Electromagnetic scattering

Graduate Course

Electrical Engineering (Communications)

1st Semester, 1388-1389

Sharif University of Technology

Contents of lecture 7

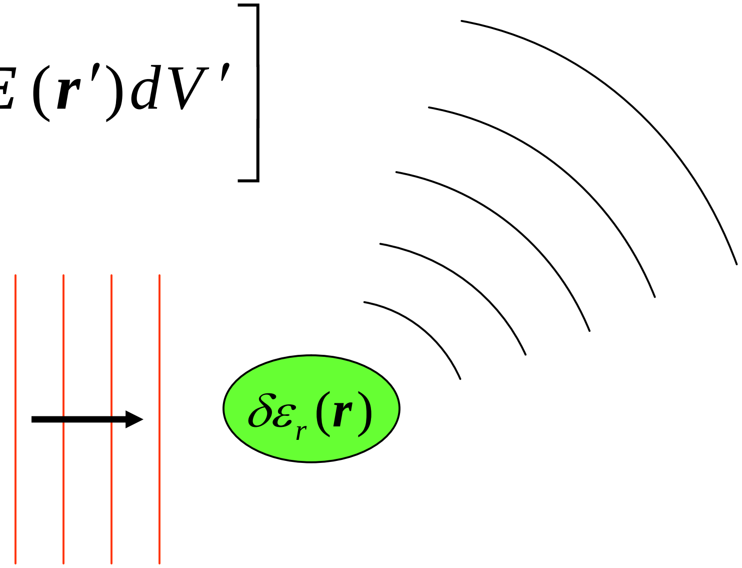
- **Contents of lecture 7:**
  - Scattering from complex objects
    - General formulation
    - Volume integral equation formalism for dielectrics
    - Huygens principle and the extinction theorem
    - Surface integral equation formalism for dielectrics
    - Scattering from conductors (surface integral equations):
      - EFIE
      - MFIE
      - Two dimensional conductors
      - One dimensional conductors (wires)

Scattering from dielectrics: general

- We have seen before that the scattered (far) field by a dielectric body with a scalar permittivity is given by

$$\boldsymbol{E}_s(\boldsymbol{r}) = -\frac{k^2 \exp(-jkr)}{4\pi\epsilon_0 r}\hat{\boldsymbol{k}}_s \times \left[\hat{\boldsymbol{k}}_s \times \int_V \exp(j\boldsymbol{k}_s \cdot \boldsymbol{r}')\,\delta\varepsilon_r(\boldsymbol{r}')\boldsymbol{E}(\boldsymbol{r}')dV'\right]$$

$$\delta\varepsilon_r(\boldsymbol{r}) = \frac{\varepsilon_p(\boldsymbol{r}) - \epsilon_0}{\epsilon_0}$$



Scattering from dielectrics: general

- Certain canonical problems (flat layers, spheres, infinite cylinders) can be exactly solved
- But, in general, the field distribution inside the scatterer has to be found by numerical means
- Remember that the equivalent electric current of a dielectric is given by

$$\boldsymbol{J}_{eq}(\boldsymbol{r}) = j\omega\left[\varepsilon_p(\boldsymbol{r}) - \epsilon_0\right]\boldsymbol{E}(\boldsymbol{r}) = j\omega\epsilon_0\delta\varepsilon(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r})$$

- But, instead of using this current to compute the far field, we use it to express the electric field everywhere as we see below

Dyadic Green's function

- We restrict ourselves to the Lorentz gauge:

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

$$\nabla^2 \varphi + k^2 \varphi = -\frac{\rho}{\epsilon}$$

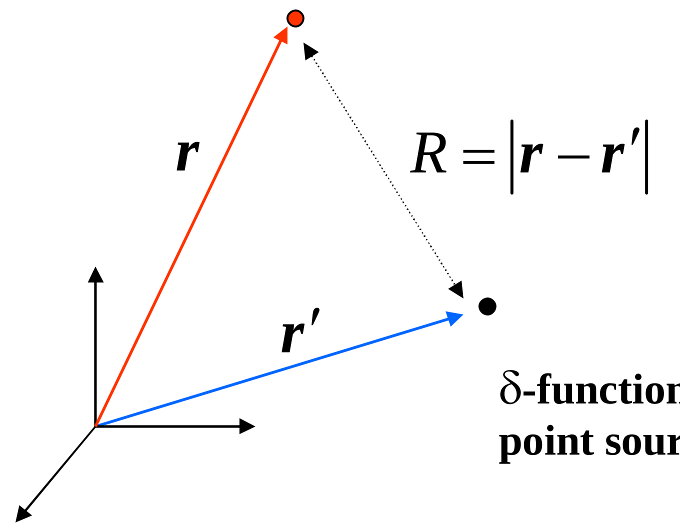
- The components of the vector potential and the scalar potential basically satisfy the same equation (Helmholtz)
- Consider the Green's function satisfying:

$$\nabla^2 G(\boldsymbol{r},\boldsymbol{r}') + k^2 G(\boldsymbol{r},\boldsymbol{r}') = -\delta\left(\boldsymbol{r}-\boldsymbol{r}'\right)$$

$$k^2 = \omega^2 \epsilon \mu$$

Dyadic Green's function

- This function gives the field at $\boldsymbol{r}$ generated by a point source at $\boldsymbol{r}$' (current or charge)
- The solution in *infinite space* is:



3D

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{\exp\left(-jk\left|\boldsymbol{r}-\boldsymbol{r}'\right|\right)}{4\pi\left|\boldsymbol{r}-\boldsymbol{r}'\right|}$$

$$\boldsymbol{r} = (x, y, z)$$

2D

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4j} H_0^{(2)}\left(k\left|\boldsymbol{r}-\boldsymbol{r}'\right|\right)$$

$$\boldsymbol{r} = (x, y)$$

Dyadic Green's function

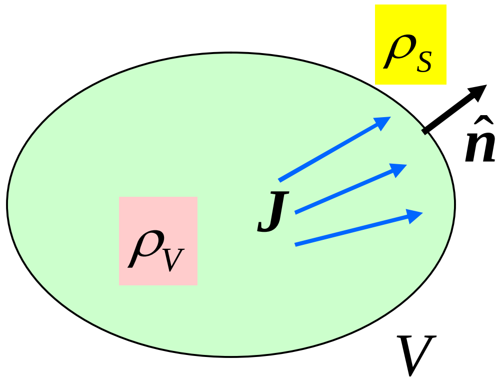
- Solution for potentials:

$$\varphi(\mathbf{r}) = \frac{1}{\epsilon}\int G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')dV' \qquad \boldsymbol{A}(\mathbf{r}) = \mu\int G(\mathbf{r},\mathbf{r}')\boldsymbol{J}(\mathbf{r}')dV'$$

- Electric field generated by currents and charges in an object

$$\boldsymbol{E}(\mathbf{r}) = -j\omega\mu\int G(\mathbf{r},\mathbf{r}')\boldsymbol{J}(\mathbf{r}')dV' - \frac{1}{\epsilon}\nabla\int G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')dV'$$

- We have:
  - Volume charges $\rho_V = -\nabla\cdot\boldsymbol{J}\,/\,j\omega$
  - Surface charges $\rho_S = \boldsymbol{J}\cdot\hat{\boldsymbol{n}}\,/\,j\omega$



Dyadic Green's function

- Using partial integration:

$$\boldsymbol{E}(\boldsymbol{r}) = -j\omega\mu\int_V G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}')dV' - \frac{1}{j\omega\epsilon}\nabla\int_V \nabla' G(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{J}(\boldsymbol{r}')dV'$$

$$\boldsymbol{E}(\boldsymbol{r}) = -j\omega\mu\int_{V_s} \overline{\overline{\boldsymbol{G}}}(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{J}(\boldsymbol{r}')dV'$$ ←

Be careful of the difficulties which may arise from literal interpretation of this result!

$$\overline{\overline{\boldsymbol{G}}}(\boldsymbol{r},\boldsymbol{r}') = G(\boldsymbol{r},\boldsymbol{r}')\overline{\overline{\boldsymbol{I}}} - \frac{1}{k^2}\nabla\nabla' G(\boldsymbol{r},\boldsymbol{r}')$$

Dyadic Green's function (matrix)

Scattering from dielectrics: general

- Returning to the dielectric object, the field generated by the equivalent current (scattered field) is everywhere

$$\boldsymbol{E}_s(\boldsymbol{r}) = -j\omega\mu_0 \int_V \overline{\overline{\boldsymbol{G}}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}_{eq}(\boldsymbol{r}')dV'$$

$$\overline{\overline{\boldsymbol{G}}}(\boldsymbol{r},\boldsymbol{r}') = G(\boldsymbol{r},\boldsymbol{r}')\overline{\overline{\boldsymbol{I}}} - \frac{1}{k^2}\nabla\nabla' G(\boldsymbol{r},\boldsymbol{r}')$$

Dyadic Green's function

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{\exp\left(-jk\left|\boldsymbol{r}-\boldsymbol{r}'\right|\right)}{4\pi\left|\boldsymbol{r}-\boldsymbol{r}'\right|}$$

3D

- Bearing in mind that the total field (everywhere) is the superposition of the incident and scattered fields, one finds a volume integral equation for the electric field inside the object

Volume integral equation

- Volume integral equation formulation

$$\boldsymbol{E}(\mathbf{r}) = \boldsymbol{E}_i(\mathbf{r}) + k_0^2 \int_V \overline{\overline{\boldsymbol{G}}}(\mathbf{r},\mathbf{r}') \cdot \left[\delta\varepsilon(\mathbf{r}')\boldsymbol{E}(\mathbf{r}')\right] dV'$$

$$k_0^2 = \omega^2 \mu_0 \epsilon_0$$

- This equation has to be solved inside the volume of the object
- Once the electric field is found, it can be used to compute the scattering far field

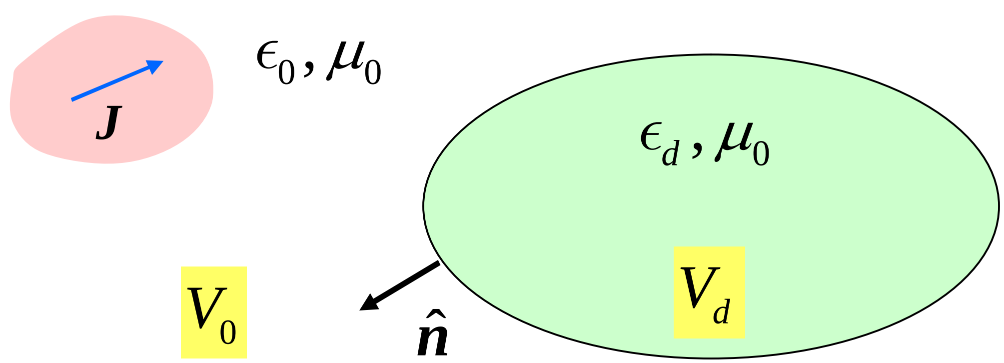
Surface integral equation

- If the permittivity of the object is a constant, the integral equation found above may appear to be an "overkill". Is there a way to simplify the formalism?
- To provide an answer we should use the Huygens principle and the extinction theorem
- To discuss these, note that the dyadic Green's function satisfies the vector wave equation (curl with respect to columns of the matrix)

$$\nabla \times \nabla \times \bar{\bar{\boldsymbol{G}}}(\boldsymbol{r},\boldsymbol{r}') - k^2 \bar{\bar{\boldsymbol{G}}}(\boldsymbol{r},\boldsymbol{r}') = \delta\left(\boldsymbol{r}-\boldsymbol{r}'\right)\bar{\bar{\boldsymbol{I}}}$$

Huygens principle and the extinction theorem

- The dyadic Green's function gives the field generated by point current sources in an infinite, homogeneous medium.
- But, what if there is also a *finite* homogeneous medium?
- Assume that there is finite, bound volume (V_d) of a dielectric. Impressed sources may be present outside this volume
- The volume outside is denoted by V_0



Huygens principle and the extinction theorem

- Inside V_0:

$$\nabla \times \nabla \times \boldsymbol{E}(\boldsymbol{r}) - k_0^2 \boldsymbol{E}(\boldsymbol{r}) = -j\omega\mu_0 \boldsymbol{J}(\boldsymbol{r})$$

- Now consider an arbitrary constant vector $\boldsymbol{v}$ and the equation

$$\nabla \times \nabla \times \boldsymbol{W}\left(\boldsymbol{r} \mid \boldsymbol{r}'\right) - k_0^2 \boldsymbol{W}\left(\boldsymbol{r} \mid \boldsymbol{r}'\right) = \delta\left(\boldsymbol{r} - \boldsymbol{r}'\right)\boldsymbol{v}$$

$$\left\{ \begin{aligned} &\boldsymbol{W}\left(\boldsymbol{r} \mid \boldsymbol{r}'\right) = \overline{\overline{\boldsymbol{G}}}_0(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{v} \\ &\overline{\overline{\boldsymbol{G}}}_0(\boldsymbol{r}, \boldsymbol{r}') = G_0(\boldsymbol{r}, \boldsymbol{r}')\overline{\overline{\boldsymbol{I}}} - \frac{1}{k_0^2}\nabla\nabla' G_0(\boldsymbol{r}, \boldsymbol{r}') \\ &G_0(\boldsymbol{r}, \boldsymbol{r}') = \frac{\exp\left(-jk_0\left|\boldsymbol{r} - \boldsymbol{r}'\right|\right)}{4\pi\left|\boldsymbol{r} - \boldsymbol{r}'\right|} \end{aligned} \right.$$

← A vector function

Huygens principle and the extinction theorem

- Theorem for any two vector fields:

$$\nabla \cdot \left[\boldsymbol{Q} \times \left(\nabla \times \boldsymbol{P} \right) - \boldsymbol{P} \times \left(\nabla \times \boldsymbol{Q} \right) \right]$$

$$= \boldsymbol{P} \cdot \left(\nabla \times \nabla \times \boldsymbol{Q} \right) - \boldsymbol{Q} \cdot \left(\nabla \times \nabla \times \boldsymbol{P} \right)$$

- Leads to

$$\nabla \cdot \left[\boldsymbol{W} \times \left(\nabla \times \boldsymbol{E} \right) - \boldsymbol{E} \times \left(\nabla \times \boldsymbol{W} \right) \right] =$$

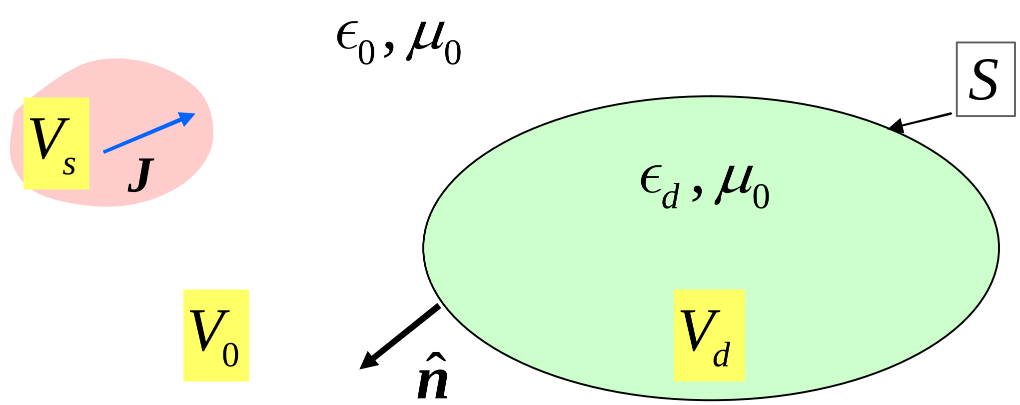
$$\delta \left(\boldsymbol{r} - \boldsymbol{r}' \right) \boldsymbol{E}(\boldsymbol{r}) \cdot \boldsymbol{v} + j\omega\mu_0 \boldsymbol{J}(\boldsymbol{r}) \cdot \boldsymbol{W} \left(\boldsymbol{r} \,|\, \boldsymbol{r}' \right)$$

Huygens principle and the extinction theorem

- Integration over V_0 leads to

$$-\oint_S \left[\boldsymbol{W} \times \left(\nabla \times \boldsymbol{E} \right) - \boldsymbol{E} \times \left(\nabla \times \boldsymbol{W} \right) \right] \cdot \hat{\boldsymbol{n}} dS =$$

$$\boldsymbol{v} \cdot \int_{V_0} \delta \left(\boldsymbol{r} - \boldsymbol{r}' \right) \boldsymbol{E}(\boldsymbol{r}) dV + j\omega\mu_0 \int_{V_s} \boldsymbol{J}(\boldsymbol{r}) \cdot \boldsymbol{W} \left(\boldsymbol{r} \,|\, \boldsymbol{r}' \right) dV$$

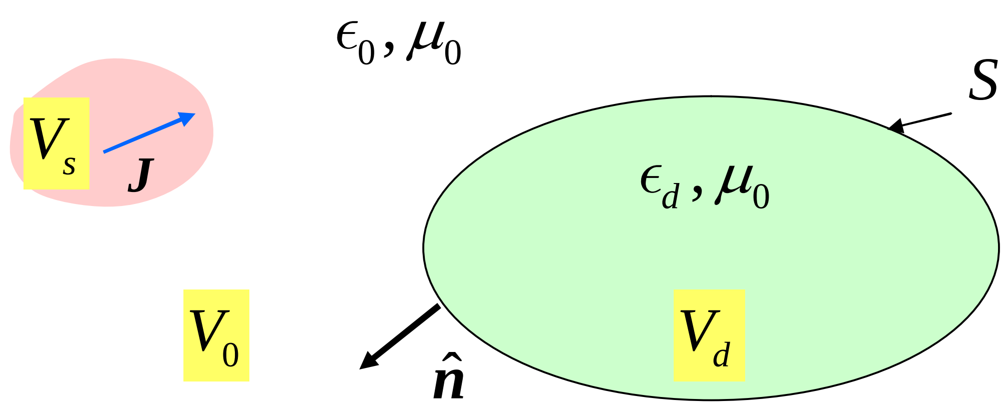


Huygens principle and the extinction theorem

- Result:

$$\oint_S \left\{ -\boldsymbol{W}(\boldsymbol{r}\,|\,\boldsymbol{r}') \times \left[\nabla \times \boldsymbol{E}(\boldsymbol{r}) \right] + \boldsymbol{E}(\boldsymbol{r}) \times \left[\nabla \times \boldsymbol{W}(\boldsymbol{r}\,|\,\boldsymbol{r}') \right] \right\} \cdot \hat{\boldsymbol{n}} \, dS$$

$$- j\omega\mu_0 \int_{V_s} \boldsymbol{J}(\boldsymbol{r}) \cdot \boldsymbol{W}(\boldsymbol{r}\,|\,\boldsymbol{r}') \, dV = \begin{cases} \boldsymbol{E}(\boldsymbol{r}') \cdot \boldsymbol{v} & \text{if } \boldsymbol{r}' \text{ inside } V_0 \\ 0 & \text{if } \boldsymbol{r}' \text{ inside } V_d \end{cases}$$



Huygens principle and the extinction theorem

- After eliminating $\boldsymbol{v}$, it follows that

$$\oint_S \left(\hat{\boldsymbol{n}}\times\left[\nabla\times\boldsymbol{E}(\boldsymbol{r})\right]\right)\cdot\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r},\boldsymbol{r}')dS+\oint_S\left[\hat{\boldsymbol{n}}\times\boldsymbol{E}(\boldsymbol{r})\right]\cdot\left[\nabla\times\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r},\boldsymbol{r}')\right]dS$$

$$-j\omega\mu_0\int_V \boldsymbol{J}(\boldsymbol{r})\cdot\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r},\boldsymbol{r}')dV=\begin{cases}\boldsymbol{E}^T(\boldsymbol{r}') & \text{if } \boldsymbol{r}' \text{ inside } V_0\\ 0 & \text{if } \boldsymbol{r}' \text{ inside } V_d\end{cases}$$

- Next, use:

$$\left[\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r},\boldsymbol{r}')\right]^T=\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r}) \qquad \left[\nabla\times\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r},\boldsymbol{r}')\right]^T=\nabla'\times\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})$$

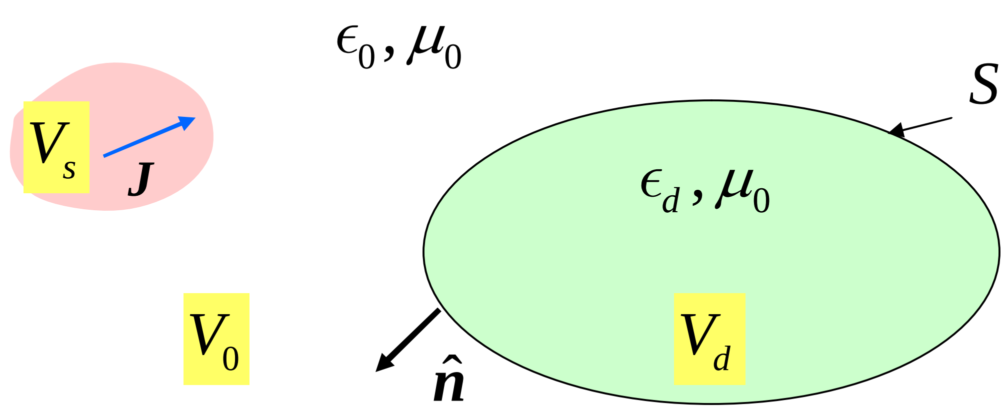
$$\nabla\times\boldsymbol{E}(\boldsymbol{r})=-j\omega\mu_0\boldsymbol{H}(\boldsymbol{r})$$

Huygens principle and the extinction theorem

- Final result:

$$-j\omega\mu_0 \oint_S \overline{\overline{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\cdot[\hat{\boldsymbol{n}}\times\boldsymbol{H}(\boldsymbol{r})]dS + \oint_S \left[\nabla'\times\overline{\overline{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\right]\cdot[\hat{\boldsymbol{n}}\times\boldsymbol{E}(\boldsymbol{r})]dS$$

$$-j\omega\mu_0 \int_V \overline{\overline{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\cdot\boldsymbol{J}(\boldsymbol{r})dV = \begin{cases} \boldsymbol{E}(\boldsymbol{r}') & \text{if } \boldsymbol{r}' \text{ inside } V_0 \\ 0 & \text{if } \boldsymbol{r}' \text{ inside } V_d \end{cases}$$



Huygens principle and the extinction theorem

$$-j\omega\mu_0 \oint_S \overline{\overline{G}}_0(\mathbf{r}',\mathbf{r}) \cdot \left[\hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r})\right] dS$$

Field of an electric *surface* current density

$$\hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r})$$

$$-\oint_S \left[\nabla' \times \overline{\overline{G}}_0(\mathbf{r}',\mathbf{r})\right] \cdot \left[\hat{\mathbf{n}} \times \mathbf{E}(\mathbf{r})\right] dS$$

Field of a magnetic *surface* current density

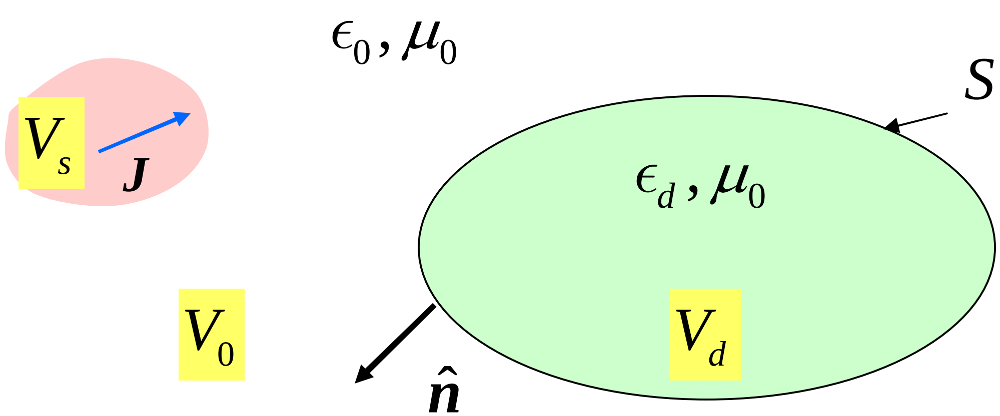
$$-\hat{\mathbf{n}} \times \mathbf{E}(\mathbf{r})$$

$$-j\omega\mu_0 \int_V \overline{\overline{G}}_0(\mathbf{r}',\mathbf{r}) \cdot \mathbf{J}(\mathbf{r})\, dV \equiv \mathbf{E}_i(\mathbf{r}')$$

Electric field of impressed sources
in an otherwise empty space

Huygens principle and the extinction theorem

- Remarks:
  - Only *inside* the volume V_0 these contributions add up to the actual electric field; inside V_d they add up to zero! (The actual field is, of course, not zero)
  - This means that inside V_d, the field generated by impressed (true) sources ($\boldsymbol{J}$) is exactly cancelled out by the fields of the fictitious surface sources on S



Huygens principle and the extinction theorem

- The result obtained can also be formulated in terms of the magnetic field by taking the curl with respect to ***r'***, using the equation for the matrix Green's function, and noting that ***r*** in on the surface while ***r'*** is *inside* the volume:

$$\oint_S \left[\nabla' \times \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\right]\cdot\left[\hat{\boldsymbol{n}}\times \boldsymbol{H}(\boldsymbol{r})\right]dS + j\omega\epsilon_0 \oint_S \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\cdot\left[\hat{\boldsymbol{n}}\times \boldsymbol{E}(\boldsymbol{r})\right]dS$$

$$+\boldsymbol{H}_i(\boldsymbol{r}') = \begin{cases} \boldsymbol{H}(\boldsymbol{r}') & \text{if } \boldsymbol{r}' \text{ inside } V_0 \\ 0 & \text{if } \boldsymbol{r}' \text{ inside } V_d \end{cases}$$

$$\boldsymbol{H}_i(\boldsymbol{r}') = \int_V \left[\nabla' \times \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\right]\cdot \boldsymbol{J}(\boldsymbol{r})dV \longleftarrow$$

Magnetic field of the impressed source in empty space

Surface integral equations for dielectrics

- In the language of scattering, the field generated by the impressed sources is just the incident field (sources may be moved far away)
- Consider the electric field formulation again. It we are inside V_d , and move towards the surface we get the relation

$$-\boldsymbol{E}_i(\boldsymbol{r}') = -j\omega\mu_0 \oint_S \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r}) \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r})\right] dS + \oint_S \left[\nabla' \times \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\right] \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r})\right] dS$$

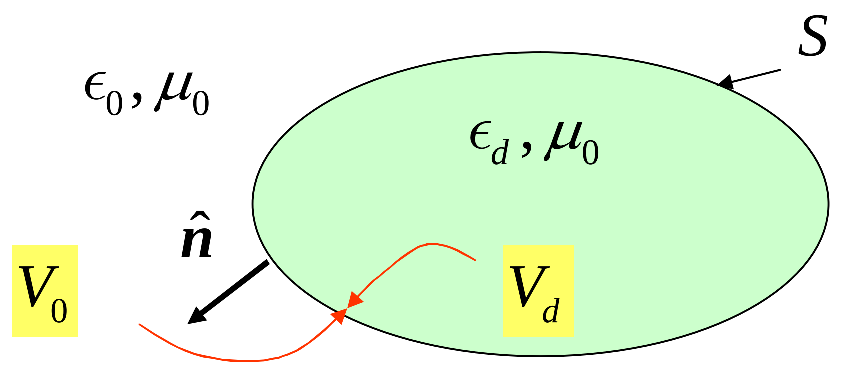
$\boldsymbol{r}' \to S$ from within V_d

Surface integral equations for dielectrics

- But, what if we had done the calculation by considering the volume V_d instead of V_0? (Note that there are no impressed sources inside V_d) We would then have

$$0 = -j\omega\mu_0 \oint_S \bar{\bar{\boldsymbol{G}}}_d(\boldsymbol{r}',\boldsymbol{r}) \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r})\right] dS + \oint_S \left[\nabla' \times \bar{\bar{\boldsymbol{G}}}_d(\boldsymbol{r}',\boldsymbol{r})\right] \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r})\right] dS$$

$\boldsymbol{r}' \to S$ from within V_0



Huygens principle and the extinction theorem

- In this equation

$$\overline{\overline{\boldsymbol{G}}}_d(\boldsymbol{r},\boldsymbol{r}') = G_d(\boldsymbol{r},\boldsymbol{r}')\overline{\overline{\boldsymbol{I}}} - \frac{1}{k_d^2}\nabla\nabla' G_d(\boldsymbol{r},\boldsymbol{r}')$$

$$G_d(\boldsymbol{r},\boldsymbol{r}') = \frac{\exp\left(-jk_d\left|\boldsymbol{r}-\boldsymbol{r}'\right|\right)}{4\pi\left|\boldsymbol{r}-\boldsymbol{r}'\right|}$$

$$k_d^2 = \omega^2\mu_0\epsilon_d$$