Electromagnetic scattering

Graduate Course
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Contents of lecture 5:

- Scattering from a conductive wedge
- The scalar wave equation
- The vector solutions
- Introducing a source
- Solving the equations
- The scattering problem
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Introduction

- An example of another canonical, exactly solvable scattering problem is that of an infinite, perfectly conducting wedge.
- The wedge is assumed to have an angle of $\beta$.
- Note that the wedge is infinitely extended and infinitely long.
- Also, an important limiting case is when $\beta \to 0$. (Why?)
Scalar equation

- This time the system has no rotational symmetry
- Nonetheless, the ‘natural’ waves of the system can still be found by solution of the wave equation in cylindrical coordinates
- As before, first consider the scalar wave equation

\[
\left(\nabla^2 + k^2\right) \psi = 0
\]

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0
\]

In cylindrical coordinates
Scalar equation

- The solution should be found for the angles

\[ 0 < \phi < 2\pi - \beta \]

- Let us consider solutions of the type

\[ \psi_{v,k_z}(\rho, \phi, z) = f_{v,k_z}(\rho) \exp(-j k_z z) \begin{cases} \sin(v\phi) \\ \cos(v\phi) \end{cases} \]

- Where the radial function \( f \) satisfies

\[ \rho \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f_{v,k_z}}{\partial \rho} \right) + \left( k^2 \rho + v^2 \right) f_{v,k_z} = 0 \]

\[ k_\rho = \sqrt{k^2 - k_z^2} \]

\[ f_{v,k_z}(\rho) = AJ_v(k_\rho \rho) + BY_v(k_\rho \rho) \]
Vector solution

- For the moment we do not impose any boundary conditions
- Solutions of the vector wave equation can once again be found:

\[ M_{v,k_z} = \frac{1}{k} \nabla \psi_{v,k_z} \times \hat{z} = -\frac{\partial \psi_{v,k_z}}{k \partial \rho} \hat{\phi} + \frac{\partial \psi_{v,k_z}}{k \rho \partial \phi} \hat{\rho} \]

\[ N_{v,k_z} = \frac{1}{k^2} \left[ -jk_z \frac{\partial \psi_{v,k_z}}{\partial \rho} \hat{\rho} - \frac{jk_z}{\rho} \frac{\partial \psi_{v,k_z}}{\partial \phi} \hat{\phi} + k^2 \rho \psi_{v,k_z} \hat{z} \right] \]

- These vectors can be used to construct the electric field in the region outside the wedge.
Vector solution

- But, which kind of function should we choose to satisfy the boundary conditions?

- For the $\mathbf{M}$ vector to be a legitimate electric field its radial component should vanish on the wedge surfaces for all $\rho$

\[
\frac{\partial \psi_{v,k_z}}{\partial \phi} = 0 \text{ for } \phi = 0, 2\pi - \beta
\]

\[
\psi^\text{TE}_{v,k_z} = f_{v,k_z}(\rho) \exp(-jk_zz)\cos(v\phi)
\]

\[
v = \frac{m\pi}{2\pi - \beta}
\]
Vector solution

- What about the $N$ vector? Now we should demand for all $\rho$

$$\frac{\partial \psi_{v,k_z}}{\partial \rho} = \psi_{v,k_z} = 0 \text{ for } \phi = 0, 2\pi - \beta$$

$$\psi_{v,k_z}^{TM} = f_{v,k_z}(\rho) \exp(-jk_zz) \sin(v\phi)$$

$$v = \frac{m\pi}{2\pi - \beta}$$
**Vector solution**

- **TE solution (horizontal polarization):**

  \[
  E^{TE} = M_{v,k_z} = -\frac{\partial \psi_{v,k_z}^{TE}}{k\partial \rho} \hat{\phi} + \frac{\partial \psi_{v,k_z}^{TE}}{k \rho \partial \phi} \hat{\rho}
  \]

  \[
  H^{TE} = -\frac{1}{j \omega \mu} \nabla \times E^{TE} = \frac{j}{\eta} N_{v,k_z}
  \]

  \[
  = \frac{j}{\eta k^2} \left[ -jk_z \frac{\partial \psi_{v,k_z}^{TE}}{\partial \rho} \hat{\rho} - \frac{j k_z}{\rho} \frac{\partial \psi_{v,k_z}^{TE}}{\partial \phi} \hat{\phi} + k^2 \rho \psi_{v,k_z}^{TE} \hat{z} \right]
  \]

- **No electric field along z, but magnetic field has a z-component**
Vector solution

- TM solution (vertical polarization)

\[
E^{TM} = N_{v,k_z} = \frac{1}{k^2} \left[ -jk_z \frac{\partial \psi^{TM}_{v,k_z}}{\partial \rho} \hat{\rho} - \frac{jk_z}{\rho} \frac{\partial \psi^{TM}_{v,k_z}}{\partial \phi} \hat{\phi} + k^2 \psi^{TM}_{v,k_z} \hat{z} \right]
\]

\[
H^{TM} = -\frac{1}{j\omega \mu} \nabla \times E^{TM} = \frac{j}{\eta} M_{v,k_z}
\]

\[
= \frac{j}{\eta} \left[ -\frac{\partial \psi^{TM}_{v,k_z}}{k \partial \rho} \hat{\phi} + \frac{\partial \psi^{TM}_{v,k_z}}{k \rho \partial \phi} \hat{\rho} \right]
\]

- No magnetic field along z, but electric field has a z component
Vector solution

- Here, unlike the problem of a conducting cylinder, the solution extends to $\rho=0$.

- To ensure the finiteness of the electromagnetic energy (not necessarily field) near the origin we should have

$$f_{v,k_z}(\rho) = J_v(k_\rho \rho)$$

$$\psi_{v,k_z}^{TE} = J_v(k_\rho \rho) \exp(-jk_z z) \cos(v\phi)$$

$$\psi_{v,k_z}^{TM} = J_v(k_\rho \rho) \exp(-jk_z z) \sin(v\phi)$$
Vector solution

- The *total* electric field which
  - satisfies boundary conditions at surfaces of conducting wedge
  - and ensures ‘correct behavior’ as $\rho \to 0$
- Is a superposition of these TE and TM vector solutions
- Remember: this representation is for the total field; for the scattering problem we cannot separate the incident and scattered waves in this representation
Vector solutions in the scattering problem

- If the incident wave has a certain $k_{i,z}$, then we know from the uniformity of the system in the $z$ direction that the total solution consists of waves with

$$k_z = k_{i,z} \quad k_\rho = k_{i,\rho} = \sqrt{k^2 - k_{i,z}^2}$$

- Then, for example for the TE case, the total field can look like

$$E^{TE} = \sum_v a_v \left[ -\frac{\partial \psi^{TE}_{v,k_i,z}}{k \partial \rho} \hat{\phi} + \frac{\partial \psi^{TE}_{v,k_i,z}}{k \rho \partial \phi} \hat{\rho} \right]$$

$$\psi^{TE}_{v,k_i,z} = J_v(k_{i,\rho} \rho) \cos(v\phi) \exp(-jk_{i,z}z)$$
Vector solutions in the scattering problem

- Or:

\[ E^{TE} = \sum_v a_v \left[ -\frac{k_{i,\rho}}{k} J'_v(k_{i,\rho}\rho) \cos(v\phi) \hat{\phi} \right. \]

\[ \left. -\nu \frac{J_v(k_{i,\rho}\rho)}{k \rho} \sin(v\phi) \hat{\rho} \right] \exp(-jk_{i,z}z) \]

- Note: electric field is not necessarily finite as \( \rho \to 0 \), it is electric energy density which should remain integrable.
Vector solutions in the scattering problem

- For TM case or vertical polarization

\[ E^{TM} = \sum_v a_v \frac{1}{k^2} \left[ -j k_{i,z} \frac{\partial \psi^{TM}_{v,k_{i,z}}}{\partial \rho} \hat{\rho} - j k_z \frac{\partial \psi^{TM}_{v,k_{i,z}}}{\partial \phi} \hat{\phi} + k^2 \rho \psi^{TM}_{v,k_{i,z}} \hat{z} \right] \]

\[ \psi^{TM}_{v,k_{i,z}} = J_v(k_{i,\rho}) \sin(v\phi) \exp(-j k_{i,z} z) \]

\[ E^{TM} = \sum_v a_v \left[ -j \frac{k_{i,z} k_{i,\rho}}{k^2} J'_v(k_{i,\rho}) \sin(v\phi) \hat{\rho} - \nu \frac{j k_{i,z}}{k^2 \rho} J_v(k_{i,\rho}) \cos(v\phi) \hat{\phi} \right. \]

\[ \left. + \frac{k_{i,\rho}^2}{k^2} J_v(k_{i,\rho}) \sin(v\phi) \hat{z} \right] \exp(-j k_{i,z} z) \]
Vector solutions in the scattering problem

- So far, it is fine, but unfortunately it is not so easy to find the constant coefficients $a_i$ to describe the scattering solution for a particular incident wave propagating in a certain direction.

- For a cylinder we could separate the incident and scattered field from the beginning and then apply the boundary condition on the cylinder surface.

- Also, it was essential to expand the incident plane wave in Bessel functions in order to be able to solve the problem. This is not possible here because of the non-integer Bessel functions used.
Introducing a source

- The traditional way to treat this problem is by solving the problem in presence of the sources.

- Once the problem is solved, the sources is moved to a point infinitely far away. The field generated by such a source is a far zone field and has a plane-wave character.

- Therefore by taking this limit of the full solution the problem is solved.
Introducing a source

- Let us consider this problem in the simpler case where \( k_z = 0 \). So, we are interested in waves which are uniform along \( z \).

- From the field expansion discussed, we expect the total electric field vector to look like

\[
E^{TE} = \sum_{\nu} a_{\nu} \left[-J'_\nu(k\rho)\cos(\nu\phi) \hat{\phi} - \nu \frac{J_\nu(k\rho)}{k\rho} \sin(\nu\phi) \hat{\rho}\right]
\]

\[
E^{TM} = \hat{z} \sum_{\nu} a_{\nu} J_\nu(k\rho) \sin(\nu\phi)
\]

- Here we used \( k_{i,z} = 0 \rightarrow k_{i,\rho} = k \)
Introducing a source

Let us return to Maxwell equations, with electric and magnetic line current sources along z, and uniform along z.

\[ \nabla \times \mathbf{E} = -j\omega \mu \mathbf{H} - \mathbf{M}^i \]

\[ \nabla \times \mathbf{H} = j\omega \varepsilon \mathbf{E} + \mathbf{J}^i \]

\[ \mathbf{J}^i = J_z \mathbf{\hat{z}} \quad \mathbf{M}^i = M_z \mathbf{\hat{z}} \]
Introducing a source

- With Maxwell equations in cylindrical coordinates, and fields uniform in z-direction:

\[
\begin{align*}
\frac{\partial E_z}{\rho \partial \phi} &= -j \omega \mu H_\rho \\
- \frac{\partial E_z}{\partial \rho} &= -j \omega \mu H_\phi \\
\frac{\partial}{\rho \partial \rho} (\rho E_\phi) - \frac{\partial E_\rho}{\rho \partial \phi} &= -j \omega \mu H_z - M^i_z \\
\frac{\partial}{\rho \partial \rho} (\rho H_\phi) - \frac{\partial H_\rho}{\rho \partial \phi} &= j \omega \epsilon E_\rho + J^i_z
\end{align*}
\]
Introducing a source

- Two separate sets for horizontal and vertical polarizations

Vertical electric field TM

\[
\frac{\partial E_z}{\rho \partial \phi} = -j \omega \mu H_\rho \\
- \frac{\partial E_z}{\partial \rho} = -j \omega \mu H_\phi \\
\frac{\partial}{\rho \partial \rho} \left( \rho H_\phi \right) - \frac{\partial H_\rho}{\rho \partial \phi} = j \omega \epsilon E_z + J^i_z
\]

Horizontal electric field TE

\[
\frac{\partial H_z}{\rho \partial \phi} = j \omega \epsilon E_\rho \\
- \frac{\partial H_z}{\partial \rho} = j \omega \epsilon E_\phi \\
\frac{\partial}{\rho \partial \rho} \left( \rho E_\phi \right) - \frac{\partial E_\rho}{\rho \partial \phi} = -j \omega \mu H_z - M^i_z
\]

Horizontal magnetic field

Vertical magnetic field

Scattering from a conductive wedge
Introducing a source

- 1\textsuperscript{st} set leads to:

\[ \frac{\partial}{\partial \rho} \left( \rho \frac{\partial E_z}{\partial \rho} \right) + \frac{\partial^2 E_z}{\rho^2 \partial \phi^2} + k^2 E_z = j \omega \mu J^i_z \]

- 2\textsuperscript{nd} set gives:

\[ \frac{\partial}{\partial \rho} \left( \rho \frac{\partial H_z}{\partial \rho} \right) + \frac{\partial^2 H_z}{\rho^2 \partial \phi^2} + k^2 H_z = j \omega \epsilon M^i_z \]

- Left hand side same as the scalar wave equation in cylindrical coordinates for uniform fields along $z$
Introducing a source

- We would like to solve these equations in presence of the conducting wedge, for infinitely thin line sources

\[ J_z^i = \frac{I_0}{\rho_0} \delta(\rho - \rho_0) \delta(\phi - \phi_0) \]

\[ M_z^i = \frac{M_0}{\rho_0} \delta(\rho - \rho_0) \delta(\phi - \phi_0) \]

- Boundary conditions:

\[ E_z = E_\rho = 0 \quad \text{for} \quad \phi = 0 \]

\[ E_z = E_\rho = 0 \quad \text{for} \quad \phi = 2\pi - \beta \]
Solving the equations (TM)

- Consider the TM case first.

Solution in the region $\rho < \rho_0$ which satisfies boundary conditions at the wedge surfaces, and shows the right behavior at the origin is

$$E_{z}^{TM}(\rho, \phi) = \sum_{\nu} a_{\nu} J_{\nu}(k \rho) \sin(\nu \phi)$$

$$\nu = \frac{m\pi}{2\pi - \beta}$$
Solving the equations (TM)

- Solution in the region $\rho > \rho_0$
  which satisfies the wedge
boundary conditions, and the
radiation condition at the
infinity is

$$E_z^{TM} (\rho, \phi) = \sum_{\nu} b_{\nu} H_{\nu}^{(2)} (k \rho) \sin (\nu \phi)$$

$$\nu = \frac{m \pi}{2 \pi - \beta}$$
Solving the equations (TM)

- Next we have to match the solutions at the source by demanding that

\[ E_{z}^{TM}(\rho_0^+, \phi) = E_{z}^{TM}(\rho_0^-, \phi) \]

\[
\frac{\partial E_{z}^{TM}}{\partial \rho} \bigg|_{\rho_0^+} - \frac{\partial E_{z}^{TM}}{\partial \rho} \bigg|_{\rho_0^-} = j\omega\mu \frac{I_0}{\rho_0} \delta(\phi - \phi_0)
\]
Solving the equations (TM)

- Resulting equations (m is shown explicitly)

\[
\sum_{m=1}^{\infty} b_{\nu_m} H_{\nu_m}^{(2)}(k\rho_0) \sin(\nu_m \phi) = \sum_{m=1}^{\infty} a_{\nu_m} J_{\nu_m}(k\rho_0) \sin(\nu_m \phi)
\]

\[
\sum_{m=1}^{\infty} b_{\nu_m} H_{\nu_m}^{(2)'}(k\rho_0) \sin(\nu_m \phi) - \sum_{m=1}^{\infty} a_{\nu_m} J_{\nu_m}'(k\rho_0) \sin(\nu_m \phi)
\]

\[= j\omega\mu \frac{I_0}{k\rho_0} \delta(\phi - \phi_0)\]

\[\nu_m = \frac{m\pi}{2\pi - \beta}\]

Scattering from a conductive wedge
Solving the equations (TM)

\[ b_{vm} H_{vm}^{(2)}(k \rho_0) = a_{vm} J_{vm}(k \rho_0) \]

\[ b_{vm} H_{vm}^{(2)'}(k \rho_0) - a_{vm} J_{vm}'(k \rho_0) = \frac{2j \omega \mu}{2\pi - \beta k \rho_0} \frac{I_0}{W_{vm}} \sin (\nu_m \phi_0) \]

\[ a_{vm} = \frac{I_0}{k \rho_0} \frac{2j \omega \mu}{2\pi - \beta} \frac{H_{vm}^{(2)}(k \rho_0)}{W_{vm}} \sin (\nu_m \phi_0) \]

\[ W_{vm} = J_{vm}(k \rho_0) H_{vm}^{(2)'}(k \rho_0) - J_{vm}'(k \rho_0) H_{vm}^{(2)}(k \rho_0) = -j \left[ J_{vm}(k \rho_0) Y_{vm}'(k \rho_0) - J_{vm}'(k \rho_0) Y_{vm}(k \rho_0) \right] = \frac{-2j}{\pi k \rho_0} \]
Solving the equations (TM)

\[ a_{vm} = -I_0 \frac{\pi \omega \mu}{2\pi - \beta} H^{(2)}_{\nu m}(k \rho_0) \sin(\nu_m \phi_0) \]

- Solution for \( \rho < \rho_0 \)

\[ E_{z}^{TM} (\rho, \phi) = -I_0 \frac{\pi \omega \mu}{2\pi - \beta} \sum_{m=0}^{\infty} H^{(2)}_{\nu m}(k \rho_0) \sin(\nu_m \phi_0) J_{\nu m}(k \rho) \sin(\nu_m \phi) \]
Plane wave solution (TM)

- What if we had no wedge?
- Then the equation to be solved for this case is the same, but has to be solved for a source in free space

\[
\frac{\partial}{\partial \rho} \left( \rho \frac{\partial E_z}{\partial \rho} \right) + \frac{\partial^2 E_z}{\partial \phi^2} + k^2 E_z = \frac{j \omega \mu I_0}{\rho_0} \delta(\rho - \rho_0) \delta(\phi - \phi_0)
\]

- The solution is

\[
E_z(\rho, \phi) = -\frac{\omega \mu I_0}{4} H_0^{(2)}(kR) \quad R = \sqrt{\rho^2 + \rho_0^2 - 2\rho_0 \rho \cos(\phi - \phi_0)}
\]
Plane wave solution (TM)

- Now, if we move the source to a point far away in the free space problem then

\[ E_z (\rho, \phi) \sim -\frac{\omega \mu I_0}{4} H_0^{(2)} (kR) \sim -\frac{\omega \mu I_0}{4} \frac{2}{\pi kR} \exp(-jkR + j\pi / 4) \]

\[ R = \sqrt{\rho^2 + \rho_0^2 - 2\rho_0 \rho \cos(\phi - \phi_0)} \approx \rho_0 - \rho \cos(\phi - \phi_0) \]

\[ E_z (\rho, \phi) \sim A_0 \exp\left[jk \rho \cos(\phi - \phi_0)\right] \]

\[ A_0 = -\frac{\omega \mu I_0}{4} \frac{2}{\pi k\rho_0} \exp(-jk\rho_0 + j\pi / 4) \]
Plane wave solution (TM)

- Apart from its amplitude this is the field of a plane wave propagating towards the origin along the angle $\phi_0$

\[
A_0 \exp\left[ jk \rho \cos(\phi - \phi_0) \right] = A_0 \exp(-jk_i \cdot r)
\]

\[
k_i = (-k \cos \phi_0, -k \sin \phi_0, 0)
\]

\[
r = (\rho \cos \phi, \rho \sin \phi, z)
\]
So, in the original problem, if we take the same limit $\rho_0 \to \infty$, while keeping the amplitude constant, the resulting solution will be that of the total field for the incident plane wave along $\phi_0$.

Note: keeping amplitude constant means that we should put

$$I_0 = -\frac{4}{\omega \mu} \sqrt{\frac{\pi k \rho_0}{2}} \exp\left( jk \rho_0 - j \frac{\pi}{4} \right) A_0$$
Plane wave solution (TM)

- Result:

\[ E_{z}^{TM}(\rho, \phi) = A_0 \frac{4\pi}{2\pi - \beta} \sum_{m=1}^{\infty} (j)^{\nu_m} \sin(\nu_m \phi_0) J_{\nu_m}(k\rho) \sin(\nu_m \phi) \]

- This way we have found the coefficients for the total electric field when the incident wave is a plane wave with

\[ \mathbf{k}_i = (-k \cos \phi_0, -k \sin \phi_0, 0) \]
Plane wave solution (TM)

- The corresponding magnetic field is

\[
H_{\phi}^{TM}(\rho, \phi) = \frac{A_0 k}{j \omega \mu} \frac{4\pi}{2\pi - \beta} \sum_{m=1}^{\infty} (j)^{\nu_m} \sin(\nu_m \phi_0) J'_{\nu_m}(k \rho) \sin(\nu_m \phi)
\]

\[
H_{\rho}^{TM}(\rho, \phi) = -\frac{A_0}{j \omega \mu} \frac{4\pi}{2\pi - \beta} \sum_{m=1}^{\infty} (j)^{\nu_m} \frac{\nu_m}{\rho} \sin(\nu_m \phi_0) J_{\nu_m}(k \rho) \cos(\nu_m \phi)
\]
**TE scattering**

- In a similar way, the TE scattering problem can be solved.

- Now we solve the equation for the magnetic field

\[
\frac{\partial}{\rho \partial \rho} \left( \rho \frac{\partial H_{z}^{TE}}{\partial \rho} \right) + \frac{\partial^2 H_z^{TE}}{\partial \phi^2} + k^2 H_z^{TE} = j\omega \varepsilon M_z^i
\]

\[
M_z^i = \frac{M_z^0}{\rho_0} \delta (\rho - \rho_0) \delta (\phi - \phi_0)
\]

- With the boundary condition

\[
\frac{\partial H_z^{TE}}{\partial \phi} = 0 \quad \text{for} \quad \phi = 0, 2\pi - \beta
\]
TE scattering

- The overall solution can be written as

\[
H_{z}^{TE}(\rho, \phi) = \sum_{m=0}^{\infty} a_{\nu_{m}} J_{\nu_{m}}(k \rho) \cos(\nu_{m} \phi) \quad \rho < \rho_0
\]

\[
H_{z}^{TE}(\rho, \phi) = \sum_{m=0}^{\infty} b_{\nu_{m}} H_{\nu_{m}}^{(2)}(k \rho) \cos(\nu_{m} \phi) \quad \rho > \rho_0
\]

- At the line source, they should be matched by

\[
H_{z}^{TE}(\rho_0^{+}, \phi) = H_{z}^{TE}(\rho_0^{-}, \phi)
\]

\[
\frac{\partial H_{z}^{TE}}{\partial \rho} \bigg|_{\rho_0^{+}} - \frac{\partial H_{z}^{TE}}{\partial \rho} \bigg|_{\rho_0^{-}} = j \omega \mu \frac{M_0}{\rho_0} \delta(\phi - \phi_0)
\]
TE scattering

- Due to the analogy with the TM case, we conclude that if the incident magnetic field is

\[ H_z^i = A_0 \exp(-jk_i \cdot r) \]

- Then the total magnetic field is given by

\[ H_{z}^{TE}(\rho, \phi) = A_0 \frac{2\pi}{2\pi - \beta} \sum_{m=0}^{\infty} \tau_m (j)^m \cos(v_m \phi_0) J_{v_m}(k\rho) \cos(v_m \phi) \]

\[ \tau_m = \begin{cases} 1 & m = 0 \\ 2 & m \geq 1 \end{cases} \]
TE scattering

- But what about the electric field?
- The (horizontal) incident field is
  \[
  E_z^i = \frac{1}{j\omega \varepsilon} \nabla \times \left( H_z^i \hat{z} \right) 
  = A_0 \eta \exp(-jk_i \cdot r) (\hat{z} \times k_i) 
  \]
- The total electric field is
  \[
  E^{TE} = \frac{1}{j\omega \varepsilon} \left( \frac{\partial H_z^{TE}}{\rho \partial \phi} \hat{\rho} - \frac{\partial H_z^{TE}}{\partial \rho} \hat{\phi} \right) 
  \]
TE scattering

\[ E_{TE}(\rho, \phi) = -j \eta A_0 \frac{2\pi}{2\pi - \beta} \sum_{m=0}^{\infty} \tau_m (j)^\nu_m \cos(\nu_m \phi_0) \]

\[ \left[ -\nu_m \frac{J_{\nu_m}(k\rho)}{k\rho} \sin(\nu_m \phi) \hat{\rho} - J'_{\nu_m}(k\rho) \cos(\nu_m \phi) \hat{\phi} \right] \]

- Compare with the general expression found before

Scattering from a conductive wedge
Special cases

- Let us consider some specific cases. First when $\beta = \pi$.
- Corresponds to reflection of a plane incident wave by a flat, infinite perfect conductor.
- Solution for vertical fields (TM)

$$
\nu_m = \frac{m\pi}{2\pi - \beta} = m
$$

$$
E^{TM}_z (\rho, \phi) = 4A_0 \sum_{m=1}^{\infty} (j)^m \sin (m\phi_0) J_m (k\rho) \sin (m\phi)
$$
Special cases

\[ E_{z}^{TM}(\rho, \phi) = 4A_{0} \sum_{m=1}^{\infty} (j)^{m} \sin(m\phi_{0}) J_{m}(k\rho) \sin(m\phi) \]

- We can use the relationship

\[ \exp(jz \cos \vartheta) = J_{0}(z) + 2 \sum_{m=1}^{\infty} (j)^{m} \cos(m\vartheta) J_{m}(z) \]

- To arrive at

\[ E_{z}^{TM}(\rho, \phi) = A_{0} \exp\left[jk\rho \cos(\phi - \phi_{0})\right] - A_{0} \exp\left[jk\rho \cos(\phi + \phi_{0})\right] \]

**Incident wave** \[ A_{0} \exp(-jk_{i} \cdot r) \]
Special cases

- The remaining (scattered) term corresponds to:

\[ A_0 \exp\left[jk\rho \cos(\phi + \phi_0)\right] = A_0 \exp(-jk_s \cdot r) \]

\[ k_s = (-k \cos \phi_0, -k \sin \phi_0, 0) \]

- This is what one would have expected.

- The TE case is similar except for the sign of the reflected (magnetic) field.
Special cases

- Another important case is the limit $\beta \to 0$
- This is the limit of a half-infinite plane which is quite important
- Total fields:

$$E_{z}^{TM}(\rho, \phi) = 2A_0 \sum_{m=1}^{\infty} (j)^{m/2} \sin\left(\frac{m\phi_0}{2}\right) J_{m/2}(k\rho) \sin\left(\frac{m\phi}{2}\right)$$

$$H_{z}^{TE}(\rho, \phi) = 2A_0 \sum_{m=0}^{\infty} \tau_m (j)^{m/2} \cos\left(\frac{m\phi_0}{2}\right) J_{m/2}(k\rho) \cos\left(\frac{m\phi}{2}\right)$$
Numerical examples

- We plot the scattered field as function of the observation angle (at a certain distance) for a certain incoming wave.

- TM scattered field:

\[
E_{z}^{TM}(\rho, \phi) = A_0 \frac{4\pi}{2\pi - \beta} \sum_{m=1}^{\infty} (j)^{\nu_m} \sin(\nu_m \phi_0) J_{\nu_m}(k\rho) \sin(\nu_m \phi)
\]

\[
E_{s,z}^{TM}(\rho, \phi) = E_{z}^{TM}(\rho, \phi) - A_0 \exp(-j k_i \cdot r)
\]

\[
= E_{z}^{TM}(\rho, \phi) - A_0 \exp\left[ jk_0 \rho \cos(\phi - \phi_0) \right]
\]
Numerical examples (TM)

\[ \beta = 45^\circ, \ k\rho = 2 \]

\[ \phi_0 = 30^\circ \]

\[ \phi_0 = 90^\circ \]
Numerical examples (TM)

\[ \beta = 45^\circ, \ k \rho = 2 \]

\[ \phi_0 = 180^\circ \quad \text{and} \quad \phi_0 = 225^\circ \]
Numerical examples (TM)

\[ \beta = 0^\circ, \ k \rho = 2 \]

\[ \phi_0 = 30^\circ \]

\[ \phi_0 = 90^\circ \]

\[ \phi_0 = 180^\circ \]
Numerical examples (TE)

- For TE scattering, we plot the scattered magnetic field as function of the observation angle (at a certain distance) for a certain incoming wave.

- TE scattered field:

\[
H_{z}^{TE}(\rho, \phi) = A_0 \frac{2\pi}{2\pi - \beta} \sum_{m=0}^{\infty} \tau_{m}(j)^{\nu_m} \cos(\nu_m \phi_0) J_{\nu_m}(k\rho) \cos(\nu_m \phi)
\]

\[
H_{s,z}^{TE}(\rho, \phi) = H_{z}^{TE}(\rho, \phi) - A_0 \exp(-j k_i \cdot r)
\]

\[
= H_{z}^{TE}(\rho, \phi) - A_0 \exp\left[j k \rho \cos(\phi - \phi_0) \right]
\]
Numerical examples (TE)

$\beta = 0^\circ, \ k\rho = 2$

$\phi_0 = 30^\circ$  $\phi_0 = 90^\circ$  $\phi_0 = 180^\circ$