k_i,z} (\rho, \phi, z) + \sum_{m=-\infty}^{\infty} v_m N^J_{m,k_i,z} (\rho, \phi, z) $$

$$ u_m = -j \left( E_i^0 \cdot \hat{h}_i \right) \frac{k}{k_{i,\rho}} (-j)^m \exp(jm\phi_i) $$

   Related to TE: horizontal electric field

$$ v_m = - \left( E_i^0 \cdot \hat{v}_i \right) \frac{k}{k_{i,\rho}} (-j)^m \exp(jm\phi_i) $$

   Related to TM: horizontal magnetic field
Exact solution of the problem

- We now consider the scattering of a plane wave by an infinitely long, perfectly conducting cylinder.
- When the incident wave hits the cylinder, surface currents (and charges) are induced.
- These currents create the ‘scattered’ field. At any point, the total electric field is

\[ E(r) = E_i(r) + E_s(r) \]

\[ E_i(r) = E_i^0 \exp(-j k_i \cdot r) \]
Exact solution of the problem

- We saw how the incident plane wave can be represented in terms of cylindrical vector solutions.

- The scattered field (outside the cylinder) can also be expanded in terms of those solutions.

- But: for the scattered field we should use vectors with the right condition at the infinity $\rho \to \infty$. 
Exact solution of the problem

- We should use the Hankel function of the 2\textsuperscript{nd} kind for these waves which satisfy the radiation condition (behave as outgoing waves at infinity)

- Also, since the system is uniform in z-direction, \( k_z \) is the same as that of the incident wave: \( k_z = k_{i,z} \). Of course the wave will be scattered along different angles (different \( \phi \)'s)

\[
E_s(r) = \sum_{m=-\infty}^{\infty} a_m M_{m,k_i,z}^H(\rho, \phi, z) + \sum_{m=-\infty}^{\infty} b_m N_{m,k_i,z}^H(\rho, \phi, z)
\]

\[
M_{m,k_i,z}^H = \frac{1}{k} \nabla \times \left[ \psi_{m,k_i,z}^H \hat{z} \right] \quad N_{m,k_i,z}^H = \frac{1}{k} \nabla \times M_{m,k_i,z}^H
\]
Exact solution of the problem

- The vector functions are explicitly given by

\[
M^H_{m,k_{i,z}} = \exp\left(-jm\phi - jk_{i,z}z\right) \\
\left[-\hat{\rho} \frac{jm}{k\rho} H_m^{(2)}(k_{i,\rho}\rho) - \hat{\phi} \frac{k_{i,\rho}}{k} H_m^{(2)'}(k_{i,\rho}\rho) \right]
\]

\[
N^H_{m,k_{i,z}}(r) = \exp\left(-jm\phi - jk_{i,z}z\right) \\
\left[-\hat{\rho} \frac{jk_{i,z}k_{i,\rho}}{k^2} H_m^{(2)'}(k_{i,\rho}\rho) - \hat{\phi} \frac{mk_{i,z}}{k^2\rho} H_m^{(2)}(k_{i,\rho}\rho) + \hat{z} \frac{k_{i,\rho}^2}{k^2} H_m^{(2)}(k_{i,\rho}\rho) \right]
\]

- Note that we have used

\[
k_{\rho} = \sqrt{k^2 - k_z^2} = \sqrt{k^2 - k_{i,z}^2} = k_{i,\rho}
\]
The next step is to choose the coefficients of the expansion such that the condition at the surface of the cylinder is satisfied.

Zero tangential component of total electric field

\[ E_z(a, \phi, z) = E_\phi(a, \phi, z) = 0 \]

1st Result:

\[ b_m = -\nu_m \frac{J_m(k_{i,\rho}a)}{H_m^{(2)}(k_{i,\rho}a)} \]

\[ = \left( E_i^0 \cdot \hat{v}_i \right) \frac{k}{k_{i,\rho}} \frac{J_m(k_{i,\rho}a)}{H_m^{(2)}(k_{i,\rho}a)} (-j)^m \exp(jm\phi_i) \]
Exact solution of the problem

- 2\textsuperscript{nd} result:

\[ a_m = -u_m \frac{J'_m(k_{i,\rho}a)}{\left[H^{(2)}_m\right]'(k_{i,\rho}a)} \]

\[ = j\left(E_i^0 \cdot \hat{h}_i\right) \frac{k}{k_{i,\rho}} \frac{J'_m(k_{i,\rho}a)}{H^{(2)}_m(k_{i,\rho}a)} (-j)^m \exp(jm\phi_i) \]

- Note that a TE incident wave excites TE polarized waves (horizontal polarization), and a TM wave excites TM polarized waves (vertical polarization)
Exact solution of the problem (TE polarization)

- TE scattering (horizontal electric field):

\[
E_{s}^{TE}(r) = \sum_{m=-\infty}^{\infty} a_m M_{m,k_i,z}^H(\rho, \phi, z)
\]

\[
= - \sum_{m=-\infty}^{\infty} \frac{u_m J'_m(k_{i,\rho} a)}{H_m^{(2)'}(k_{i,\rho} a)} M_{m,k_i,z}^H(\rho, \phi, z)
\]

\[
= \left( E_0^0 \cdot \hat{n}_i \right) \frac{jk}{k_{i,\rho}} \sum_{m=-\infty}^{\infty} \frac{J'_m(k_{i,\rho} a)}{H_m^{(2)'}(k_{i,\rho} a)} (-j)^m \exp(j m \phi) M_{m,k_i,z}^H(\rho, \phi, z)
\]
Exact solution of the problem (TE polarization)

- Explicit form in components

\[
E^{TE}_{s,\rho}(\mathbf{r}) = \left( E^0_i \cdot \hat{n}_i \right) \frac{\exp(-jk_{i,z}z)}{k_{i,\rho}\rho} \sum_{m=-\infty}^{\infty} \frac{(-j)^m J'_m(k_{i,\rho}a)}{H^{(2)'}_m(k_{i,\rho}a)} H^{(2)}_m(k_{i,\rho}\rho) \exp[-jm(\phi-\phi_i)]
\]

\[
E^{TE}_{s,\phi}(\mathbf{r}) = -j \left( E^0_i \cdot \hat{n}_i \right) \exp(-jk_{i,z}z) \sum_{m=-\infty}^{\infty} \frac{(-j)^m J'_m(k_{i,\rho}a)}{H^{(2)'}_m(k_{i,\rho}a)} H^{(2)'}_m(k_{i,\rho}\rho) \exp[-jm(\phi-\phi_i)]
\]
Exact solution of the problem (TE polarization)

- The magnetic field in the TE case

\[ H_{s,TE}^{(m)}(r) = - \frac{j}{\eta} \sum_{m=-\infty}^{\infty} \frac{u_m J'_m(k_{i,\rho} a)}{H_m^{(2)'}(k_{i,\rho} a)} N^H_{m,k_{i,z}}(\rho, \phi, z) \]

- In components:

\[ H_{s,\rho,TE}^{(m)}(r) = \left( E_i^0 \cdot \hat{n}_i \right) \left( \frac{jk_{i,z}}{k\eta} \right) \exp(-jk_{i,z}z) \]

\[ \sum_{m=-\infty}^{\infty} (-j)^m J'_m(k_{i,\rho} a) \frac{1}{H_m^{(2)'}(k_{i,\rho} a)} H_m^{(2)'}(k_{i,\rho} \rho) \exp[-jm(\phi - \phi_i)] \]
Exact solution of the problem (TE polarization)

\[ H_{s,\phi}^{TE}(r) = \left( E_i^0 \cdot \hat{h}_i \right) \left( \frac{k_{i,z}}{\eta k} \right) \exp(-jk_{i,z}z) \]

\[ \sum_{m=-\infty}^{\infty} m \frac{(-j)^m J'_m(k_{i,\rho}a)}{H^{(2)'}_m(k_{i,\rho}a)} H_m^{(2)}(k_{i,\rho}\rho) \exp[-jm(\phi - \phi_i)] \]

\[ H_{s,z}^{TE}(r) = -\left( E_i^0 \cdot \hat{h}_i \right) \left( \frac{k_{i,\rho}}{\eta k} \right) \exp(-jk_{i,z}z) \]

\[ \sum_{m=-\infty}^{\infty} \frac{(-j)^m J'_m(k_{i,\rho}a)}{H^{(2)'}_m(k_{i,\rho}a)} H_m^{(2)}(k_{i,\rho}\rho) \exp[-jm(\phi - \phi_i)] \]
Exact solution of the problem (TM polarization)

For the TM case we have

\[
E_{s}^{TM}(r) = \sum_{m=-\infty}^{\infty} b_{m} N_{m,k_{i,\rho},z}^{H}(\rho, \phi, z)
\]

\[
= -\sum_{m=-\infty}^{\infty} v_{m} \frac{J_{m}(k_{i,\rho} a)}{H_{m}^{(2)}(k_{i,\rho} a)} N_{m,k_{i,\rho},z}^{H}(\rho, \phi, z)
\]

\[
= (E_{i}^{0} \cdot \hat{v}_{i}) \frac{k}{k_{i,\rho}} \sum_{m=-\infty}^{\infty} \frac{J_{m}(k_{i,\rho} a)}{H_{m}^{(2)}(k_{i,\rho} a)} (-j)^{m} \exp(jm\phi_{i}) N_{m,k_{i,\rho},z}^{H}(\rho, \phi, z)
\]
Exact solution of the problem (TM polarization)

- Explicit form in components

\[
E_{s,\rho}^{TM}(\mathbf{r}) = (E_0^i \cdot \hat{v}_i) \left( -\frac{jk_{i,z}}{k} \right) \exp(-jk_{i,z}z)
\]

\[
\sum_{m=-\infty}^{\infty} \frac{(-j)^m J_m(k_{i,\rho}a)}{H_m^{(2)}(k_{i,\rho}a)} H_m^{(2)*}(k_{i,\rho}\rho) \exp[-jm(\phi - \phi_i)]
\]

\[
E_{s,\phi}^{TM}(\mathbf{r}) = (E_0^i \cdot \hat{v}_i) \left( -\frac{k_{i,z}}{k} \right) \frac{1}{k_{i,\rho}\rho} \exp(-jk_{i,z}z)
\]

\[
\sum_{m=-\infty}^{\infty} \frac{(-j)^m J_m(k_{i,\rho}a)}{H_m^{(2)}(k_{i,\rho}a)} H_m^{(2)}(k_{i,\rho}\rho) \exp[-jm(\phi - \phi_i)]
\]
Exact solution of the problem (TM polarization)

- Explicit form in components

\[ E^{TM}_{s,z}(r) = (E^0_i \cdot \hat{v}_i) \left( \frac{k_{i,\rho}}{k} \right) \exp(-jk_{i,z}z) \]

\[ \sum_{m=-\infty}^{\infty} \frac{(-j)^m J_m(k_{i,\rho}a)}{H^{(2)}_m(k_{i,\rho}a)} H^{(2)}_m(k_{i,\rho}\rho) \exp[-jm(\phi - \phi_i)] \]

- The magnetic field

\[ H^{TM}_s(r) = -\frac{j}{\eta} \sum_{m=-\infty}^{\infty} \nu_m \frac{J_m(k_{i,\rho}a)}{H^{(2)}_m(k_{i,\rho}a)} M^H_{m,k_{i,z}}(\rho, \phi, z) \]
Exact solution of the problem (TM polarization)

\[ H_{s,\rho}^{TM}(r) = \left( E_i^0 \cdot \hat{v}_i \right) \frac{\exp(-jk_{i,z}z)}{\eta k_{i,\rho} \rho} \]

\[ \sum_{m=-\infty}^{\infty} m \frac{(-j)^m J_m(k_{i,\rho}a)}{H_{m}^{(2)}(k_{i,\rho}a)} H_{m}^{(2)}(k_{i,\rho}\rho) \exp[-jm(\phi - \phi_i)] \]

\[ H_{s,\phi}^{TM}(r) = -\left( E_i^0 \cdot \hat{v}_i \right) \frac{j}{\eta} \exp(-jk_{i,z}z) \]

\[ \sum_{m=-\infty}^{\infty} \frac{(-j)^m J_m(k_{i,\rho}a)}{H_{m}^{(2)}(k_{i,\rho}a)} H_{m}^{(2)'}(k_{i,\rho}\rho) \exp[-jm(\phi - \phi_i)] \]
Scattering by a perfectly conducting infinite cylinder

- Remember that this is the full solution everywhere. We are actually interested in the scattering in the far field limit.

- We again use the asymptotic relationship

\[ H_m^{(2)}(k_\rho \rho \gg 1) \sim \sqrt{\frac{2}{\pi}} \exp\left(\frac{j \pi}{4}\right) \frac{j^m \exp(-jk_\rho \rho)}{\sqrt{k_\rho \rho}} = -jC_0 \frac{j^m \exp(-jk_\rho \rho)}{\sqrt{k_\rho \rho}} \]

- We find that in the far field limit the fields are parallel to:

\[ E_s^{TE}(k_{i,\rho} \rho \gg 1) \| \hat{\phi} \quad E_s^{TM}(k_{i,\rho} \rho \gg 1) \| \left( -\frac{k_{i,z}}{k} \hat{\rho} + \frac{k_{i,\rho}}{k} \hat{z} \right) \]

Same as \( \hat{\theta} \)
Scattering by a perfectly conducting infinite cylinder

- More specifically:

\[
E_{s}^{TE} \rightarrow \hat{\phi} \left( E_{i}^{0} \cdot \hat{h}_{i} \right) \frac{jC_{0} \exp(-jk_{\rho}\rho - jk_{i,z}z)}{\sqrt{k_{\rho}\rho}} \sum_{m=-\infty}^{\infty} \frac{J_{m}'(k_{i,\rho}a)}{H_{m}^{(2)'}(k_{i,\rho}a)} \exp\left[ jm(\phi_{i} - \phi) \right]
\]

\[
E_{s}^{TM} \rightarrow \left[ -\frac{k_{i,z}}{k} \hat{\rho} + \frac{k_{i,\rho}}{k} \hat{z} \right] \left( E_{i}^{0} \cdot \hat{v}_{i} \right) \frac{(-jC_{0}) \exp(-jk_{i,\rho}\rho - jk_{i,z}z)}{\sqrt{k_{i,\rho}\rho}} \sum_{m=-\infty}^{\infty} \frac{J_{m}(k_{i,\rho}a)}{H_{m}^{(2)}(k_{i,\rho}a)} \exp\left[ jm(\phi_{i} - \phi) \right]
\]
Scattering by a perfectly conducting infinite cylinder

- The interesting point here is the behavior of far field as function of the angle $\phi$, which is governed by the functions

$$\sum_{m=-\infty}^{\infty} \frac{J'_m(k_{i,\rho}a)}{H^{(2)'}_m(k_{i,\rho}a)} \exp[ jm(\phi_i - \phi) ]$$

and

$$\sum_{m=-\infty}^{\infty} \frac{J_m(k_{i,\rho}a)}{H^{(2)}_m(k_{i,\rho}a)} \exp[ jm(\phi_i - \phi) ]$$

- Remember that

$$H^{(2)}_{-m} = (-1)^m H^{(2)}_m \quad J_{-m} = (-1)^m J_m$$
TE Scattering by a thin conducting cylinder

- To get some insight let us consider the limit of a thin cylinder

\[ k_{i, \rho} a < ka \ll 1 \]

- For small arguments (sufficient to consider positive or zero m’s)

\[
J'_0(z) \sim -\frac{z}{2} \quad J'_{m>0}(z) = -J_{m+1}(z) + \frac{m}{z} J_m(z) \sim \frac{z^{m-1}}{2^m (m-1)!} \]

\[
H^{(2)'}_0(z) \sim -\frac{j 2}{\pi z} \quad H^{(2)'}_{m>0}(z) = -H^{(2)}_{m+1}(z) + \frac{m}{z} H^{(2)}_m(z) \sim -\frac{j}{\pi} \frac{2^m m!}{z^{m+1}} \]
To the lowest order we have for the TE case

$$\sum_{m=-\infty}^{\infty} \frac{J'_m(k_{i,\rho}a)}{H^{(2)'}_m(k_{i,\rho}a)} \exp\left[jm(\phi_i - \phi)\right]$$

$$\sim -j \frac{\pi (k_{i,\rho}a)^2}{4} \left[1 - 2\cos(\phi_i - \phi)\right]$$

$$E^{TE}_s(\rho, \phi, z) \rightarrow \hat{\phi} \frac{\pi C_0}{4} \frac{(E^0_i \cdot \hat{h}_i)(k_{i,\rho}a)^2}{\sqrt{k_{i,\rho}\rho}} \times \exp(-jk_{i,\rho}\rho - jk_{i,z}z) \left[1 - 2\cos(\phi_i - \phi)\right]$$
Consider the scattered electric field. It’s phase does not depend on angle. It’s amplitude is given below

\[ E_{s, \rho, \phi, z}^{TE} = \hat{\phi} \tilde{E}_{s,\phi}^{TE}(\phi) \exp(-jk_{i,\rho} \rho - jk_{i,z} z) \]

\[ \tilde{E}_{s,\phi}^{TE}(\phi) = \frac{\pi C_0}{4} \frac{(E_i^0 \cdot \hat{h}_i)(k_{i,\rho} a)^2}{\sqrt{k_{i,\rho} \rho}} \left[ 1 - 2\cos(\phi - \phi_i) \right] \]
TE Scattering by a thin conducting cylinder

- Note that scattering is largest when
  \[ \cos(\phi - \phi_i) = -1 \Rightarrow \phi = \phi_i + \pi \]
- This is the backward scattering case
- It is zero when \( \phi = \phi_i \pm \frac{\pi}{2} \)

![Graph showing the scattering behavior](image-url)
Numerical experiments

- Let us return to the exact result for TE case and again focus on the azimuthal component of the electric field

\[ E_{s,\phi}^{TE}(r) = -j \left( E_i^0 \cdot \hat{h}_i \right) \exp(-jk_{i,z}z) \]

\[ \sum_{m=-\infty}^{\infty} \frac{(-j)^m J_m'(k_i\rho a)}{H_m^{(2)'}(k_i\rho a)} H_m^{(2)'}(k_i\rho \rho) \exp\left[-jm(\phi - \phi_i)\right] \]

- We would like to compare this with the asymptotic result

- This a difficult function to compute because of ‘bad’ behavior of Bessel functions of large order
Scattering by a perfectly conducting infinite cylinder

- So let us consider the exact numerical result for the azimuthal component of the electric field.

- We plot this function when

\[ \phi_i = 0 \rightarrow k_i = \left(k_{i,x}, 0, k_{i,z}\right), k_{i,\rho} = k_{i,x} \]

\[ \hat{h}_i = \hat{\phi} = \hat{y} \quad E_i^0 \cdot \hat{h}_i = E_{i,y}^0 \]

\[ E_i(r) = E_{i,y}^0 \hat{y} \exp(-jk_i \cdot r) = E_{i,y}^0 \hat{y} \exp(-jk_{i,x}x - jk_{i,z}z) \]
Numerical experiments

- For a thin cylinder we have the amplitude profile

\[ |E_i| = \frac{k_i}{3} \]

\[ k_i,\rho a = 0.25 \]
Numerical experiments

- It is also instructive to look at constant phase fronts, at which

\[ \text{phase}(Q) = 0 \]

\[ k_{i,\rho} a = 0.25 \]
Numerical experiments

- So in the far field, the phase fronts are almost circular in the x-y plane: there is no angle-dependence of the phase of the scattered field, as found from the thin-film approximation.

- Also the amplitude is largest in case of backscattered waves, as found from the same approximation.

- The amplitude becomes nearly zero when $\phi = \phi_i \pm \frac{\pi}{3}$.

- Again this is in line with what we found before.

- So this approximation is quite accurate.
TM Scattering by a thin conducting cylinder

- For the TM case from a thin cylinder we have
  \[ k_{i,\rho}a < ka \ll 1 \]

- For small arguments (consider positive or zero m’s)
  \[ J_m(z) \sim \frac{1}{m!} \left( \frac{z}{2} \right)^m \]
  \[ H^{(2)}_0(z) \sim -\frac{2j}{\pi} \ln z \]
  \[ H^{(2)}_{m>0}(z) \sim -\frac{j}{\pi} \frac{2^m(m-1)!}{z^m} \]

- Lowest order in \( k_{i,\rho}a \):
  \[ \sum_{m=-\infty}^{\infty} \frac{J_m(k_{i,\rho}a)}{H^{(2)}_m(k_{i,\rho}a)} \exp\left[ jm(\phi_i - \phi) \right] \sim \frac{j\pi}{2 \ln(k_{i,\rho}a)} \]
TM Scattering by a thin conducting cylinder

- Resulting far scattered field for a thin conducting cylinder

\[
E_{s}^{TM}(\rho, \phi, z) = E_{s,v}^{TM} \left( -\frac{k_{i,z}}{k} \hat{\rho} + \frac{k_{i,\rho}}{k} \hat{z} \right) \exp(-jk_{i,\rho} \rho - jk_{i,z} z)
\]

\[
E_{s,v}^{TM} = \left( E_{i}^{0} \cdot \hat{v}_{i} \right) \frac{C_{0}}{\sqrt{k_{i,\rho} \rho}} \frac{\pi}{2 \ln(k_{i,\rho} a)}
\]

- In the lowest order approximation there is no angle dependence for the TM scattered wave!

- The amplitude in all directions is the same
TM Scattering by a thin conducting cylinder

- Let us compare the scattering strength in the two cases

\[
|E_{s,\phi}^{TE}(\phi)| = \frac{k}{k_{i,\rho}} \frac{\pi |C_0| |E_i^0 \cdot \hat{h}_i|(k_{i,\rho}a)^2}{4 \sqrt{k_{i,\rho}} \rho} \left|1 - 2\cos(\phi - \phi_i)\right|
\]

\[
|E_{s,v}^{TM}| = |E_i^0 \cdot \hat{v}_i| \frac{|C_0| \pi}{\sqrt{k_{i,\rho}} \rho} \frac{2 \ln(k_{i,\rho}a)}{2 \ln(k_{i,\rho}a)}
\]

- For equal horizontal and vertical components of the incident field we have

\[
\frac{|E_{s,\phi}^{TE}(\phi)|}{|E_{s,v}^{TM}|} < \frac{3}{2} \frac{k}{k_{i,\rho}} (k_{i,\rho}a)^2 \left|\ln(k_{i,\rho}a)\right|
\]
TM Scattering by a thin conducting cylinder

- This ratio is quite small for thin cylinders
- But this result is to be expected: a vertically polarized incident electric field has a component along the ‘wire’ and easily induces electric currents along the wire. These currents, in turn, generate the scattered field.
- A horizontally polarized incident wave has no longitudinal components and cannot excite such currents
TE Scattering by a thick conducting cylinder

- Now, let us investigate the other limit, that of a thick conducting cylinder which satisfies

\[
k_{i,\rho} a = a \sqrt{k_i^2 - k_{i,z}^2} \gg 1
\]

- Remember: scattered electric field

\[
E_{s}^{TE}(r) = - \sum_{m=-\infty}^{\infty} \frac{u_m J'_m(k_{i,\rho}a)}{H_{m}^{(2)'}(k_{i,\rho}a)} M_{m,k_{i,z}}^{H}(\rho, \phi, z)
\]

- Far field limit

\[
E_{s}^{TE} \rightarrow \hat{\phi} \left( E_i^0 \cdot \hat{n}_i \right) \frac{j C_0 \exp(-jk_{i,\rho} \rho - jk_{i,z} z)}{\sqrt{k_{i,\rho} \rho}} \sum_{m=-\infty}^{\infty} \frac{J'_m(k_{i,\rho}a)}{H_{m}^{(2)'}(k_{i,\rho}a)} \exp[jm(\phi - \phi)]
\]
TE Scattering by a thick conducting cylinder

- Let us approximate the denominator

\[ z \gg 1 \rightarrow H_m^{(2)'}(z) \sim -jH_m^{(2)}(z) \sim -j^{m+1} \sqrt{\frac{2}{\pi z}} \exp \left( -jz + \frac{j\pi}{4} \right) \]

\[ \sum_{m=-\infty}^{\infty} \frac{J_m'(k_{i,\rho} a)}{H_m^{(2)'}(k_{i,\rho} a)} \exp \left[ jm(\phi_i - \phi) \right] \sim j \sqrt{\frac{\pi k_{i,\rho} a}{2}} \exp \left( jk_{i,\rho} a - \frac{j\pi}{4} \right) \]

\[ \sum_{m=-\infty}^{\infty} (-j)^m J_m'(k_{i,\rho} a) \exp \left[ jm(\phi_i - \phi) \right] \]

- Next, use the relation

\[ -j \cos \theta \exp \left( -jz \cos \theta \right) = \sum_{m=-\infty}^{\infty} (-j)^m J_m'(z) \exp \left( -jm\theta \right) \]
The TE far scattered field becomes

$$E_{s}^{TE}(\rho, \phi, z) \sim -\hat{\phi}(E_{i}^{0} \cdot \hat{h}_{i}) \sqrt{\frac{a}{\rho}} \cos(\phi - \phi_{i})$$

$$\times \exp\left[ -j k_{i,\rho}(\rho - a) - j k_{i,\rho}a \cos(\phi - \phi_{i}) - j k_{i,z}z \right]$$

Remember that this result is also based on the far-field behavior of the vector solutions (used here), which in turn, was based on the asymptotic behavior of the Hankel function when

$$k_{\rho} \rho \gg 1$$
Numerical results for a thick cylinder

- So let us again look at some numerical results for the azimuthal component of the electric field.

- We again plot this function when

\[ \phi_i = 0 \rightarrow k_i = (k_{i,x}, 0, k_{i,z}), k_{i,\rho} = k_{i,x} \]

\[ \hat{h}_i = \hat{\phi} = \hat{y} \quad E_i^0 \cdot \hat{h}_i = E_{i,y}^0 \]

\[ E_i(r) = E_{i,y}^0 \hat{y} \exp(-jk_i \cdot r) = E_{i,y}^0 \hat{y} \exp(-jk_{i,x}x - jk_{i,z}z) \]
Numerical results for a thick cylinder

- The amplitude profile is shown below

\[ k_{i,x} a = 10 \]
Numerical results for a thick cylinder

- Phase fronts:

\[ k_{i,x} a = 10 \]
Numerical results for a thick cylinder

- Note that in front of the cylinder the scattered wave looks normal, it has a circular phase front.
- But behind the cylinder, at a not too far distance, the phase fronts are flat!
- Besides, its amplitude is much larger than the waves scattered back (to the left).
- But this is just a misconception. To understand this point, one should realize that the total field is

\[ E_i(r) + E_{sTE}(r) \]
Numerical results for a thick cylinder

- Let plot the $\phi$ component of the total field.
- The amplitude of the scattered wave behind the cylinder is large because it has to partially cancel the incident wave, to form a shadow region.
Numerical results for a thick cylinder

- Nevertheless, the numerical results are very different from the asymptotic result we found earlier if we only demand $k_\rho \rho \gg 1$.

- To understand why, consider the exact electric field

$$E_{s,\phi}^{TE} = -j \left( E_i^0 \cdot \hat{h}_i \right) \exp(-jk_{i,z}z)$$

$$\sum_{m=-\infty}^{\infty} \frac{(-j)^m J'_m(k_{i,\rho}a)H^{(2)'}_m(k_{i,\rho}\rho)}{H^{(2)'}_m(k_{i,\rho}a)} \exp[-jm(\phi-\phi_i)]$$

- Asymptotic behavior of Hankel function depends on whether argument is larger, smaller, or comparable to order $m$. 

Scattering from cylindrical objects
TE Scattering by a thick conducting cylinder

- The correct asymptotic result is

\[ k_{i,\rho} \rho \gg |m|^2 \rightarrow H_m^{(2)}(k_{i,\rho} \rho) \sim j^m \sqrt{\frac{2}{\pi k_{i,\rho} \rho}} \exp\left(-jk_{i,\rho} \rho + j\pi/4\right) \]

- For a thin cylinder, \( k_{i,\rho} a \ll 1 \) only small \( m \)'s contribute (\(|m|=0,1\))

- Then, far field asymptotic is valid since \( k_{i,\rho} \rho \gg 1 \)
TE Scattering by a thick conducting cylinder

- But, for a thick cylinder, all orders with $|m| \leq k_{i,\rho}a$ have a contribution so that large orders are also important.

- The used asymptotic is then only accurate when

$$k_{i,\rho} \rho \gg \left( k_{i,\rho}a \right)^2 \rightarrow \rho \gg \frac{2\pi a^2}{\lambda_{i,\rho}}$$

- This is similar to the ‘definition’ of the far field region (why?)

- So our earlier result (and the earlier discussion of the far-field behavior of vector wave solutions) is, strictly speaking, only applicable in this limit.
TE Scattering by a thick conducting cylinder

- Similar discussion applies to TM scattering: whereas in the far field limit the scattered wave behaves as TEM waves with a conical wave front (circular in $x$-$y$ plane), the behavior outside this limit can be totally different.

- One can have flat wave fronts, a shadow region, etc.

- In particular, note that when the wavelength is very short compared to the dimensions of the object, the far field result is only valid so far away that it may be useless for practical applications.
Scattering cross section (scattering width)

- We saw in the beginning how a scattering cross section is defined for a finite scatterer in terms of the scattered power.
- An infinite cylinder, however, is not a finite object.
- The field radiated by sources inside the cylinder (scattered field) does not drop as $1/r$, but as $1/\sqrt{\rho}$.
- We have to reformulate our definition of scattering cross section for infinite cylindrical objects.
Scattering cross section (scattering width)

- Again consider an incident wave

\[
E_i = e_i E_0 \exp(-j k_i \cdot r)
\]

- Now, the scattered *far field* is always of the type

\[
E_s = e_s E_0 \frac{f(\hat{k}_s, \hat{k}_i)}{\sqrt{\rho}} \exp(-j k_{i,r} \rho - j k_{i,z} z)
\]

Note: scattering amplitude may depend on the polarization of the incident wave!
Scattering cross section (scattering width)

- Corresponding far field Poynting vectors are
  \[ S_i = \frac{|E_i^0|^2}{2\eta} \hat{k}_i \]
  \[ S_s = \frac{|E_s^0|^2}{2\eta} \hat{k}_s = \frac{|f(\hat{k}_s, \hat{k}_i)|^2}{\rho} \frac{|E_i^0|^2}{2\eta} \hat{k}_s \]

- It is important to bear in mind that
  \[ \hat{k}_s = \frac{k_{i,\rho}}{k} \hat{\rho} + \frac{k_{i,z}}{k} \hat{z} \]
Scattering cross section (scattering width)

- The Poynting vector is independent of $z$, it is the same for all vertical coordinates.
- We, therefore, cannot define the total scattered power since it is infinite.
- But, we can define the scattered power per unit length by integration over the angle.

\[ P_L = \rho \int_0^{2\pi} S_s \cdot \hat{\rho} \, d\phi = \frac{k_{i,\rho}}{k} |S_i| \int_0^{2\pi} |f(\phi, \phi_i)|^2 \, d\phi \]
Scattering cross section (scattering width)

- Remember that
  \[ |\mathbf{k}_i| = k \quad \frac{k_{i,\rho}}{k} = \cos \theta_i \]

- Total scattering width is now defined as
  \[ \sigma_W = \frac{P_L}{|S_i|} = \frac{k_{i,\rho}}{k} \int_0^{2\pi} \left| f(\phi, \phi_i) \right|^2 d\phi \]
Scattering cross section (scattering width)

- As an example consider the TE case

\[ E_i = \hat{h}_i E_0 \exp(-j k_i \cdot r) \quad \hat{h}_i = \phi_i \]

- The scattered far field:

\[ E_s^{TE} \rightarrow \hat{\phi} \left( E_i^0 \cdot \hat{h}_i \right) \frac{j C_0 \exp(-j k_{i,\rho} \rho - j k_z z)}{\sqrt{k_{i,\rho} \rho}} \]

\[ \sum_{m=-\infty}^{\infty} \frac{J'_m(k_{i,\rho} a)}{H_m^{(2)'}(k_{i,\rho} a)} \exp\left[ j m (\phi_i - \phi) \right] \]
Scattering cross section (scattering width)

- The scattering amplitude

\[ |f^{TE}(\phi, \phi_i)|^2 = \frac{2}{\pi k_{i,\rho}} \left| \sum_{m=-\infty}^{\infty} \frac{J'_m(k_{i,\rho} a)}{H^{(2)'}_m(k_{i,\rho} a)} \exp[jm(\phi_i - \phi)] \right|^2 \]

- Scattering width for TE waves:

\[ \sigma^{TE}_W = \frac{k_{i,\rho} 2\pi}{k} \int_0^{2\pi} |f^{TE}(\phi_s, \phi_i)|^2 \, d\phi_s = \frac{4}{k} \sum_{m=-\infty}^{\infty} \left| \frac{J'_m(k_{i,\rho} a)}{H^{(2)'}_m(k_{i,\rho} a)} \right|^2 \]
Scattering cross section (scattering width)

- **TM case:**

  \[ E_i = \hat{v}_i E_0 \exp(-jk_i \cdot r) \]

  \[ \hat{v}_i = \phi_i \times \hat{k}_i = \frac{k_{i,z}}{k} \hat{\rho}_i - \frac{k_{i,\rho}}{k} \hat{z} \]

- **Scattered field:**

  \[ E_s^{TM} \rightarrow \left[ -\frac{k_{i,z}}{k} \hat{\rho} + \frac{k_{i,\rho}}{k} \hat{z} \right] \left( E_i^0 \cdot \hat{v}_i \right) \left( -jC_0 \right) \exp(-jk_{i,\rho} \rho - jk_{i,z} z) \frac{1}{\sqrt{k_{i,\rho} \rho}} \sum_{m=-\infty}^{\infty} \frac{J_m(k_{i,\rho} a)}{H_m^{(2)}(k_{i,\rho} a)} \exp\left[ jm(\phi_i - \phi) \right] \]
Scattering from cylindrical objects

Scattering cross section (scattering width)

- Scattering amplitude:

\[
|f^{TM} (\phi, \phi_i)|^2 = \frac{2}{\pi k_i, \rho} \left| \sum_{m=-\infty}^{\infty} \frac{J_m(k_i, \rho a)}{H_m^{(2)}(k_i, \rho a)} \exp\left[jm(\phi_i - \phi_s)\right] \right|^2
\]

- Scattering width:

\[
\sigma_{w}^{TM} = \frac{k_{i, \rho}}{k} \int_{0}^{2\pi} |f^{TM} (\phi, \phi_i)|^2 \, d\phi = \frac{4}{k} \sum_{m=-\infty}^{\infty} \left| \frac{J_m(k_i, \rho a)}{H_m^{(2)}(k_i, \rho a)} \right|^2
\]
Scattering cross section (scattering width)

- Example: thin cylinder

\[
\sigma_{w}^{TE} = \frac{4}{k} \sum_{m=-\infty}^{\infty} \left| \frac{J'_m(k_i, \rho a)}{H_m^{(2)'}(k_i, \rho a)} \right|^2 = \frac{3\pi^2}{4k} (k_i, \rho a)^4 + O((k_i, \rho a)^8)
\]

\[
\sigma_{w}^{TM} = \frac{4}{k} \sum_{m=-\infty}^{\infty} \left| \frac{J_m(k_i, \rho a)}{H_m^{(2)}(k_i, \rho a)} \right|^2 = \frac{\pi^2}{k} \left[ \frac{1}{\ln(k_i, \rho a)} \right]^2 + O((k_i, \rho a)^4)
\]
Scattering cross section (scattering width)

- Numerical results for the TE case

\[ k \sigma_{TE}^W \]

\[ k_{i, \rho} a \]
Scattering cross section (scattering width)

- Plotted in a different way

\[
\frac{\sigma_{TE}^{W}}{2a} = k_{i,\rho} a
\]
Scattering cross section (scattering width)

- For the TM case

\[ k \sigma_{W}^{TM} \]

\[ k_{i, \rho} a \]
Scattering by an infinitely long dielectric cylinder

- So far we discussed a (perfectly) conducting cylinder.
- What about a dielectric cylinder? We can actually use the same machinery to solve the problem.
- Incident plane wave:

\[
E_i(r) = \sum_{m=-\infty}^{\infty} u_m M^J_{m,k_i,z}(\rho, \phi, z)
\]

\[
+ \sum_{m=-\infty}^{\infty} v_m N^J_{m,k_i,z}(\rho, \phi, z)
\]
Scattering by an infinitely long dielectric cylinder

- For the scattered wave (outside cylinder) we use the same expansion as before

\[ E_s(r) = \sum_{m=-\infty}^{\infty} a_m M^H_{m,k_i,z}(\rho, \phi, z) + \sum_{m=-\infty}^{\infty} b_m N^H_{m,k_i,z}(\rho, \phi, z) \]

- Inside the cylinder we use solutions based on Bessel function of 1\textsuperscript{st} kind (why?)

\[ E_c(r) = \sum_{m=-\infty}^{\infty} c_m M^J_{m,k_i,z}(\rho, \phi, z) + \sum_{m=-\infty}^{\infty} d_m N^J_{m,k_i,z}(\rho, \phi, z) \]
Scattering by an infinitely long dielectric cylinder

- Next step is to match the fields at the boundary:
  \[ \hat{\phi} \cdot (E_i + E_s) = \hat{\phi} \cdot E_c \]
  \[ \hat{z} \cdot (E_i + E_s) = \hat{z} \cdot E_c \]

- This again leads to algebraic equations for coefficients which can be solved

- Note that inside the cylinder

\[ k_{i,\rho} \rightarrow k_{c,\rho} = \sqrt{k_d^2 - k_{i,z}^2} \Rightarrow k_d^2 = \omega^2 \mu_0 \epsilon_d \]