Surface integral equations for dielectrics

- We can now derive a set of coupled surface integral equations for the tangential components of the electric and magnetic fields (they are continuous across interface)

$$0 = \boldsymbol{E}_i(\boldsymbol{r}') - j\omega\mu_0 \oint_S \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r}) \cdot [\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r})] dS$$

$$+ \oint_S \left[\nabla' \times \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\right] \cdot [\hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r})] dS$$

$\boldsymbol{r}' \to S$ from V_d

$$0 = -j\omega\mu_0 \oint_S \bar{\bar{\boldsymbol{G}}}_d(\boldsymbol{r}',\boldsymbol{r}) \cdot [\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r})] dS$$

$$+ \oint_S \left[\nabla' \times \bar{\bar{\boldsymbol{G}}}_d(\boldsymbol{r}',\boldsymbol{r})\right] \cdot [\hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r})] dS$$

$\boldsymbol{r}' \to S$ from V_0

Surface integral equations for dielectrics

- There seems to be a discrepancy here. There are 4 unknown fields (tangential components of E and H) but six equations!
- But actually two of those are redundant. To see this take the take the right hand side of the 2^{nd} equation.
- For $\boldsymbol{r}' \in V_0$, this is the field of equivalent surface electric and magnetic currents on S, generated in V_0 *if* V_0 is filled with the same material as the object. But if the tangential component of this electric field is zero on S (as $\boldsymbol{r}' \to S$) then all the components of the field (including normal component) will be zero in V_0

Surface integral equations for dielectrics

- Hence, we only need to consider the tangential part:

$$0 = -j\omega\mu_0 \hat{\boldsymbol{n}}' \times \oint_S \bar{\bar{\boldsymbol{G}}}_d(\boldsymbol{r}',\boldsymbol{r}) \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r})\right] dS$$
$$+ \hat{\boldsymbol{n}}' \times \oint_S \left[\nabla' \times \bar{\bar{\boldsymbol{G}}}_d(\boldsymbol{r}',\boldsymbol{r})\right] \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r})\right] dS$$

- A similar argument applies to the 1st equation so that

$$-\hat{\boldsymbol{n}}' \times \boldsymbol{E}_i(\boldsymbol{r}') = -j\omega\mu_0 \hat{\boldsymbol{n}}' \times \oint_S \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r}) \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r})\right] dS$$
$$+ \hat{\boldsymbol{n}}' \times \oint_S \left[\nabla' \times \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\right] \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r})\right] dS$$

Surface integral equations for dielectrics

- The same can be done using the magnetic field formulation

$$-\hat{\boldsymbol{n}}' \times \boldsymbol{H}_i(\boldsymbol{r}') = \hat{\boldsymbol{n}}' \times \oint_S \left[\nabla' \times \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}', \boldsymbol{r}) \right] \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r}) \right] dS$$

$$+ j\omega\epsilon_0 \hat{\boldsymbol{n}}' \times \oint_S \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}', \boldsymbol{r}) \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r}) \right] dS$$

$\boldsymbol{r}' \to S$ from V_d

$$0 = \hat{\boldsymbol{n}}' \times \oint_S \left[\nabla' \times \bar{\bar{\boldsymbol{G}}}_d(\boldsymbol{r}', \boldsymbol{r}) \right] \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r}) \right] dS$$

$$+ j\omega\epsilon_d \hat{\boldsymbol{n}}' \times \oint_S \bar{\bar{\boldsymbol{G}}}_d(\boldsymbol{r}', \boldsymbol{r}) \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r}) \right] dS$$

$\boldsymbol{r}' \to S$ from V_0

Surface integral equations for dielectrics

- Let us be more specific. Note that in terms of the scalar Green's function

$$\bar{\bar{\boldsymbol{G}}}(\boldsymbol{r}',\boldsymbol{r}) = G(\boldsymbol{r}',\boldsymbol{r})\bar{\bar{\boldsymbol{I}}} - \frac{1}{k^2}\nabla'\nabla G(\boldsymbol{r}',\boldsymbol{r})$$

$$\left[\nabla'\times\bar{\bar{\boldsymbol{G}}}(\boldsymbol{r}',\boldsymbol{r})\right]\cdot\left[\hat{\boldsymbol{n}}\times\boldsymbol{E}(\boldsymbol{r})\right] = \nabla' G(\boldsymbol{r}',\boldsymbol{r})\times\left[\hat{\boldsymbol{n}}\times\boldsymbol{E}(\boldsymbol{r})\right]$$
$$= \nabla'\times\left[G(\boldsymbol{r}',\boldsymbol{r})\hat{\boldsymbol{n}}\times\boldsymbol{E}(\boldsymbol{r})\right]$$

$$\bar{\bar{\boldsymbol{G}}}(\boldsymbol{r}',\boldsymbol{r})\cdot\left[\hat{\boldsymbol{n}}\times\boldsymbol{H}(\boldsymbol{r})\right] = G(\boldsymbol{r}',\boldsymbol{r})\left[\hat{\boldsymbol{n}}\times\boldsymbol{H}(\boldsymbol{r})\right]$$
$$-\frac{1}{k^2}\nabla'\left\{\nabla G(\boldsymbol{r}',\boldsymbol{r})\cdot\left[\hat{\boldsymbol{n}}\times\boldsymbol{H}(\boldsymbol{r})\right]\right\}$$

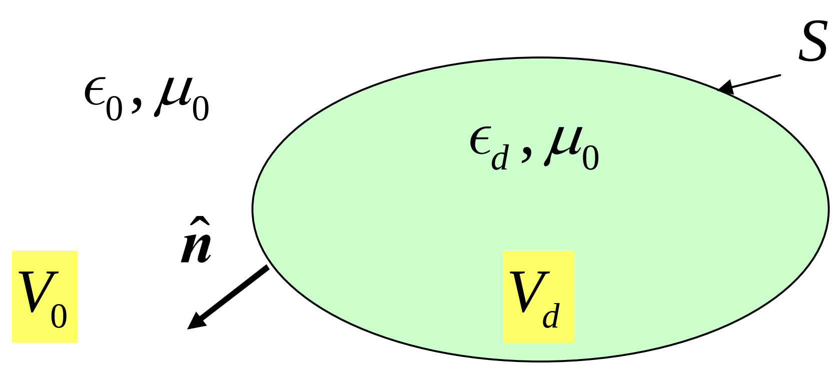
Surface integral equations for dielectrics

- Once these equations are solved, the surface fields can be used to evaluate the scattered field everywhere outside the object by using the Huygens principle:

$$\boldsymbol{E}(\boldsymbol{r}') = \boldsymbol{E}_i(\boldsymbol{r}') - j\omega\mu_0 \oint_S \overline{\overline{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r}) \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r})\right] dS + \oint_S \left[\nabla' \times \overline{\overline{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\right] \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r})\right] dS$$

Scattered field

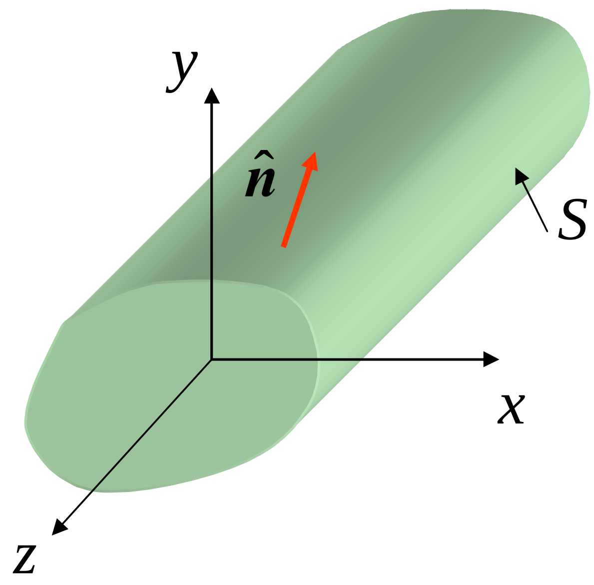
$$\boldsymbol{r}' \in V_0$$



2D surface integral equations

- Let us give an example: we consider a cylindrical object extended in the z-direction and restrict ourselves to solutions *uniform* in the z-direction (2D problem)
- We can again distinguish the TE and TM solutions:
- TE: H has only a z-component and E lies on the x-y plane
- TM: E has only a z-component and H lies on the x-y plane

$$\boldsymbol{\rho} = (x, y)$$



2D surface integral equations (TM)

- For the 2D TM problem the electric field equations can be written as

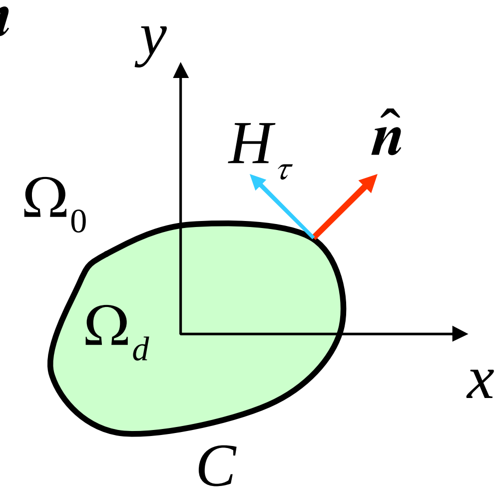
$$-E_{i,z}(\boldsymbol{\rho}') = -j\omega\mu_0 \oint_C G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho}) H_\tau(\boldsymbol{\rho}) dl - \oint_C \partial_n^- G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho}) E_z(\boldsymbol{\rho}) dl$$

$$\partial_n^- G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho}) \equiv -\nabla G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho})\cdot\hat{\boldsymbol{n}} = \nabla' G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho})\cdot\hat{\boldsymbol{n}} \qquad \boldsymbol{\rho}' \to C \ \text{ from } \Omega_d$$

$$= -\lim_{\zeta\to 0} \nabla G_0^{(2D)}(\boldsymbol{\rho}' - \zeta\hat{\boldsymbol{n}}',\boldsymbol{\rho})\cdot\hat{\boldsymbol{n}}$$

- Here we have introduced the magnetic field tangential to the boundary curve as

$$H_\tau(\boldsymbol{\rho}) = \left[\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{\rho})\right]\cdot\hat{\boldsymbol{z}}$$



2D surface integral equations (TM)

- The other equation

$$0 = -j\omega\mu_0 \oint_C G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) H_\tau(\boldsymbol{\rho}) dl + \oint_C \partial_n^+ G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) E_z(\boldsymbol{\rho}) dl$$

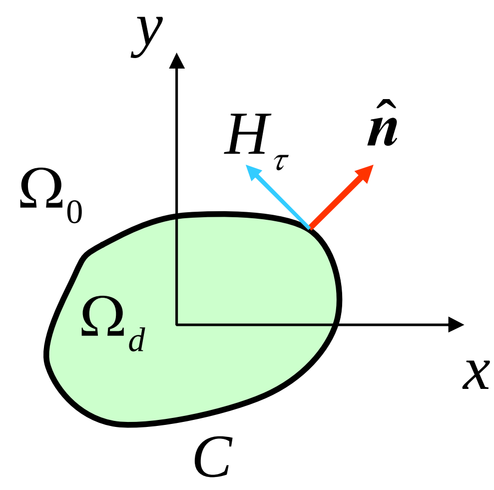
$$\partial_n^+ G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) \equiv \nabla G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) \cdot \hat{\boldsymbol{n}} = -\nabla' G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) \cdot \hat{\boldsymbol{n}} \quad \boldsymbol{\rho}' \to C \ \text{ from } \Omega_0$$

$$= \lim_{\zeta \to 0} \nabla G_d^{(2D)}(\boldsymbol{\rho}' + \zeta \hat{\boldsymbol{n}}', \boldsymbol{\rho}) \cdot \hat{\boldsymbol{n}}$$

- In all these equations

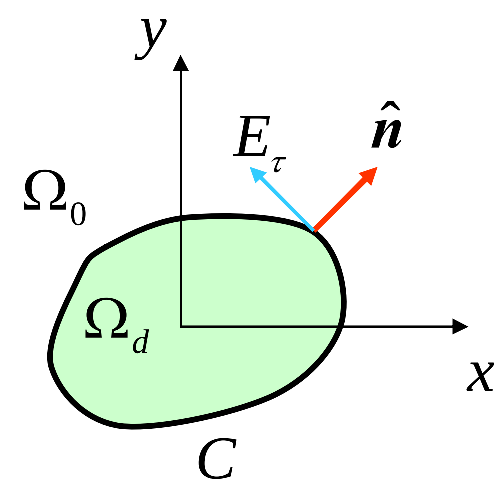
$$G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) = \frac{1}{4j} H_0^{(2)}\left(k_0 \left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right)$$

$$G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) = \frac{1}{4j} H_0^{(2)}\left(k_d \left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right)$$



2D surface integral equations (TE)

- Now, for the TE case we use the magnetic field formulation. We now have a magnetic field along z and an electric field on the x-y plane (uniformity is again assumed along z)



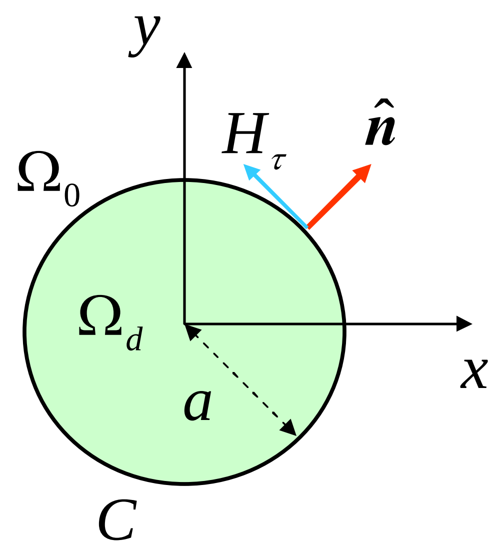
$$-H_{i,z}(\boldsymbol{\rho}') = j\omega\epsilon_0 \oint_C G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho}) E_\tau(\boldsymbol{\rho}) dl - \oint_C \partial_n^- G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho}) H_z(\boldsymbol{\rho}) dl$$

$$0 = j\omega\epsilon_d \oint_C G_d^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho}) E_\tau(\boldsymbol{\rho}) dl + \oint_C \partial_n^+ G_d^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho}) H_z(\boldsymbol{\rho}) dl$$

- Now we can define surface *impedance* functions and an equivalent surface magnetic current for the incident field

2D surface integral equations (TE)

- Let us treat a circular dielectric cylinder. Of course we can exactly solve it using the usual cylindrical vector functions. But here we apply the surface integral technique.



- For the TM case:

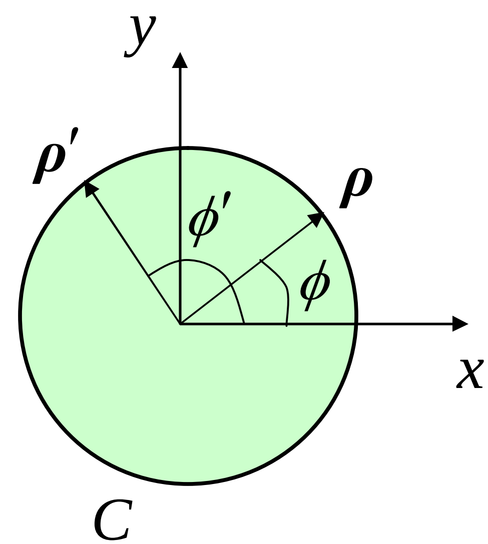
$$-E_{i,z}(\boldsymbol{\rho}') = -j\omega\mu_0 \oint_C G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) H_\tau(\boldsymbol{\rho}) dl - \oint_C \partial_n^- G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) E_z(\boldsymbol{\rho}) dl$$

$$0 = -j\omega\mu_0 \oint_C G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) H_\tau(\boldsymbol{\rho}) dl + \oint_C \partial_n^+ G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) E_z(\boldsymbol{\rho}) dl$$

Dielectric cylinder

- Let us see what the Green's functions are on the surface

$$G^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) = \frac{1}{4j} H_0^{(2)}\left(k\left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right)$$



- We use the addition theorem

$$H_0^{(2)}\left(\sqrt{u^2 + v^2 - 2uv\cos\beta}\right) = \sum_{m=-\infty}^{\infty} H_m^{(2)}(u) J_m(v) \exp\left(-jm\beta\right) \qquad u \geq v$$

$$G^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) = \frac{1}{4j} \sum_{m=-\infty}^{\infty} H_m^{(2)}\left(ka\right) J_m\left(ka\right) \exp\left[-jm\left(\phi' - \phi\right)\right]$$

$$k = k_0, k_d$$

Dielectric cylinder

- Now for the other functions:

$$\partial_n^- G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) = -\nabla G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) \cdot \hat{\boldsymbol{n}} = -\frac{\partial G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho})}{\partial \rho}$$

$\boldsymbol{\rho}' \to C$ from Ω_d

$$\partial_n^- G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) = -\frac{k_0}{4j} \sum_{m=-\infty}^{\infty} H_m^{(2)\prime}(k_0 a) J_m(k_0 a) \exp\left[-jm(\phi' - \phi)\right]$$

- Moreover:

$$\partial_n^+ G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) = \nabla G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) \cdot \hat{\boldsymbol{n}} = \frac{\partial G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho})}{\partial \rho}$$

$\boldsymbol{\rho}' \to C$ from Ω_0

$$\partial_n^+ G_d^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) = \frac{k_d}{4j} \sum_{m=-\infty}^{\infty} H_m^{(2)}(k_d a) J_m'(k_d a) \exp\left[-jm(\phi' - \phi)\right]$$

Dielectric cylinder

- Resulting equations

$$-4j\mathrm{e}_{i,m} = -j\omega\mu_0 2\pi a H_m^{(2)}(k_0 a) J_m(k_0 a)\mathrm{h}_m + 2\pi k_0 a H_m^{(2)\prime}(k_0 a) J_m(k_0 a)\mathrm{e}_m$$

$$0 = -j\omega\mu_0 2\pi a H_m^{(2)}(k_d a) J_m(k_d a)\mathrm{h}_m + 2\pi k_d a H_m^{(2)}(k_d a) J_m'(k_d a)\mathrm{e}_m$$

$$\begin{Bmatrix} \mathrm{h}_m \\ \mathrm{e}_m \\ \mathrm{e}_{i,m} \end{Bmatrix} = \frac{1}{2\pi}\int_0^{2\pi} \begin{Bmatrix} H_\tau(\phi') \\ E_z(\phi') \\ E_{i,z}(\phi') \end{Bmatrix} \exp(jm\phi')\, d\phi'$$

Dielectric cylinder

$$\mathrm{h}_m = \frac{1}{j\omega\mu_0 a}\frac{(k_d a)J'_m(k_d a)}{J_m(k_d a)}\mathrm{e}_m$$

$$-\frac{2j}{\pi}\mathrm{e}_{i,m} = H_m^{(2)}(k_0 a)J_m(k_0 a)\left[\frac{(k_0 a)H_m^{(2)\prime}(k_0 a)}{H_m^{(2)}(k_0 a)} - \frac{(k_d a)J'_m(k_d a)}{J_m(k_d a)}\right]\mathrm{e}_m$$

- The solved coefficients can then be used to evaluate the surface fields and, next, the overall field outside the object by using the Huygens principle

Surface integral equations for dielectrics

- In general, however, the solution to the line 2D or surface (3D) boundary integral equations should be obtained numerically, e.g. by using the method of moments