Electromagnetic scattering

Graduate Course

Electrical Engineering (Communications)

1st Semester, 1388-1389

Sharif University of Technology

Contents of lecture 6

- **Contents of lecture 6:**
  - Scattering from spherical objects
  - Scalar waves in spherical coordinates
    - Spherical harmonics
    - Spherical Bessel functions
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  - Far field behavior of solutions
  - Expansion of a plane wave
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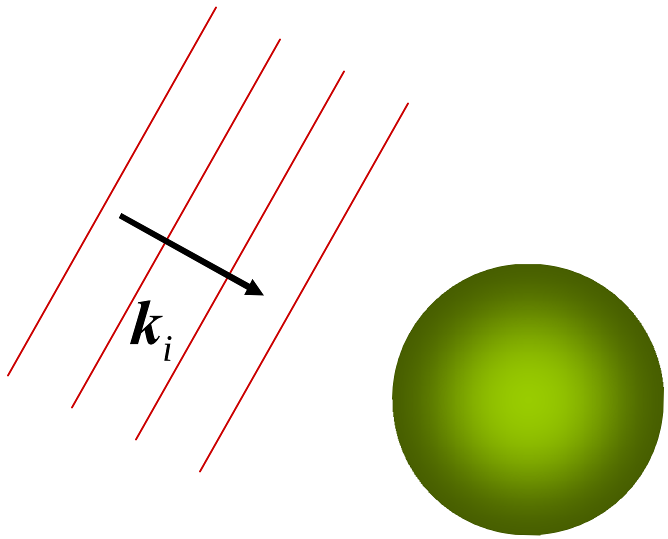
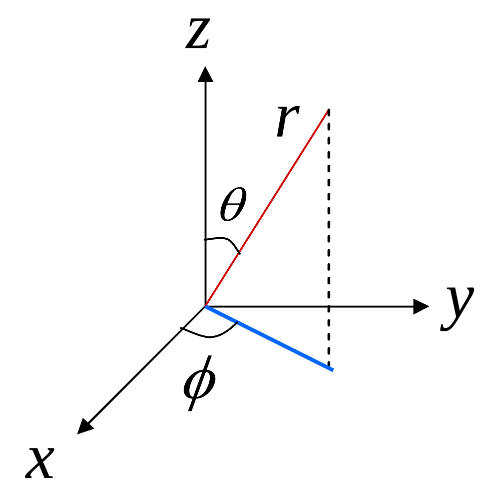
Introduction

- The previously analyzed canonical problems were exactly solvable.
- But they were all related to objects which were infinitely extended in one (cylinder, wedge) or two (layered media) directions. (Although approximations can be made for finite structures.)
- A conducting or dielectric sphere is one of the (few) problems which are exactly solvable and involve a finite object



Introduction

- Since the system has spherical symmetry, we analyze the problem in terms of the 'natural' solutions of the wave equation in spherical coordinates
- We first consider the scalar wave equation in a homogeneous medium



$$\left(\nabla^2 + k^2\right)\psi = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} + k^2\psi = 0$$

Wave equation in spherical coordinates

Scalar waves in spherical coordinates

- This is a classic problem. Let us represent the solution as

$$\psi(r,\theta,\phi) = f(r)Y(\theta,\phi)$$

- The functions should satisfy the equations

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2} = -\lambda Y(\theta,\phi)$$

$$\frac{d}{dr}\left(r^2\frac{df(r)}{dr}\right)+\left(k^2r^2-\lambda\right)f(r) = 0$$

λ: constant to be specified

Spherical harmonics

- The solutions of the first problem (eigenfunctions) are called the spherical harmonics
- Since we are going to deal with fields which are periodic functions of the angle ϕ, we look for solutions of the type

$$Y(\theta,\phi) = g(\theta)\exp(jm\phi)$$

$$\Rightarrow \quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) - \frac{m^2}{\sin^2\theta}g = -\lambda g$$

$$u = \cos\theta \quad \Rightarrow \quad (1-u^2)\frac{d^2g}{du^2} - 2u\frac{dg}{du} - \frac{m^2}{1-u^2}g + \lambda g = 0$$

Spherical harmonics

- This is the differential equation for Legendre functions
- Requiring analytic properties, solutions are only possible when

$$\lambda = \ell(\ell+1) \qquad \ell = 0, 1, 2, \cdots$$

: nonnegative integer

- Corresponding solutions: associated Legendre functions of the first kind (2^{nd} kind is not analytic when $\theta \to 0, \pi$ or $u \to \pm 1$

$$g(\theta) = P_\ell^m(\cos\theta)$$

$$P_\ell^m(u) = \frac{(-1)^m}{2^\ell \ell!}(1-u^2)^{m/2}\frac{d^{\ell+m}(u^2-1)^\ell}{du^{\ell+m}} \qquad -\ell \le m \le \ell$$

Spherical harmonics

- Note that m=0 corresponds to the so called Legendre polynomials. Even values of m result in polynomials in general, but odd values do not.

- Here are some examples:

$$P_0^0(\cos\theta) = 1$$

$$\ell = 0$$

$$P_1^{-1}(\cos\theta) = \frac{1}{2}\sin\theta$$

$$P_1^0(\cos\theta) = \cos\theta$$

$$P_1^1(\cos\theta) = -\sin\theta$$

$$\ell = 1$$

$$P_2^{-2}(\cos\theta) = \frac{1}{8}\sin^2\theta$$

$$P_2^{-1}(\cos\theta) = \frac{1}{2}\sin\theta\cos\theta$$

$$P_2^0(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right)$$

$$P_2^1(\cos\theta) = -3\sin\theta\cos\theta$$

$$P_2^2(\cos\theta) = 3\sin^2\theta$$

$$\ell = 2$$

Spherical harmonics

- Note that for positive and negative values of m these functions are not independent from each other

$$P_\ell^{-m}(u) = (-1)^m \frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(u)$$

- Orthogonal properties of Legendre functions

$$\int_{-1}^{1} P_\ell^m(u) P_{\ell'}^m(u) du = \int_0^\pi P_\ell^m(\cos\theta) P_{\ell'}^m(\cos\theta) \sin\theta d\theta$$

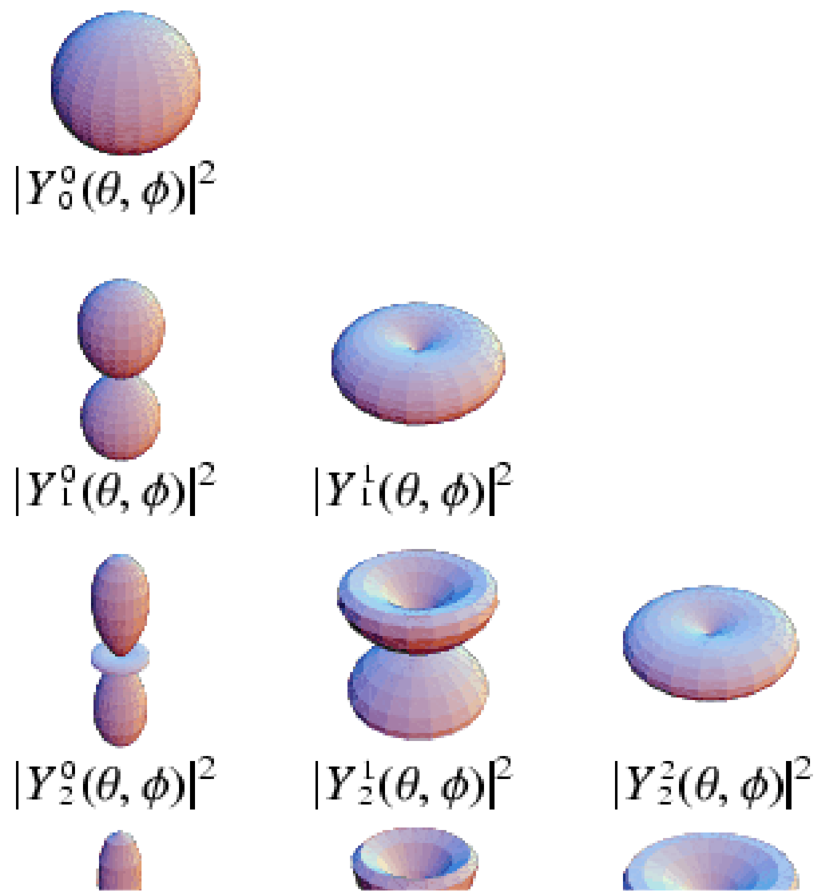
$$= \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'}$$

Spherical harmonics

- Summarizing: $Y_\ell^m(\theta,\phi) = P_\ell^m(\cos\theta)\exp(jm\phi)$ $-\ell \le m \le \ell$

- Visualization: here is the amplitude of these functions

- Negative *m*'s have not been shown as they have the same distribution as positive *m*'s



Spherical harmonics

- Orthogonality of spherical harmonics:

$$\int_{4\pi} Y_\ell^m(\theta,\phi) Y_{\ell'}^{-m'}(\theta,\phi) d\Omega = \int_0^\pi \int_0^{2\pi} Y_\ell^m(\theta,\phi) Y_{\ell'}^{-m'}(\theta,\phi) \sin\theta d\theta d\phi$$

$$= (-1)^m \frac{4\pi}{2\ell+1} \delta_{\ell\ell'} \delta_{mm'}$$

- Completeness of spherical harmonics:

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (-1)^m (2\ell+1) Y_\ell^m(\theta,\phi) Y_\ell^{-m}(\theta',\phi')$$

$$= \delta(\phi-\phi')\delta(\cos\theta - \cos\theta')$$

Spherical Bessel functions

- Now, consider the equation for radial distance

$$\frac{d}{dr}\left(r^2\frac{df}{dr}\right)+\left[k^2r^2-\ell\left(\ell+1\right)\right]f=0$$

- Solution: linear combinations of the spherical Bessel functions

$$j_\ell(kr) \text{ and } y_\ell(kr)$$

$$j_\ell(z)=\sqrt{\frac{\pi}{2z}}\,J_{\ell+\frac{1}{2}}(z) \qquad y_\ell(z)=\sqrt{\frac{\pi}{2z}}\,Y_{\ell+\frac{1}{2}}(z)$$

Spherical Bessel functions

- Examples:

$$j_0(z) = \frac{\sin z}{z}$$

$$j_1(z) = -\frac{\cos z}{z} + \frac{\sin z}{z^2}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z$$

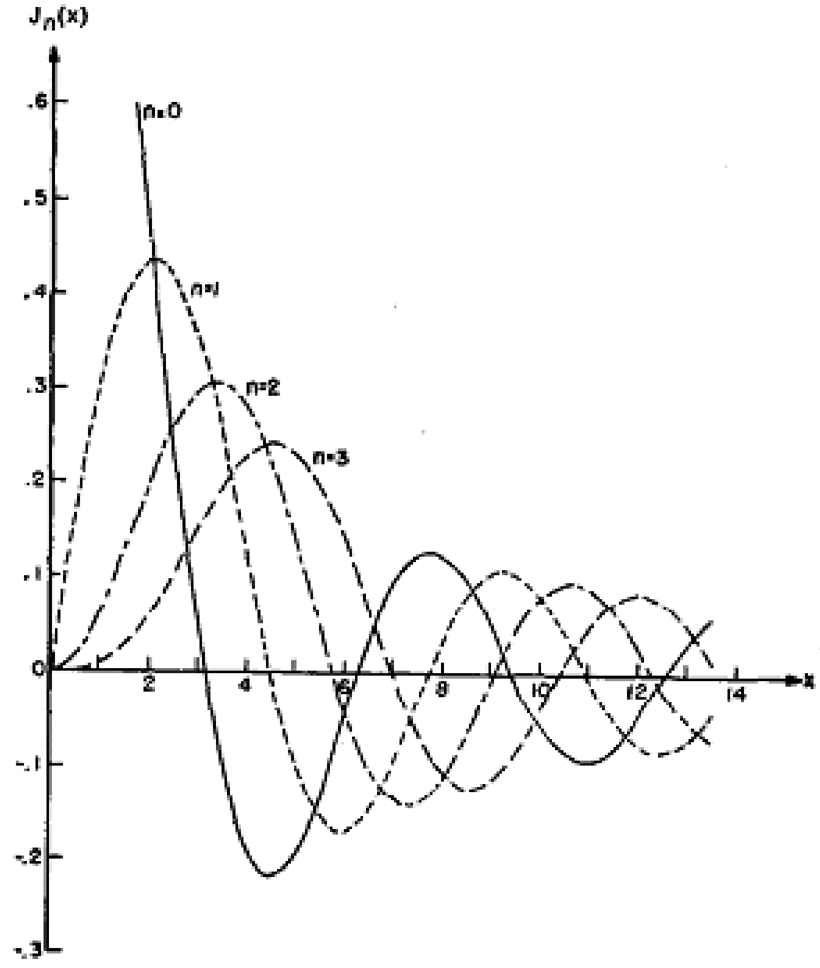


FIGURE 10.1. $j_n(x)$. $n=0(1)3$.

Spherical Bessel functions

- Examples:

$$y_0(z) = -\frac{\cos z}{z}$$

$$y_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}$$

$$y_2(z) = \left(-\frac{3}{z^3} + \frac{1}{z}\right)\cos z - \frac{3}{z^2}\sin z$$

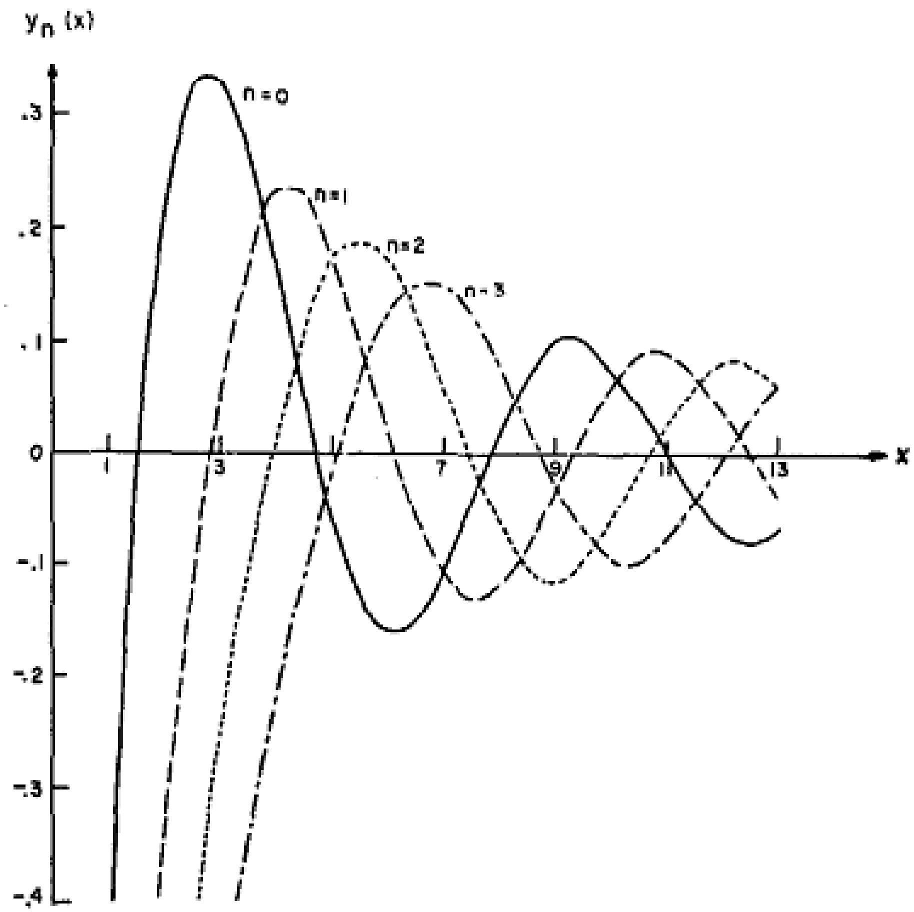


FIGURE 10.2. $y_n(x)$. $n=0(1)3$.

Spherical Bessel functions

- The function f may also be represented as a linear combination of the spherical Hankel functions: $h_\ell^{(1)}(kr)$ and $h_\ell^{(2)}(kr)$

$$h_\ell^{(1)}(z) = j_\ell(z) + jy_\ell(z) = \sqrt{\frac{\pi}{2z}} H_{\ell+1/2}^{(1)}(z)$$

$$h_\ell^{(2)}(z) = j_\ell(z) - jy_\ell(z) = \sqrt{\frac{\pi}{2z}} H_{\ell+1/2}^{(2)}(z)$$

- Overall solution

$$\psi(r,\theta,\phi) = f_\ell(r) \underbrace{P_\ell^m(\cos\theta)\exp(jm\phi)}_{Y_\ell^m(\theta,\phi)}$$

Vector wave equation

- Aim: solutions of the *vector* wave equation

$$\nabla\times\left(\nabla\times\boldsymbol{E}\right)-k^2\boldsymbol{E}=0$$

- Assume that $\psi(\boldsymbol{r})$ is a solution of the ‘scalar’ wave equation; also consider a known vector field $\boldsymbol{a}(\boldsymbol{r})$ whose curl is zero

- Then solutions of the vector wave equation are

$$\boldsymbol{M}(\boldsymbol{r})=\frac{1}{k}\nabla\times\left[\psi(\boldsymbol{r})\boldsymbol{a}(\boldsymbol{r})\right] \qquad \boldsymbol{N}(\boldsymbol{r})=\frac{1}{k}\nabla\times\boldsymbol{M}(\boldsymbol{r})$$

- Provided that

$$\nabla\times\boldsymbol{a}=0 \qquad \nabla\times\left[\nabla\psi\nabla\cdot\boldsymbol{a}-2\left(\nabla\psi\cdot\nabla\right)\boldsymbol{a}\right]=0$$

Vector wave equation

- It is customary to choose $\boldsymbol{a}(\boldsymbol{r}) = \boldsymbol{r}$ which satisfies both equations (why?). It then follows that

$$\boldsymbol{M}(\boldsymbol{r}) = \nabla \times \left[\psi(\boldsymbol{r})\boldsymbol{r}\right] = \nabla\psi(\boldsymbol{r}) \times \boldsymbol{r}$$

Unlike cylindrical functions, the factor 1/k has not been included here in order to obtain dimensionless quantities.

$$\boldsymbol{N}(\boldsymbol{r}) = \frac{1}{k}\nabla \times \boldsymbol{M}(\boldsymbol{r})$$

$$\psi(\boldsymbol{r}) = \psi_{\ell,m}(r,\theta,\phi) = f_\ell(r)\, Y_\ell^{-m}(\theta,\phi)$$

We have used –m instead of m to preserve the convention used for the cylindrical case.

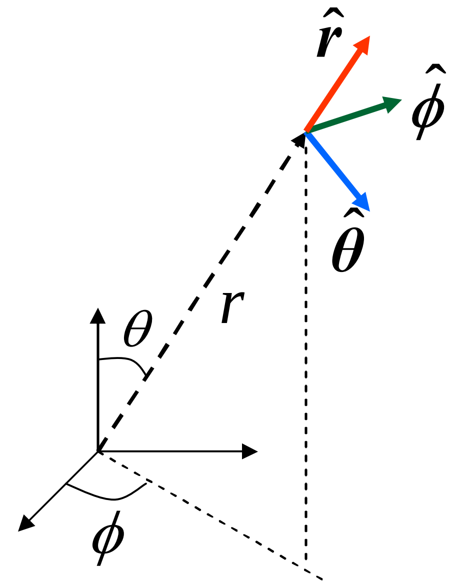
Vector wave equation

- These two vector fields (***M*** and ***N***) are linearly independent from each other as for the cylindrical vector functions.
- They satisfy the wave equation and have zero divergence.
- Similar to the cylindrical case, very solution of the vector wave equation in a homogeneous medium which has zero divergence can be written as a combination of these vectors for different solutions of the scalar wave equation.

Vector wave equation

- We next use spherical coordinates

$$\nabla \psi = \frac{\partial \psi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{\boldsymbol{\phi}}$$



$$\boldsymbol{M}_{\ell,m} = \nabla \psi_{\ell,m} \times \boldsymbol{r} = \frac{1}{\sin\theta}\frac{\partial \psi_{\ell,m}}{\partial \phi}\hat{\boldsymbol{\theta}} - \frac{\partial \psi_{\ell,m}}{\partial \theta}\hat{\boldsymbol{\phi}}$$

$$= -\frac{jm\psi_{\ell,m}}{\sin\theta}\hat{\boldsymbol{\theta}} - \frac{\partial \psi_{\ell,m}}{\partial \theta}\hat{\boldsymbol{\phi}}$$

Vector wave equation

- The 2[nd] solution use $\nabla\times(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{a}(\nabla\cdot\boldsymbol{b})-\boldsymbol{b}(\nabla\cdot\boldsymbol{a})+(\boldsymbol{b}\cdot\nabla)\boldsymbol{a}-(\boldsymbol{a}\cdot\nabla)\boldsymbol{b}$

$$\boldsymbol{N}_{\ell,m}=\frac{1}{k}\nabla\times\boldsymbol{M}_{\ell,m}=\hat{\boldsymbol{r}}\frac{1}{kr\sin\theta}\left[-\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi_{\ell,m}}{\partial\theta}\right)+\frac{m^2\psi_{\ell,m}}{\sin\theta}\right]$$
$$+\hat{\boldsymbol{\theta}}\left[\frac{1}{kr}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_{\ell,m}}{\partial\theta}\right)\right]-\hat{\boldsymbol{\phi}}\frac{jm}{kr\sin\theta}\frac{\partial}{\partial r}\left(r\psi_{\ell,m}\right)$$

- Leads to

$$\boldsymbol{N}_{\ell,m}=\hat{\boldsymbol{r}}\frac{\ell(\ell+1)\psi_{\ell,m}}{kr}+\hat{\boldsymbol{\theta}}\left[\frac{1}{kr}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_{\ell,m}}{\partial\theta}\right)\right]-\hat{\boldsymbol{\phi}}\frac{jm}{kr\sin\theta}\frac{\partial}{\partial r}\left(r\psi_{\ell,m}\right)$$

Vector wave equation

- Note that if the electric field is given by $\boldsymbol{M}_{\ell,m}$ then the magnetic field is necessarily given by $(j/\eta)\boldsymbol{N}_{\ell,m}$
- Similarly if the electric field is given by $\boldsymbol{N}_{\ell,m}$ then the magnetic field is necessarily given by $(j/\eta)\boldsymbol{M}_{\ell,m}$
- Note that also in this case the electric and magnetic fields are normal to each other for each mode (why?)

$$\boldsymbol{M}_{\ell,m}\cdot\boldsymbol{N}_{\ell,m}=\frac{jm}{kr\sin\theta}\left[\frac{\partial\psi_{\ell,m}}{\partial\theta}\frac{\partial}{\partial r}\left(r\psi_{\ell,m}\right)-\psi_{\ell,m}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_{\ell,m}}{\partial\theta}\right)\right]=0$$

$$\rightarrow \boldsymbol{E}\cdot\boldsymbol{H}=0$$

Far field behavior of solutions

- For later use, let us consider the vector functions when

$$f_\ell(r) = h_\ell^{(2)}(kr) = \sqrt{\frac{\pi}{2kr}} H_{\ell+1/2}^{(2)}(kr)$$

- Consider the functions at a large distance from the center of the sphere. By this we mean that $kr \gg \ell$

- Asymptotic relation for the spherical Hankel function

$$h_\ell^{(2)}(kr) = \sqrt{\frac{\pi}{2kr}} H_{\ell+1/2}^{(2)}(kr) \sim \frac{j^{\ell+1}}{kr} \exp(-jkr)$$

Far field behavior of solutions

- The fields have the asymptotic behavior:

$$\boldsymbol{M}_{\ell,m} \sim j^{\ell+1} \frac{\exp(-jkr)}{kr}\left[-\frac{jmY_\ell^{-m}}{\sin\theta}\hat{\boldsymbol{\theta}} - \frac{\partial Y_\ell^{-m}}{\partial\theta}\hat{\boldsymbol{\phi}}\right]$$

$$\boldsymbol{N}_{\ell,m} \sim j^{\ell} \frac{\exp(-jkr)}{kr}\left[\frac{\partial Y_\ell^{-m}}{\partial\theta}\hat{\boldsymbol{\theta}} - \frac{jm}{\sin\theta}Y_\ell^{-m}\hat{\boldsymbol{\phi}}\right]$$

- For each mode the resulting Poynting vector is along the radial direction: these are spherical waves (in far field)

Far field behavior of solutions

- Note that for the particular case of m=0 we have linearly polarized electric fields in both cases

$$\boldsymbol{M}_{\ell,0} \sim -j^{\ell+1} \frac{\exp(-jkr)}{kr} \frac{\partial Y_\ell^0}{\partial \theta} \hat{\boldsymbol{\phi}}$$

$$\boldsymbol{N}_{\ell,0} \sim j^{\ell} \frac{\exp(-jkr)}{kr} \frac{\partial Y_\ell^0}{\partial \theta} \hat{\boldsymbol{\theta}}$$

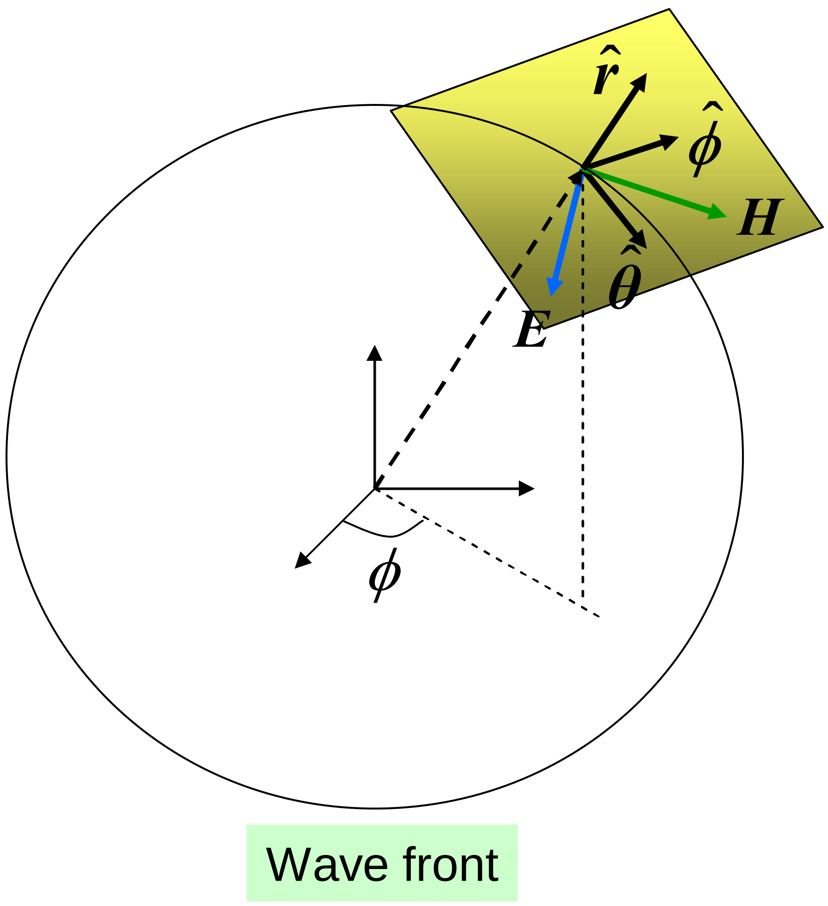
- But, in general, the fields have more complicated polarizations. They are elliptically polarized on the plane described by the unit vectors $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\theta}}$. Hence, it is wrong to talk of horizontal and vertical modes like for a cylinder.

Far field behavior of solutions

- On these planes the elliptic polarization is determined by the ratio between

$$\frac{P_\ell^{-m}(\cos\theta)}{\sin\theta} \qquad \frac{dP_\ell^{-m}(\cos\theta)}{d\theta}$$

- It does not depend on the angle ϕ

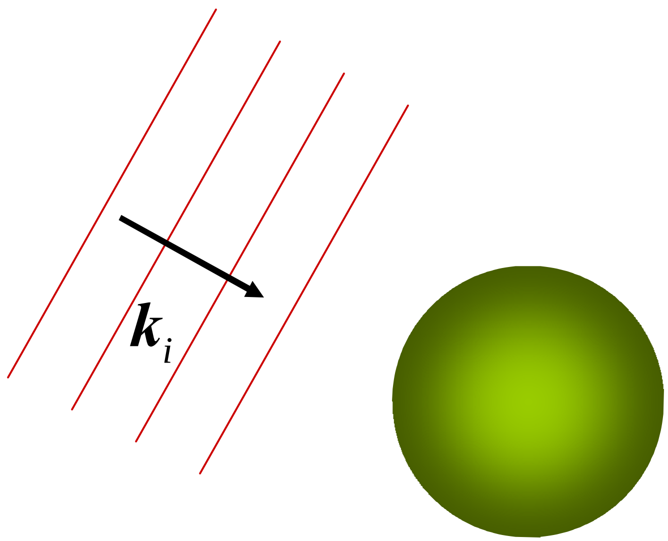


Expansion of a plane wave

- We have analyzed the ‘natural’ solutions to the vector wave equation in spherical coordinates.
- These solutions behave as TEM waves at large distance
- But the actual problem we are interested is not the scattering of these waves, but the scattering of a simple plane wave such as

$$\boldsymbol{E}_i(\boldsymbol{r}) = \boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right)$$

$$\left|\boldsymbol{k}_i\right| = k$$

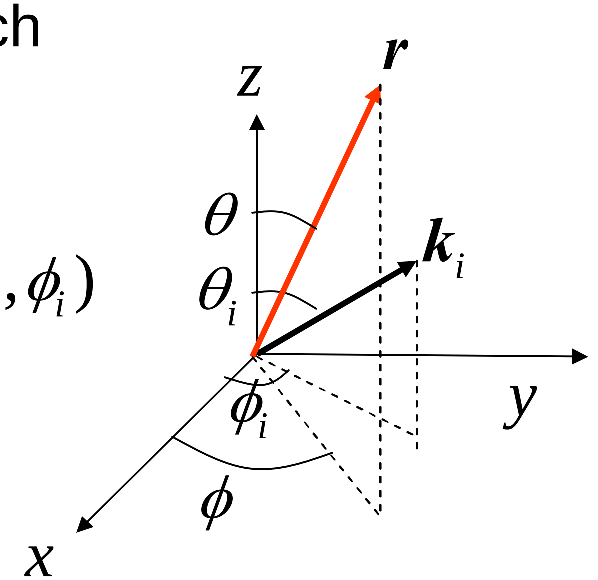


Expansion of a plane wave

- As for cylindrical waves we have to find the appropriate plane wave expansion for the incident wave vector

$$\boldsymbol{k}_i = \left(k_{i,x}, k_{i,y}, k_{i,z}\right) = k\left(\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i\right)$$

- Again, 1st consider a scalar plane wave which can be expanded as



$$\exp(-j\boldsymbol{k}_i \cdot \boldsymbol{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} w_{\ell m} j_\ell(kr) Y_\ell^{-m}(\theta,\phi) Y_\ell^{m}(\theta_i,\phi_i)$$

$$w_{\ell m} = (-1)^m (-j)^\ell (2\ell+1)$$

$$\boldsymbol{r} = r\left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)$$

Expansion of a plane wave

- Now, look at the following;

$$\nabla \times \left[\boldsymbol{r} \exp(-j\boldsymbol{k}_i \cdot \boldsymbol{r}) \right] = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} w_{\ell m} \boldsymbol{M}_{\ell,m}^{J}(r,\theta,\phi) Y_\ell^m(\theta_i,\phi_i)$$

$$\nabla \times \left[\boldsymbol{r} \exp(-j\boldsymbol{k}_i \cdot \boldsymbol{r}) \right] = -j\boldsymbol{k}_i \times \boldsymbol{r} \exp(-j\boldsymbol{k}_i \cdot \boldsymbol{r}) =$$

$$\boldsymbol{k}_i \times \nabla_{k_i} \exp(-j\boldsymbol{k}_i \cdot \boldsymbol{r}) = -\nabla_{k_i} \times \left[\boldsymbol{k}_i \exp(-j\boldsymbol{k}_i \cdot \boldsymbol{r}) \right]$$

- We have defined

$$\boldsymbol{M}_{\ell,m}^{J} = \nabla \times \left(\psi_{\ell,m}^{J} \boldsymbol{r} \right) = -\hat{\boldsymbol{\theta}} \frac{jm\psi_{\ell,m}^{J}}{\sin\theta} - \hat{\boldsymbol{\phi}} \frac{\partial \psi_{\ell,m}^{J}}{\partial \theta}$$

$$\psi_{\ell,m}^{J}(r,\theta,\phi) = j_\ell\left(kr\right) Y_\ell^{-m}\left(\theta,\phi\right)$$

Expansion of a plane wave

- It follows from the orthogonality of spherical harmonics that

$$\boldsymbol{M}_{\ell,m}^{J}(r,\theta,\phi)=\frac{j^{l}}{4\pi}\int_{4\pi}Y_{\ell}^{-m}(\theta_i,\phi_i)\boldsymbol{k}_i\times\nabla_{k_i}\exp(-j\boldsymbol{k}_i\cdot\boldsymbol{r})d\Omega_i$$

$$d\Omega_i=\sin\theta_i d\theta_i d\phi_i \qquad \boldsymbol{k}_i\times\nabla_{k_i}=\hat{\boldsymbol{\phi}}_i\frac{\partial}{\partial\theta_i}-\hat{\boldsymbol{\theta}}_i\frac{1}{\sin\theta_i}\frac{\partial}{\partial\phi_i}$$

- Partial integration yields

$$\boldsymbol{M}_{\ell,m}^{J}(r,\theta,\phi)=\frac{j^{l}}{4\pi}\int_{4\pi}\boldsymbol{C}_{\ell,m}\left(\theta_i,\phi_i\right)\exp(-j\boldsymbol{k}_i\cdot\boldsymbol{r})d\Omega_i$$

Expansion of a plane wave

$$\boldsymbol{C}_{\ell,m}\left(\theta_i,\phi_i\right)=\frac{1}{\sin\theta_i}\frac{\partial}{\partial\phi_i}\left[\hat{\boldsymbol{\theta}}_i\; Y_\ell^{-m}(\theta_i,\phi_i)\right]$$

$$-\frac{1}{\sin\theta_i}\frac{\partial}{\partial\theta_i}\left[\hat{\boldsymbol{\phi}}_i\; Y_\ell^{-m}(\theta_i,\phi_i)\sin\theta_i\right]$$

- Note that

$$\frac{\partial}{\partial\theta_i}\hat{\boldsymbol{\phi}}_i=0 \qquad \frac{\partial}{\partial\phi_i}\hat{\boldsymbol{\theta}}_i=\cos\theta_i\;\;\hat{\boldsymbol{\phi}}_i$$

⇒

$$\boldsymbol{C}_{\ell,m}\left(\theta_i,\phi_i\right)=\frac{1}{\sin\theta_i}\hat{\boldsymbol{\theta}}_i\;\frac{\partial Y_\ell^{-m}(\theta_i,\phi_i)}{\partial\phi_i}-\hat{\boldsymbol{\phi}}_i\frac{\partial Y_\ell^{-m}(\theta_i,\phi_i)}{\partial\theta_i}$$

$$=\nabla_{k_i}\times\left[\boldsymbol{k}_iY_\ell^{-m}(\theta_i,\phi_i)\right]=\nabla_{k_i}Y_\ell^{-m}(\theta_i,\phi_i)\times\boldsymbol{k}_i$$

Expansion of a plane wave

- It then follows that

$$\boldsymbol{N}_{\ell,m}^{J}(r,\theta,\phi) = \frac{1}{k}\nabla\times\boldsymbol{M}_{\ell,m}^{J}(r,\theta,\phi)$$

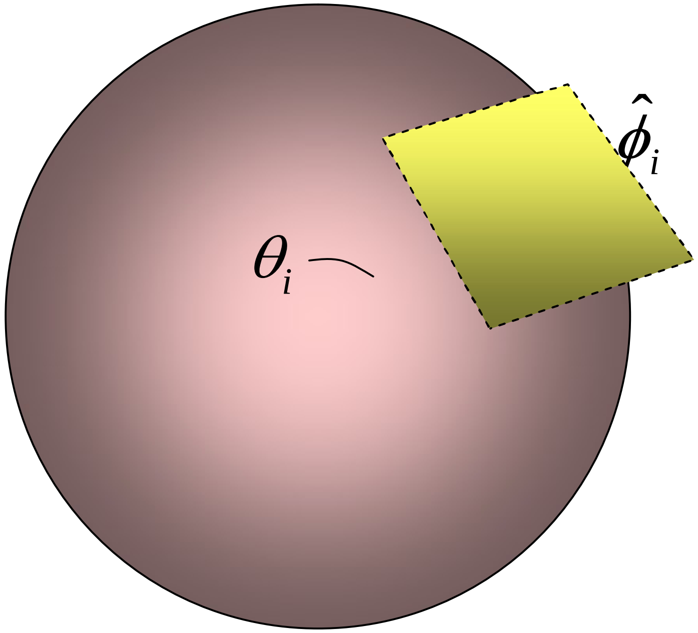
$$= \frac{j^{\ell-1}}{4\pi}\int_{4\pi}\boldsymbol{B}_{\ell,m}\left(\theta_i,\phi_i\right)\exp(-j\boldsymbol{k}_i\cdot\boldsymbol{r})d\Omega_i$$

$$\boldsymbol{B}_{\ell,m}\left(\theta_i,\phi_i\right) = \hat{\boldsymbol{k}}_i\times\boldsymbol{C}_{\ell,m}\left(\theta_i,\phi_i\right) = k\nabla_{k_i}Y_\ell^{-m}(\theta_i,\phi_i)$$

$$= \hat{\boldsymbol{\theta}}_i\,\frac{\partial Y_\ell^{-m}(\theta_i,\phi_i)}{\partial\theta_i} + \hat{\boldsymbol{\phi}}_i\frac{1}{\sin\theta_i}\frac{\partial Y_\ell^{-m}(\theta_i,\phi_i)}{\partial\phi_i}$$

Expansion of a plane wave

- The vectors ***B*** and ***C*** are called vector spherical harmonics
- They are functions defined on a sphere in the ***k***-space with the radius *k*, i.e., they are functions of θ_i, ϕ_i
- At each point on this sphere they are normal to wave vector, hence are tangential to sphere
- And they are normal to each other for each mode



K-space

Expansion of a plane wave

- Now, we introduce a new vector spherical harmonic in the k-space. Consider the radial vector function

$$\boldsymbol{A}_{\ell,m}\left(\theta_i,\phi_i\right)=\hat{\boldsymbol{k}}_i Y_\ell^{-m}(\theta_i,\phi_i)$$

- From the scalar plane wave expansion it follows that

$$\boldsymbol{L}_{\ell,m}^{J}(r,\theta,\phi)=\frac{(j)^{\ell-1}}{4\pi}\int_{4\pi}\boldsymbol{A}_{\ell,m}(\theta_i,\phi_i)\exp(-j\boldsymbol{k}_i\cdot\boldsymbol{r})d\Omega_i$$

$$\boldsymbol{L}_{\ell,m}^{J}=\frac{1}{k}\nabla\psi_{\ell,m}^{J}$$

Expansion of a plane wave

- The vector spherical harmonics constitute an orthogonal set on the space of vector functions on the k-sphere

$$\int_{4\pi} \boldsymbol{C}_{\ell,m}(\theta_i,\phi_i)\cdot\boldsymbol{C}_{\ell',-m'}(\theta_i,\phi_i)\,d\Omega_i = \delta_{\ell\ell'}\delta_{mm'}(-1)^m\frac{4\pi\ell(\ell+1)}{2\ell+1}$$

$$\int_{4\pi} \boldsymbol{B}_{\ell,m}(\theta_i,\phi_i)\cdot\boldsymbol{B}_{\ell',-m'}(\theta_i,\phi_i)\,d\Omega_i = \delta_{\ell\ell'}\delta_{mm'}(-1)^m\frac{4\pi\ell(\ell+1)}{2\ell+1}$$

$$\int_{4\pi} \boldsymbol{A}_{\ell,m}(\theta_i,\phi_i)\cdot\boldsymbol{A}_{\ell',-m'}(\theta_i,\phi_i)\,d\Omega_i = \delta_{\ell\ell'}\delta_{mm'}(-1)^m\frac{4\pi}{2\ell+1}$$

Expansion of a plane wave

- They are also mutually orthogonal

$$\int_{4\pi} \boldsymbol{B}_{\ell,m}(\theta_i,\phi_i)\cdot \boldsymbol{C}_{\ell',-m'}(\theta_i,\phi_i)\,d\Omega_i = 0$$

$$\int_{4\pi} \boldsymbol{B}_{\ell,m}(\theta_i,\phi_i)\cdot \boldsymbol{A}_{\ell',-m'}(\theta_i,\phi_i)\,d\Omega_i = 0$$

$$\int_{4\pi} \boldsymbol{A}_{\ell,m}(\theta_i,\phi_i)\cdot \boldsymbol{C}_{\ell',-m'}(\theta_i,\phi_i)\,d\Omega_i = 0$$

Expansion of a plane wave

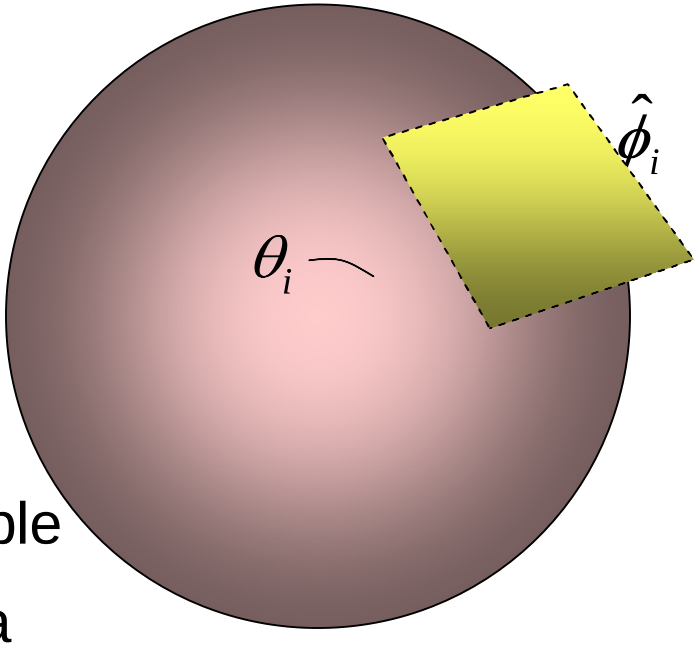
- Consider a vector field defined in the ***k***-space on a sphere with the radius k, i.e., a vector function of the angles θ_i, ϕ_i
- Then the vector field can be expanded in the spherical vector functions which are also defined on this sphere
- In other words: spherical vectors are complete in this space

Expansion of a plane wave

- Consider the incident field

$$\boldsymbol{E}_i(\boldsymbol{r}) = \boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right)$$

- Now, let us keep $\boldsymbol{E}_i^0$ constant while considering the direction of ***k*** as a variable (of course this field does not represent a true plane wave any more)



K-sphere

- The incident wave is now a vector function defined on the k-sphere since

$$\boldsymbol{k}_i \cdot \boldsymbol{r} = kr\left[\sin\theta_i \sin\theta \cos\left(\phi - \phi_i\right) + \cos\theta_i \cos\theta\right]$$

Expansion of a plane wave

- Expansion in terms of vector spherical harmonics:

$$\boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right) = \sum_{\ell,m}\left[a_{\ell m}\left(\boldsymbol{r}\right)\boldsymbol{A}_{\ell,m}\left(\theta_i,\phi_i\right) + b_{\ell m}\left(\boldsymbol{r}\right)\boldsymbol{B}_{\ell,m}\left(\theta_i,\phi_i\right) + c_{\ell m}\left(\boldsymbol{r}\right)\boldsymbol{C}_{\ell,m}\left(\theta_i,\phi_i\right)\right]$$

$$a_{\ell m}\left(\boldsymbol{r}\right) = \left[(-1)^m \frac{2\ell+1}{4\pi}\right]\boldsymbol{E}_i^0 \cdot \int_{4\pi} \boldsymbol{A}_{\ell,-m}\left(\theta_i,\phi_i\right)\exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right) d\Omega_i$$

$$b_{\ell m}\left(\boldsymbol{r}\right) = \left[(-1)^m \frac{2\ell+1}{4\pi\ell\left(\ell+1\right)}\right]\boldsymbol{E}_i^0 \cdot \int_{4\pi} \boldsymbol{B}_{\ell,-m}\left(\theta_i,\phi_i\right)\exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right) d\Omega_i$$

$$c_{\ell m}\left(\boldsymbol{r}\right) = \left[(-1)^m \frac{2\ell+1}{4\pi\ell\left(\ell+1\right)}\right]\boldsymbol{E}_i^0 \cdot \int_{4\pi} \boldsymbol{C}_{\ell,-m}\left(\theta_i,\phi_i\right)\exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right) d\Omega_i$$

Expansion of a plane wave

- Result:

$$a_{\ell m}(\boldsymbol{r}) = \left[(-1)^m(-j)^{\ell-1}(2\ell+1)\right]\boldsymbol{E}_i^0 \cdot \boldsymbol{L}_{\ell,-m}^J(r,\theta,\phi)$$

$$b_{\ell m}(\boldsymbol{r}) = \left[(-1)^m(-j)^{\ell-1}\frac{2\ell+1}{\ell(\ell+1)}\right]\boldsymbol{E}_i^0 \cdot \boldsymbol{N}_{\ell,-m}^J(r,\theta,\phi)$$

$$c_{\ell m}(\boldsymbol{r}) = \left[(-1)^m(-j)^{\ell}\frac{2\ell+1}{\ell(\ell+1)}\right]\boldsymbol{E}_i^0 \cdot \boldsymbol{M}_{\ell,-m}^J(r,\theta,\phi)$$

Expansion of a plane wave

- It follows that

$$\boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right) = \boldsymbol{E}_i^0 \cdot \sum_{\ell,m} \Big[v_{\ell,m}^{(a)} \boldsymbol{L}_{\ell,m}^J(r,\theta,\phi) \boldsymbol{A}_{\ell,-m}\left(\theta_i,\phi_i\right) + v_{\ell,m}^{(b)} \boldsymbol{M}_{\ell,m}^J(r,\theta,\phi) \boldsymbol{B}_{\ell,-m}\left(\theta_i,\phi_i\right) + v_{\ell,m}^{(c)} \boldsymbol{N}_{\ell,m}^J(r,\theta,\phi) \boldsymbol{C}_{\ell,-m}\left(\theta_i,\phi_i\right) \Big]$$

$$v_{\ell,m}^{(a)} = (-1)^m (-j)^{\ell-1} \left(2\ell+1\right)$$

$$v_{\ell,m}^{(b)} = (-1)^m (-j)^{\ell-1} \frac{2\ell+1}{\ell\left(\ell+1\right)}$$

$$v_{\ell,m}^{(c)} = (-1)^m (-j)^{\ell} \frac{2\ell+1}{\ell\left(\ell+1\right)}$$

Expansion of a plane wave

- Since this holds for any constant vector $\boldsymbol{E}_i^0$ we must have

$$\bar{\bar{\boldsymbol{I}}}\exp\left(-j\boldsymbol{k}_i\cdot\boldsymbol{r}\right)=\sum_{\ell,m}\Big[v_{\ell,m}^{(a)}\boldsymbol{L}_{\ell,m}^J(r,\theta,\phi)\boldsymbol{A}_{\ell,-m}\left(\theta_i,\phi_i\right)+$$

$$v_{\ell,m}^{(b)}\boldsymbol{M}_{\ell,m}^J(r,\theta,\phi)\boldsymbol{B}_{\ell,-m}\left(\theta_i,\phi_i\right)+v_{\ell,m}^{(c)}\boldsymbol{N}_{\ell,m}^J(r,\theta,\phi)\boldsymbol{C}_{\ell,-m}\left(\theta_i,\phi_i\right)\Big]$$

$$\Rightarrow\quad \boldsymbol{E}_i^0\exp\left(-j\boldsymbol{k}_i\cdot\boldsymbol{r}\right)=\sum_{\ell,m}\Big[v_{\ell,m}^{(a)}\boldsymbol{L}_{\ell,m}^J(r,\theta,\phi)\boldsymbol{A}_{\ell,-m}\left(\theta_i,\phi_i\right)\cdot\boldsymbol{E}_i^0$$

$$+v_{\ell,m}^{(b)}\boldsymbol{M}_{\ell,m}^J(r,\theta,\phi)\boldsymbol{B}_{\ell,-m}\left(\theta_i,\phi_i\right)\cdot\boldsymbol{E}_i^0$$

$$+v_{\ell,m}^{(c)}\boldsymbol{N}_{\ell,m}^J(r,\theta,\phi)\boldsymbol{C}_{\ell,-m}\left(\theta_i,\phi_i\right)\cdot\boldsymbol{E}_i^0\Big]$$

Expansion of a plane wave

- Now, for a true plane wave the electric field has no component along the wave vector so that

$$\boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i\cdot\boldsymbol{r}\right)=\sum_{\ell,m}\Big[v_{\ell,m}^{(b)}\boldsymbol{M}_{\ell,m}^J(r,\theta,\phi)\boldsymbol{B}_{\ell,-m}\left(\theta_i,\phi_i\right)\cdot\boldsymbol{E}_i^0 + v_{\ell,m}^{(c)}\boldsymbol{N}_{\ell,m}^J(r,\theta,\phi)\boldsymbol{C}_{\ell,-m}\left(\theta_i,\phi_i\right)\cdot\boldsymbol{E}_i^0\Big]$$

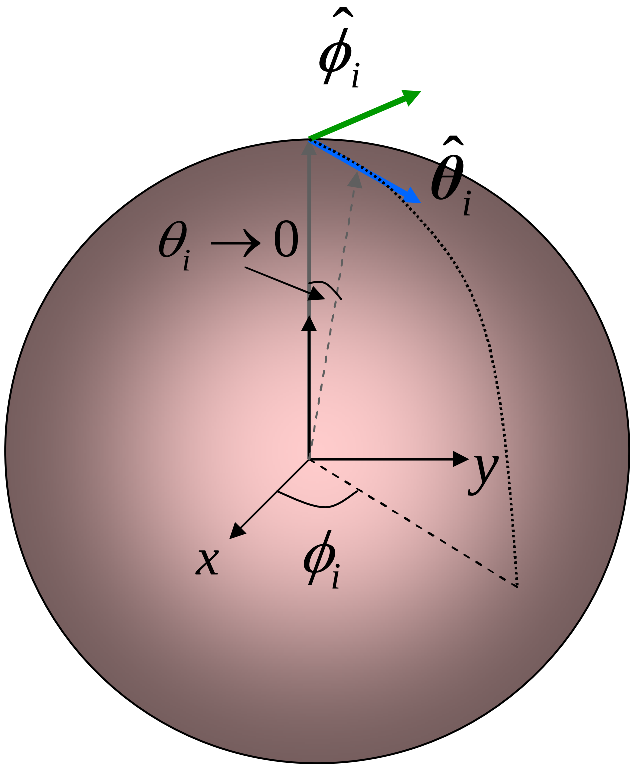
$$\boldsymbol{B}_{\ell,-m}\left(\theta_i,\phi_i\right)=\hat{\boldsymbol{\theta}}_i\ \frac{\partial Y_\ell^m(\theta_i,\phi_i)}{\partial\theta_i}+\hat{\boldsymbol{\phi}}_i\frac{1}{\sin\theta_i}\frac{\partial Y_\ell^m(\theta_i,\phi_i)}{\partial\phi_i}$$

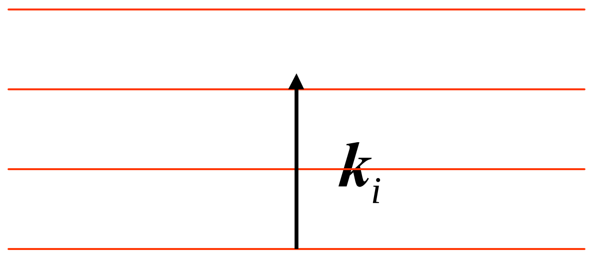
$$\boldsymbol{C}_{\ell,-m}\left(\theta_i,\phi_i\right)=\frac{1}{\sin\theta_i}\hat{\boldsymbol{\theta}}_i\ \frac{\partial Y_\ell^m(\theta_i,\phi_i)}{\partial\phi_i}-\hat{\boldsymbol{\phi}}_i\frac{\partial Y_\ell^m(\theta_i,\phi_i)}{\partial\theta_i}$$

Expansion of a plane wave

- Example: wave propagating along +z-axis: $\theta_i \to 0$
- This case is general enough as the sphere is symmetric
- Specify the angle ϕ_i before taking the limit, otherwise the direction of unit vectors cannot be determined
- Any angle will do, again because the sphere is symmetric, we can later take

$$\phi_i = 0 \to \hat{\boldsymbol{\theta}}_i = \hat{\boldsymbol{x}},\ \boldsymbol{\phi}_i = \hat{\boldsymbol{y}}$$





Expansion of a plane wave

- We then use the relationships (m>0)

$$P_\ell^m(u) = (-1)^m \left(1-u^2\right)^{m/2} \frac{d^m P_\ell(u)}{du^m}$$

$$P_\ell^{-m}(u) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(u) = \frac{(\ell-m)!}{(\ell+m)!} \left(1-u^2\right)^{m/2} \frac{d^m P_\ell(u)}{du^m}$$

- It can be shown that all B's become zero except

$$\boldsymbol{B}_{\ell,-1} = -\left.\frac{dP_\ell(u)}{du}\right|_{u=1} \left(\hat{\boldsymbol{\theta}}_i + j\hat{\boldsymbol{\phi}}_i\right) \exp\left(j\phi_i\right)$$

Left polarized

$$\boldsymbol{B}_{\ell,1} = \frac{1}{\ell(\ell+1)} \left.\frac{dP_\ell(u)}{du}\right|_{u=1} \left(\hat{\boldsymbol{\theta}}_i - j\hat{\boldsymbol{\phi}}_i\right) \exp\left(-j\phi_i\right)$$

Right polarized

Expansion of a plane wave

- Similarly, all C's become zero except

$$\boldsymbol{C}_{\ell,-1} = -j \left.\frac{dP_\ell(u)}{du}\right|_{u=1} \left(\hat{\boldsymbol{\theta}}_i + j\hat{\boldsymbol{\phi}}_i\right)\exp\left(j\phi_i\right)$$

Left polarized

$$\boldsymbol{C}_{\ell,1} = -\frac{j}{\ell\left(\ell+1\right)} \left.\frac{dP_\ell(u)}{du}\right|_{u=1} \left(\hat{\boldsymbol{\theta}}_i - j\hat{\boldsymbol{\phi}}_i\right)\exp\left(-j\phi_i\right)$$

Right polarized

- All vector functions are zero for $\ell = 0$

- Also, note that $\left.\frac{dP_\ell(u)}{du}\right|_{u=1} = \frac{\ell\left(\ell+1\right)}{2}$

Expansion of a plane wave

- Collecting the results: $\left(\phi_i = 0\right)$

$$\boldsymbol{B}_{\ell,-1} = -\ell\left(\ell+1\right)\frac{1}{2}\left(\hat{\boldsymbol{x}} + j\hat{\boldsymbol{y}}\right)$$

Left polarized

$$\boldsymbol{C}_{\ell,-1} = -\ell\left(\ell+1\right)\frac{j}{2}\left(\hat{\boldsymbol{x}} + j\hat{\boldsymbol{y}}\right)$$

Left polarized

$$\boldsymbol{B}_{\ell,1} = \frac{1}{2}\left(\hat{\boldsymbol{x}} - j\hat{\boldsymbol{y}}\right)$$

Right polarized

$$\boldsymbol{C}_{\ell,1} = -\frac{j}{2}\left(\hat{\boldsymbol{x}} - j\hat{\boldsymbol{y}}\right)$$

Right polarized

Expansion of a plane wave

- Returning to the plane wave

$$\boldsymbol{E}_i^0 \exp\left(-jkz\right) =$$

$$\frac{1}{2}\left(\hat{\boldsymbol{x}} + j\hat{\boldsymbol{y}}\right)\cdot \boldsymbol{E}_i^0 \sum_{\ell=1}^{\infty}(-j)^{\ell-1}\left(2\ell+1\right)\left[\boldsymbol{M}_{\ell,1}^J(r,\theta,\phi)+\boldsymbol{N}_{\ell,1}^J(r,\theta,\phi)\right]+$$

$$\frac{1}{2}\left(\hat{\boldsymbol{x}} - j\hat{\boldsymbol{y}}\right)\cdot \boldsymbol{E}_i^0 \sum_{\ell=1}^{\infty}(-j)^{\ell-1}\frac{\left(2\ell+1\right)}{\ell\left(\ell+1\right)}\left[\boldsymbol{M}_{\ell,-1}^J(r,\theta,\phi)-\boldsymbol{N}_{\ell,-1}^J(r,\theta,\phi)\right]$$

Expansion of a plane wave

- More specifically

$$\boldsymbol{M}_{\ell,-1}^{J} = j_\ell^{(2)}(kr)\exp(j\phi)\left(\hat{\boldsymbol{\theta}}\frac{j}{\sin\theta} - \hat{\boldsymbol{\phi}}\frac{d}{d\theta}\right)P_\ell^1(\cos\theta)$$

$$= j_\ell^{(2)}(kr)\exp(j\phi)\left[-j\hat{\boldsymbol{\theta}}\pi_\ell(\cos\theta) + \hat{\boldsymbol{\phi}}\tau_\ell(\cos\theta)\right]$$

$$\pi_\ell(\cos\theta) \equiv -\frac{P_\ell^1(\cos\theta)}{\sin\theta} \qquad \tau_\ell(\cos\theta) \equiv -\frac{dP_\ell^1(\cos\theta)}{d\theta}$$

$$\boldsymbol{M}_{\ell,1}^{J} = \frac{j_\ell(kr)}{\ell(\ell+1)}\exp(-j\phi)\left(-\hat{\boldsymbol{\theta}}\frac{j}{\sin\theta} - \hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}\right)P_\ell^{-1}(\cos\theta)$$

$$= \frac{j_\ell(kr)}{\ell(\ell+1)}\exp(-j\phi)\left[-j\hat{\boldsymbol{\theta}}\pi_\ell(\cos\theta) - \hat{\boldsymbol{\phi}}\tau_\ell(\cos\theta)\right]$$

Expansion of a plane wave

$$\boldsymbol{N}_{\ell,-1}^{J} = \exp(j\phi)\left\{\hat{\boldsymbol{r}}\frac{\ell(\ell+1)}{kr}j_\ell(kr) - \frac{d\left[rj_\ell(kr)\right]}{krdr}\left[\hat{\boldsymbol{\theta}}\tau_\ell(\cos\theta) + j\hat{\boldsymbol{\phi}}\pi_\ell(\cos\theta)\right]\right\}$$

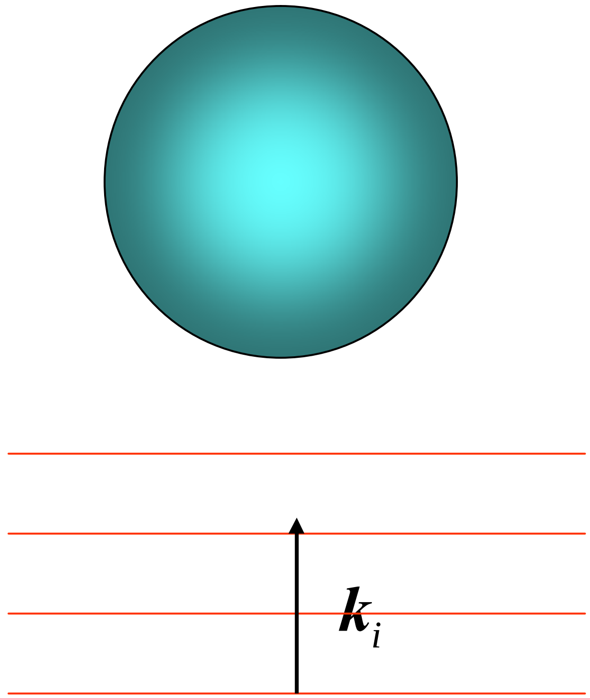
$$\boldsymbol{N}_{\ell,1}^{J} = \frac{\exp(-j\phi)}{\ell(\ell+1)}\left\{-\hat{\boldsymbol{r}}\frac{\ell(\ell+1)}{kr}j_\ell(kr) - \frac{d\left[rj_\ell(kr)\right]}{krdr}\left[\hat{\boldsymbol{\theta}}\tau_\ell(\cos\theta) - j\hat{\boldsymbol{\phi}}\pi_\ell(\cos\theta)\right]\right\}$$

Scattering by a perfectly conducting sphere

- We now consider the scattering of a plane wave by a perfectly conducting sphere
- When the incident wave hits the cylinder, surface currents (and charges) are induced
- These currents create the 'scattered' field. At any point, the total electric field is

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_i(\boldsymbol{r}) + \boldsymbol{E}_s(\boldsymbol{r})$$

$$\boldsymbol{E}_i(\boldsymbol{r}) = \boldsymbol{E}_i^0 \exp(-jkz)$$



Scattering by a perfectly conducting sphere

- We saw how the incident plane wave can be represented in terms of spherical vector solutions
- The scattered field (outside the sphere) can also be expanded in terms of those solutions
- But: for the scattered field we should use vectors with the right condition at the infinity $r \to \infty$
- We should use the spherical Hankel function of the 2nd kind for these waves which satisfy the radiation condition (behave as outgoing waves at infinity)

Scattering by a perfectly conducting sphere

- Expansion of the scattered field:

$$\boldsymbol{E}_s(\boldsymbol{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[b_{\ell m} \boldsymbol{M}_{\ell,m}^{h}(r,\theta,\phi) + c_{\ell m} \boldsymbol{N}_{\ell,m}^{h}(r,\theta,\phi) \right]$$

$$\boldsymbol{M}_{\ell,m}^{h} = -\hat{\boldsymbol{\theta}} \frac{jm}{\sin\theta} \psi_{\ell,m}^{h} - \hat{\boldsymbol{\phi}} \frac{\partial \psi_{\ell,m}^{h}}{\partial \theta}$$

$$\boldsymbol{N}_{\ell,m}^{h} = \hat{\boldsymbol{r}} \frac{\ell(\ell+1)\psi_{\ell,m}^{h}}{kr} + \hat{\boldsymbol{\theta}} \frac{\partial}{\partial \theta} \left[\frac{1}{kr} \frac{\partial}{\partial r} \left(r\psi_{\ell,m}^{h} \right) \right] - \hat{\boldsymbol{\phi}} \frac{jm}{\sin\theta} \frac{1}{kr} \frac{\partial}{\partial r} \left(r\psi_{\ell,m}^{h} \right)$$

$$\psi_{\ell,m}^{h}(r,\theta,\phi) = h_{\ell}^{(2)}(kr) Y_{\ell}^{-m}(\theta,\phi)$$

Scattering by a perfectly conducting sphere

- More specifically

$$\boldsymbol{M}_{\ell,m}^{h} = h_{\ell}^{(2)}(kr)\exp(-jm\phi)\left(-\hat{\boldsymbol{\theta}}\frac{jm}{\sin\theta} - \hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}\right)P_{\ell}^{-m}(\cos\theta)$$

$$\boldsymbol{N}_{\ell,m}^{h} = \exp(-jm\phi)$$

$$\left\{\hat{\boldsymbol{r}}\frac{\ell(\ell+1)}{kr}h_{\ell}^{(2)}(kr) + \frac{1}{kr}\frac{\partial}{\partial r}\left[rh_{\ell}^{(2)}(kr)\right]\left(\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta} - \hat{\boldsymbol{\phi}}\frac{jm}{\sin\theta}\right)\right\}P_{\ell}^{-m}(\cos\theta)$$

- From matching at the surface of the sphere, it directly follows that only the m = -1 and m = +1 terms can contribute to the series for the scattered field

Scattering by a perfectly conducting sphere

- It then follows that

$$\hat{\boldsymbol{r}}\times E_{i,-}^{0}\sum_{\ell=1}^{\infty}(-j)^{\ell-1}\left(2\ell+1\right)\left[\boldsymbol{M}_{\ell,1}^{J}(R_0,\theta,\phi)-\boldsymbol{N}_{\ell,1}^{J}(R_0,\theta,\phi)\right]=$$

$$-\;\hat{\boldsymbol{r}}\times\sum_{\ell=1}^{\infty}\left[b_{\ell,1}\boldsymbol{M}_{\ell,1}^{h}(R_0,\theta,\phi)+c_{\ell,1}\boldsymbol{N}_{\ell,1}^{h}(R_0,\theta,\phi)\right]$$

$$\hat{\boldsymbol{r}}\times E_{i,+}^{0}\sum_{\ell=1}^{\infty}(-j)^{\ell-1}\frac{\left(2\ell+1\right)}{\ell\left(\ell+1\right)}\left[\boldsymbol{M}_{\ell,-1}^{J}(R_0,\theta,\phi)-\boldsymbol{N}_{\ell,-1}^{J}(R_0,\theta,\phi)\right]=$$

$$-\;\hat{\boldsymbol{r}}\times\sum_{\ell=1}^{\infty}\left[b_{\ell,-1}\boldsymbol{M}_{\ell,-1}^{h}(R_0,\theta,\phi)+c_{\ell,-1}\boldsymbol{N}_{\ell,-1}^{h}(R_0,\theta,\phi)\right]$$

$$E_{i,-}^{0}=\frac{1}{2}\left(\hat{\boldsymbol{x}}+j\hat{\boldsymbol{y}}\right)\cdot\boldsymbol{E}_{i}^{0}\qquad E_{i,+}^{0}=\frac{1}{2}\left(\hat{\boldsymbol{x}}-j\hat{\boldsymbol{y}}\right)\cdot\boldsymbol{E}_{i}^{0}$$

Scattering by a perfectly conducting sphere

- It can be shown that ***M*** and ***N*** functions do not mix up
- Besides, different values of ℓ do not mix up

$$b_{\ell,1} = -E_{i,-}^{0}(-j)^{\ell-1}(2\ell+1)\frac{j_\ell(kR_0)}{h_\ell^{(2)}(kR_0)}$$

$$c_{\ell,1} = E_{i,-}^{0}(-j)^{\ell-1}(2\ell+1)\frac{\left[kR_0 j_\ell(kR_0)\right]'}{\left[kR_0 h_\ell^{(2)}(kR_0)\right]'}$$

$$b_{\ell,-1} = -E_{i,+}^{0}(-j)^{\ell-1}\frac{(2\ell+1)}{\ell(\ell+1)}\frac{j_\ell(kR_0)}{h_\ell^{(2)}(kR_0)}$$

$$c_{\ell,-1} = E_{i,+}^{0}(-j)^{\ell-1}\frac{(2\ell+1)}{\ell(\ell+1)}\frac{\left[kR_0 j_\ell(kR_0)\right]'}{\left[kR_0 h_\ell^{(2)}(kR_0)\right]'}$$

Scattering by a perfectly conducting sphere

- Collecting the results, the far field behavior is given by

$$\boldsymbol{E}_s(\boldsymbol{r}) = \frac{\exp(-jkr)}{jkr}\sum_{\ell=1}^{\infty}\frac{(2\ell+1)}{\ell(\ell+1)}$$

$$E_{i,+}^{0}\Omega_{\ell}\left[\hat{\boldsymbol{\theta}}\pi_{\ell}(\cos\theta)+j\hat{\boldsymbol{\phi}}\tau_{\ell}(\cos\theta)\right]\exp(j\phi)+$$

$$E_{i,+}^{0}\bar{\Omega}_{\ell}\left[\hat{\boldsymbol{\theta}}\tau_{\ell}(\cos\theta)+j\hat{\boldsymbol{\phi}}\pi_{\ell}(\cos\theta)\right]\exp(j\phi)+$$

$$E_{i,-}^{0}\Omega_{\ell}\left[\hat{\boldsymbol{\theta}}\pi_{\ell}(\cos\theta)-j\hat{\boldsymbol{\phi}}\tau_{\ell}(\cos\theta)\right]\exp(-j\phi)+$$

$$E_{i,-}^{0}\bar{\Omega}_{\ell}\left[\hat{\boldsymbol{\theta}}\tau_{\ell}(\cos\theta)-j\hat{\boldsymbol{\phi}}\pi_{\ell}(\cos\theta)\right]\exp(-j\phi)$$

$$\Omega_{\ell} = \frac{j_{\ell}(kR_0)}{h_{\ell}^{(2)}(kR_0)} \qquad \bar{\Omega}_{\ell} = \frac{\left[kR_0 j_{\ell}(kR_0)\right]'}{\left[kR_0 h_{\ell}^{(2)}(kR_0)\right]'}$$

Scattering by a perfectly conducting sphere

- Example: $\boldsymbol{E}_i^0 = E_0\hat{\boldsymbol{x}} \rightarrow E_{i,+}^0 = E_{i,-}^0 = \frac{1}{2}E_0$

$$\boldsymbol{E}_s(\boldsymbol{r}) = E_0 \frac{\exp(-jkr)}{jkr} \sum_{\ell=1}^{\infty} \frac{(2\ell+1)}{\ell(\ell+1)}$$

$$\left\{ \Omega_\ell \left[\hat{\boldsymbol{\theta}} \pi_\ell(\cos\theta)\cos\phi - \hat{\boldsymbol{\phi}} \tau_\ell(\cos\theta)\sin\phi \right] + \bar{\Omega}_\ell \left[\hat{\boldsymbol{\theta}} \tau_\ell(\cos\theta)\cos\phi - \hat{\boldsymbol{\phi}} \pi_\ell(\cos\theta)\sin\phi \right] \right\}$$

$$\pi_\ell(\cos\theta) = -\frac{P_\ell^1(\cos\theta)}{\sin\theta} \qquad \tau_\ell(\cos\theta) = -\frac{dP_\ell^1(\cos\theta)}{d\theta}$$

Scattering by a perfectly conducting sphere

- Components of the scattered far field

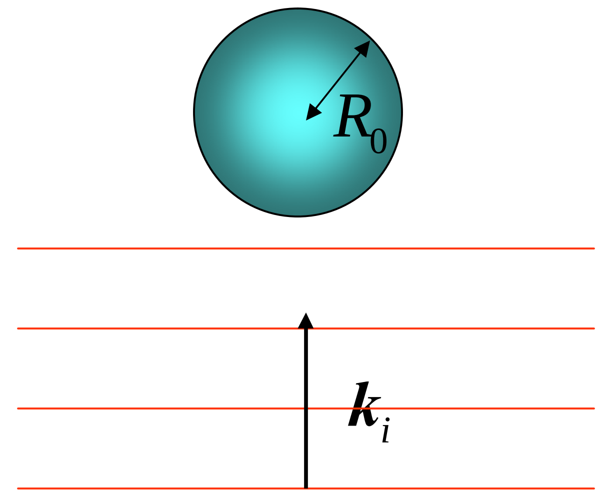
$$E_{s,\theta}(\boldsymbol{r}) = E_0 \frac{\exp(-jkr)}{jkr} \cos\phi \sum_{\ell=1}^{\infty} \frac{(2\ell+1)}{\ell(\ell+1)} \left[\Omega_\ell \pi_\ell(\cos\theta) + \bar{\Omega}_\ell \tau_\ell(\cos\theta) \right]$$

$$E_{s,\phi}(\boldsymbol{r}) = -E_0 \frac{\exp(-jkr)}{jkr} \sin\phi \sum_{\ell=1}^{\infty} \frac{(2\ell+1)}{\ell(\ell+1)} \left[\Omega_\ell \tau_\ell(\cos\theta) + \bar{\Omega}_\ell \pi_\ell(\cos\theta) \right]$$

Scattering by a small conducting sphere

- To get some insight let us consider the limit of a small sphere

$$kR_0 \ll 1$$



$$\Omega_\ell = \frac{j_\ell(kR_0)}{h_\ell^{(2)}(kR_0)} \approx j\frac{j_\ell(kR_0)}{y_\ell(kR_0)} \approx j\frac{(kR_0)^{2\ell+1}}{\left[1.3.5\ldots(2\ell-1)\right]^2(2\ell+1)}$$

$$\bar{\Omega}_\ell = \frac{\left[kR_0 j_\ell(kR_0)\right]'}{\left[kR_0 h_\ell^{(2)}(kR_0)\right]'} \approx -j\frac{\ell+1}{\ell}\frac{(kR_0)^{2\ell+1}}{\left[1.3.5\ldots(2\ell-1)\right]^2(2\ell+1)}$$

Scattering by a small conducting sphere

- Keeping the lowest order terms:

$$\Omega_1 \approx \frac{j}{3}(kR_0)^3 \qquad \bar{\Omega}_1 = -\frac{2j}{3}(kR_0)^3$$

$$\pi_1(\cos\theta) = -\frac{P_1^1(\cos\theta)}{\sin\theta} = 1 \qquad \tau_1(\cos\theta) = -\frac{dP_1^1(\cos\theta)}{d\theta} = \cos\theta$$

$$\boldsymbol{E}_s(\boldsymbol{r}) \sim E_0 (kR_0)^3 \frac{\exp(-jkr)}{2kr} \left[\hat{\boldsymbol{\theta}}\cos\phi(1-2\cos\theta) + \hat{\boldsymbol{\phi}}\sin\phi(2-\cos\theta)\right]$$

Scattering by a small conducting sphere

- Consider the amplitude of the scattered electric field

$$\left|\boldsymbol{E}_s(\boldsymbol{r})\right|^2 \sim \left|E_0\right|^2 \frac{k^4 R_0^6}{4r^2}\left[\cos^2\phi(1-2\cos\theta)^2+\sin^2\phi(2-\cos\theta)^2\right]$$

- Differential cross section

$$\sigma_d \sim \frac{k^4 R_0^6}{4}\left[\cos^2\phi(1-2\cos\theta)^2+\sin^2\phi(2-\cos\theta)^2\right]$$

- Total cross section

$$\sigma = \int_{4\pi} \sigma_d \, d\Omega \sim \frac{10\pi k^4 R_0^6}{3}$$