Electromagnetic scattering

Graduate Course
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Contents of lecture 7:

- Scattering from complex objects
  - General formulation
  - Volume integral equation formalism for dielectrics
  - Huygens principle and the extinction theorem
  - Surface integral equation formalism for dielectrics
  - Scattering from conductors (surface integral equations):
    - EFIE
    - MFIE
    - Two dimensional conductors
    - One dimensional conductors (wires)
Scattering from dielectrics: general

We have seen before that the scattered (far) field by a dielectric body with a scalar permittivity is given by

\[
E_s(r) = -\frac{k^2 \exp(-jkr)}{4\pi\epsilon_0 r} \hat{k}_s \times \left[ \hat{k}_s \times \int_V \exp(jk_s \cdot r') \delta\varepsilon_r(r') E(r') dV' \right]
\]

\[
\delta\varepsilon_r(r) = \frac{\varepsilon_p(r) - \epsilon_0}{\epsilon_0}
\]
Scattering from dielectrics: general

- Certain canonical problems (flat layers, spheres, infinite cylinders) can be exactly solved.
- But, in general, the field distribution inside the scatterer has to be found by numerical means.
- Remember that the equivalent electric current of a dielectric is given by

\[ J_{eq}(r) = j\omega \left[ \varepsilon_p(r) - \varepsilon_0 \right] E(r) = j\omega\varepsilon_0 \delta\varepsilon(r) E(r) \]

- But, instead of using this current to compute the far field, we use it to express the electric field everywhere as we see below.
Dyadic Green’s function

- We restrict ourselves to the Lorentz gauge:

\[ \nabla^2 A + k^2 A = -\mu J \]

\[ \nabla^2 \phi + k^2 \phi = -\frac{\rho}{\epsilon} \]

- The components of the vector potential and the scalar potential basically satisfy the same equation (Helmholtz)

- Consider the Green’s function satisfying:

\[ \nabla^2 G(r, r') + k^2 G(r, r') = -\delta(r - r') \]

\[ k^2 = \omega^2 \epsilon \mu \]
Dyadic Green’s function

- This function gives the field at $r$ generated by a point source at $r'$ (current or charge).
- The solution in infinite space is:

\[
G(r, r') = \frac{\exp\left(-jk|r - r'|\right)}{4\pi|r - r'|}
\]

**3D**

\[
G(r, r') = \frac{\exp\left(-jk|r - r'|\right)}{4\pi|r - r'|}
\]

$r = (x, y, z)$

**2D**

\[
G(r, r') = \frac{1}{4j} H_0^{(2)}(k|r - r'|)
\]

$r = (x, y)$
Dyadic Green’s function

- Solution for potentials:

\[
\varphi(r) = \frac{1}{\epsilon} \int G(r,r') \rho(r') dV' \quad A(r) = \mu \int G(r,r') J(r') dV'
\]

- Electric field generated by currents and charges in an object

\[
E(r) = -j \omega \mu \int G(r,r') J(r') dV' - \frac{1}{\epsilon} \nabla \int G(r,r') \rho(r') dV'
\]

- We have:
  - Volume charges \( \rho_V = -\nabla \cdot J / j \omega \)
  - Surface charges \( \rho_S = J \cdot \hat{n} / j \omega \)
Dyadic Green’s function

- Using partial integration:

\[ E(r) = -j\omega\mu\int_V G(r, r')J(r')dV' - \frac{1}{j\omega\varepsilon} \nabla \int_V \nabla' G(r, r') \cdot J(r')dV' \]

\[ E(r) = -j\omega\mu\int_{V_s} \bar{G}(r, r') \cdot J(r')dV' \]

\[ \bar{G}(r, r') = G(r, r') \bar{I} - \frac{1}{k^2} \nabla \nabla' G(r, r') \]

Be careful of the difficulties which may arise from literal interpretation of this result!
Scattering from dielectrics: general

- Returning to the dielectric object, the field generated by the equivalent current (scattered field) is everywhere

\[
E_s(r) = -j\omega \mu_0 \int_V \overline{G}(r, r') \cdot \mathbf{J}_{eq}(r') dV'
\]

\[
\overline{G}(r, r') = G(r, r') \bar{I} - \frac{1}{k^2} \nabla \nabla' G(r, r')
\]

- Bearing in mind that the total field (everywhere) is the superposition of the incident and scattered fields, one finds a volume integral equation for the electric field inside the object

\[
G(r, r') = \frac{\exp(-jk|\mathbf{r} - \mathbf{r}'|)}{4\pi|\mathbf{r} - \mathbf{r}'|}
\]
Volume integral equation

- Volume integral equation formulation

\[ E(r) = E_i(r) + k_0^2 \int_{V} \vec{G}(r,r') \cdot [\delta \varepsilon(r') E(r')] \, dV' \]

\[ k_0^2 = \omega^2 \mu_0 \varepsilon_0 \]

- This equation has to be solved inside the volume of the object
- Once the electric field is found, it can be used to compute the scattering far field
Surface integral equation

- If the permittivity of the object is a constant, the integral equation found above may appear to be an “overkill”. Is there a way to simplify the formalism?

- To provide an answer we should use the Huygens principle and the extinction theorem

- To discuss these, note that the dyadic Green’s function satisfies the vector wave equation (curl with respect to columns of the matrix)

\[
\nabla \times \nabla \times \mathcal{G}(r,r') - k^2 \mathcal{G}(r,r') = \delta(r-r') \mathcal{I}
\]
Huygens principle and the extinction theorem

- The dyadic Green’s function gives the field generated by point current sources in an infinite, homogeneous medium.
- But, what if there is also a *finite* homogeneous medium?
- Assume that there is finite, bound volume ($V_d$) of a dielectric. Impressed sources may be present outside this volume.
- The volume outside is denoted by $V_0$
Huygens principle and the extinction theorem

- Inside $V_0$:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k_0^2 \mathbf{E}(\mathbf{r}) = -j \omega \mu_0 \mathbf{J}(\mathbf{r})$$

- Now consider an arbitrary constant vector $\mathbf{v}$ and the equation

$$\nabla \times \nabla \times \mathbf{W}(\mathbf{r} | \mathbf{r}') - k_0^2 \mathbf{W}(\mathbf{r} | \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \mathbf{v}$$

$$\mathbf{W}(\mathbf{r} | \mathbf{r}') = \mathbf{\overline{G}}_0(\mathbf{r}, \mathbf{r}') \cdot \mathbf{v} \quad \text{A vector function}$$

$$\mathbf{\overline{G}}_0(\mathbf{r}, \mathbf{r}') = \mathbf{G}_0(\mathbf{r}, \mathbf{r}') \mathbf{I} - \frac{1}{k_0^2} \nabla \nabla' \mathbf{G}_0(\mathbf{r}, \mathbf{r}')$$

$$\mathbf{G}_0(\mathbf{r}, \mathbf{r}') = \frac{\exp(-jk_0|\mathbf{r} - \mathbf{r}'|)}{4\pi|\mathbf{r} - \mathbf{r}'|}$$
Huygens principle and the extinction theorem

- Theorem for any two vector fields:

\[
\nabla \cdot \left[ Q \times (\nabla \times P) - P \times (\nabla \times Q) \right] = P \cdot (\nabla \times \nabla \times Q) - Q \cdot (\nabla \times \nabla \times P)
\]

- Leads to

\[
\nabla \cdot \left[ W \times (\nabla \times E) - E \times (\nabla \times W) \right] = \delta(r - r') E(r) \cdot v + j \omega \mu_0 J(r) \cdot W(r \mid r')
\]
Huygens principle and the extinction theorem

- Integration over $V_0$ leads to

$$-\oint_S \left[ W \times (\nabla \times E) - E \times (\nabla \times W) \right] \cdot \hat{n} dS =$$

$$\mathbf{v} \cdot \int_{V_0} \delta (\mathbf{r} - \mathbf{r}') E(\mathbf{r}) dV + j \omega \mu_0 \int_{V_s} J(\mathbf{r}) \cdot W(\mathbf{r} | \mathbf{r}') dV$$
Huygens principle and the extinction theorem

- Result:

\[
\oint_S \left\{ -W(r | r') \times \left[ \nabla \times E(r) \right] + E(r) \times \left[ \nabla \times W(r | r') \right] \right\} \cdot \hat{n} dS
\]

\[
-j \omega \mu_0 \int_{V_s} J(r) \cdot W(r | r') dV = \begin{cases} E(r') \cdot v & \text{if } r' \text{ inside } V_0 \\ 0 & \text{if } r' \text{ inside } V_d \end{cases}
\]
Huygens principle and the extinction theorem

- After eliminating \( \nu \), it follows that

\[
\oint_S \left( \hat{n} \times [\nabla \times E(r)] \right) \cdot \vec{G}_0(r,r')dS + \oint_S [\hat{n} \times E(r)] \cdot \left[ \nabla \times \vec{G}_0(r,r') \right] dS
\]

\[
-j \omega \mu_0 \int_V J(r) \cdot \vec{G}_0(r,r')dV = \begin{cases} 
E^T(r') & \text{if } r' \text{ inside } V_0 \\
0 & \text{if } r' \text{ inside } V_d
\end{cases}
\]

- Next, use:

\[
\left[ \vec{G}_0(r,r') \right]^T = \vec{G}_0(r',r) \quad \left[ \nabla \times \vec{G}_0(r,r') \right]^T = \nabla' \times \vec{G}_0(r',r)
\]

\[
\nabla \times E(r) = -j \omega \mu_0 H(r)
\]
Huygens principle and the extinction theorem

- Final result:

\[-j\omega\mu_0 \oint_S \overline{G}_0(r',r) \cdot [\hat{n} \times H(r)] \, dS + \oint_S \left[ \nabla' \times \overline{G}_0(r',r) \right] \cdot [\hat{n} \times E(r)] \, dS\]

\[-j\omega\mu_0 \int_V \overline{G}_0(r',r) \cdot J(r) \, dV = \begin{cases} E(r') & \text{if } r' \text{ inside } V_0 \\ 0 & \text{if } r' \text{ inside } V_d \end{cases}\]
Huygens principle and the extinction theorem

\[-j\omega\mu_0 \int_s \vec{G}_0(\mathbf{r}', \mathbf{r}) \cdot [\hat{n} \times \mathbf{H}(\mathbf{r})]\,dS\]

Field of an electric *surface* current density

\[\hat{n} \times \mathbf{H}(\mathbf{r})\]

\[-\oint_s \left[ \nabla' \times \vec{G}_0(\mathbf{r}', \mathbf{r}) \right] \cdot [\hat{n} \times \mathbf{E}(\mathbf{r})]\,dS\]

Field of a magnetic *surface* current density

\[-\hat{n} \times \mathbf{E}(\mathbf{r})\]

\[-j\omega\mu_0 \int_v \vec{G}_0(\mathbf{r}', \mathbf{r}) \cdot \mathbf{J}(\mathbf{r})\,dV \equiv \mathbf{E}_i(\mathbf{r}')\]

Electric field of impressed sources in an otherwise empty space
Huygens principle and the extinction theorem

Remarks:

- Only inside the volume $V_0$ these contributions add up to the actual electric field; inside $V_d$ they add up to zero! (The actual field is, of course, not zero)

- This means that inside $V_d$, the field generated by impressed (true) sources ($J$) is exactly cancelled out by the fields of the fictitious surface sources on $S$
Huygens principle and the extinction theorem

- The result obtained can also be formulated in terms of the magnetic field by taking the curl with respect to \( r' \), using the equation for the matrix Green’s function, and noting that \( r \) in on the surface while \( r' \) is inside the volume:

\[
\oint_S \left[ \nabla' \times \mathcal{G}_0(r', r) \right] \cdot (\hat{n} \times \mathbf{H}(r)) \, dS + j \omega \varepsilon_0 \oint_S \mathcal{G}_0(r', r) \cdot (\hat{n} \times \mathbf{E}(r)) \, dS
\]

\[
+ \mathbf{H}_i(r') = \begin{cases} 
\mathbf{H}(r') & \text{if } r' \text{ inside } V_0 \\
0 & \text{if } r' \text{ inside } V_d 
\end{cases}
\]

\[
\mathbf{H}_i(r') = \int_V \left[ \nabla' \times \mathcal{G}_0(r', r) \right] \cdot \mathbf{J}(r) \, dV
\]

Magnetic field of the impressed source in empty space

Scattering from complex objects
Surface integral equations for dielectrics

- In the language of scattering, the field generated by the impressed sources is just the incident field (sources may be moved far away)

- Consider the electric field formulation again. If we are inside $V_d$, and move towards the surface we get the relation

$$-E_i(r') = -j\omega\mu_0 \oint_S \overline{G}_0(r',r) \cdot [\hat{n} \times H(r)] dS$$

$$+ \oint_S \left[ \nabla' \times \overline{G}_0(r',r) \right] \cdot [\hat{n} \times E(r)] dS$$

$r' \rightarrow S$ from within $V_d$
Surface integral equations for dielectrics

- But, what if we had done the calculation by considering the volume \( V_d \) instead of \( V_0 \)? (Note that there are no impressed sources inside \( V_d \)) We would then have

\[
0 = -j \omega \mu_0 \oint_S \overline{G}_d (r', r) \cdot [\hat{n} \times H (r)] \, dS \\
+ \oint_S \left[ \nabla' \times \overline{G}_d (r', r) \right] \cdot [\hat{n} \times E (r)] \, dS
\]
Huygens principle and the extinction theorem

- In this equation

\[
\vec{G}_d (r, r') = G_d (r, r') \vec{I} - \frac{1}{k_d^2} \nabla \nabla' G_d (r, r')
\]

\[
G_d (r, r') = \exp \left( - jk_d |r - r'| \right) \frac{k_d^2}{4\pi |r - r'|} = \omega^2 \mu_0 \epsilon_d
\]