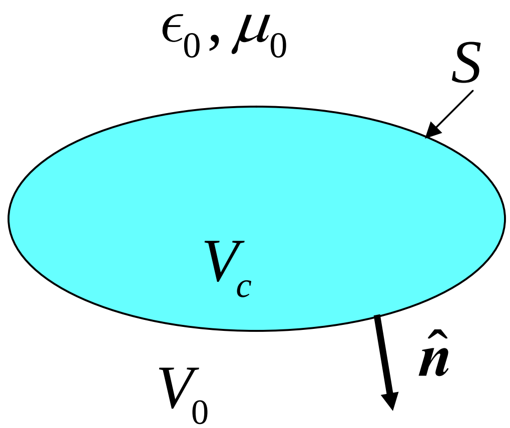
Surface integral equations for conductors

- Let us return to our surface integral equations, but now assume that the scattering object is a perfect conductor. The immediate result is the surface integral equation



$$-\hat{\boldsymbol{n}}' \times \boldsymbol{E}_i(\boldsymbol{r}') = -j\omega\mu_0 \hat{\boldsymbol{n}}' \times \oint_S \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}', \boldsymbol{r}) \cdot [\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r})] dS$$

or

$$-\hat{\boldsymbol{n}}' \times \boldsymbol{H}_i(\boldsymbol{r}') = \hat{\boldsymbol{n}}' \times \oint_S \left[\nabla' \times \bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}', \boldsymbol{r})\right] \cdot [\hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r})] dS$$

- In evaluating the Green's functions note that $\boldsymbol{r}' \to S \;\; \text{from } V_c$

Surface integral equations for conductors

- Let us first focus on the 1st equation (EFIE). Note that

$$\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\cdot[\hat{\boldsymbol{n}}\times\boldsymbol{H}(\boldsymbol{r})]=\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\cdot\boldsymbol{J}_s(\boldsymbol{r})$$

$$=G_0(\boldsymbol{r}',\boldsymbol{r})\boldsymbol{J}_s(\boldsymbol{r})-\frac{1}{k^2}\nabla'\left\{\nabla G_0(\boldsymbol{r}',\boldsymbol{r})\cdot\boldsymbol{J}_s(\boldsymbol{r})\right\}$$

$$\boldsymbol{J}_s(\boldsymbol{r})=\hat{\boldsymbol{n}}\times\boldsymbol{H}(\boldsymbol{r})$$

Surface electric current

- By partial integration over S, it's right hand side becomes

$$-j\omega\mu_0\hat{\boldsymbol{n}}'\times\oint_S\left[G_0(\boldsymbol{r}',\boldsymbol{r})\boldsymbol{J}_s(\boldsymbol{r})+\frac{1}{k^2}\nabla'G_0(\boldsymbol{r}',\boldsymbol{r})\nabla\cdot\boldsymbol{J}_s(\boldsymbol{r})\right]dS$$

Surface integral equations for conductors

- Using the continuity relation for surface current

$$\rightarrow \hat{\boldsymbol{n}}' \times \left\{ -j\omega\mu_0 \oint_S G_0(\boldsymbol{r}',\boldsymbol{r})\boldsymbol{J}_s(\boldsymbol{r})dS - \frac{1}{\epsilon_0}\nabla' \oint_S G_0(\boldsymbol{r}',\boldsymbol{r})\rho_s(\boldsymbol{r})dS \right\}$$

$$\rho_s(\boldsymbol{r}) = -\frac{1}{j\omega}\nabla \cdot \boldsymbol{J}_s(\boldsymbol{r})$$

Surface current density

- This is the tangential electric field generated inside V_c by surface currents and charges. The 1st equation states that the total electric field should have a zero tangential component as we approach S from the conductor side.
- It remains true as we move across S.

Surface integral equations for conductors

- Now consider the 2nd equation (MFIE). Note that

$$\left[\nabla'\times\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\right]\cdot\left[\hat{\boldsymbol{n}}\times\boldsymbol{H}(\boldsymbol{r})\right]=\left[\nabla'\times\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\right]\cdot\boldsymbol{J}_s(\boldsymbol{r})$$
$$=\nabla'\times\left[G_0(\boldsymbol{r}',\boldsymbol{r})\boldsymbol{J}_s(\boldsymbol{r})\right]$$

$$\hat{\boldsymbol{n}}'\times\oint_S\left[\nabla'\times\bar{\bar{\boldsymbol{G}}}_0(\boldsymbol{r}',\boldsymbol{r})\right]\cdot\left[\hat{\boldsymbol{n}}\times\boldsymbol{H}(\boldsymbol{r})\right]dS=\hat{\boldsymbol{n}}'\times\left[\nabla'\times\oint_S G_0(\boldsymbol{r}',\boldsymbol{r})\boldsymbol{J}_s(\boldsymbol{r})dS\right]$$

- This is the tangential magnetic field induced inside V_c by surface currents. The 2nd equation states that the total magnetic field should have no tangential component as we approach S from the conductor side. But this does not remain true across S!

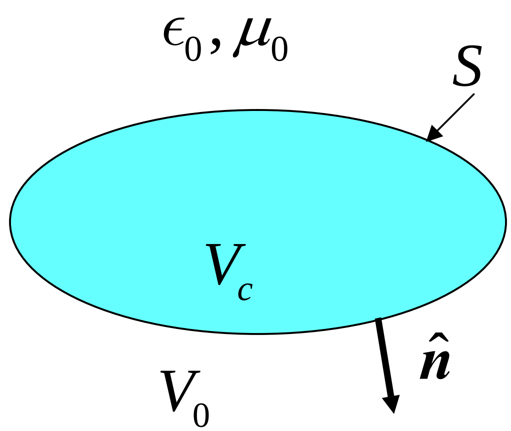
Surface integral equations for conductors

- Once solved, the scattered (far) field is found using the calculated surface magnetic field (3D case only)

$$\boldsymbol{E}_s(\boldsymbol{r}) = jk_0\eta_0 \frac{\exp(-jk_0 r)}{4\pi r} \hat{\boldsymbol{k}}_s \times \left[\hat{\boldsymbol{k}}_s \times \boldsymbol{F}\left(\hat{\boldsymbol{k}}_s\right) \right]$$

$$\boldsymbol{H}_s(\boldsymbol{r}) = -\frac{jk_0 \exp(-jk_0 r)}{4\pi r} \hat{\boldsymbol{k}}_s \times \boldsymbol{F}\left(\hat{\boldsymbol{k}}_s\right)$$

$$\boldsymbol{F}\left(\hat{\boldsymbol{k}}_s\right) = \oint_S \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}' \right) \boldsymbol{J}_s(\boldsymbol{r}') dS'$$



$$\boldsymbol{J}_s(\boldsymbol{r}) = \hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r})$$

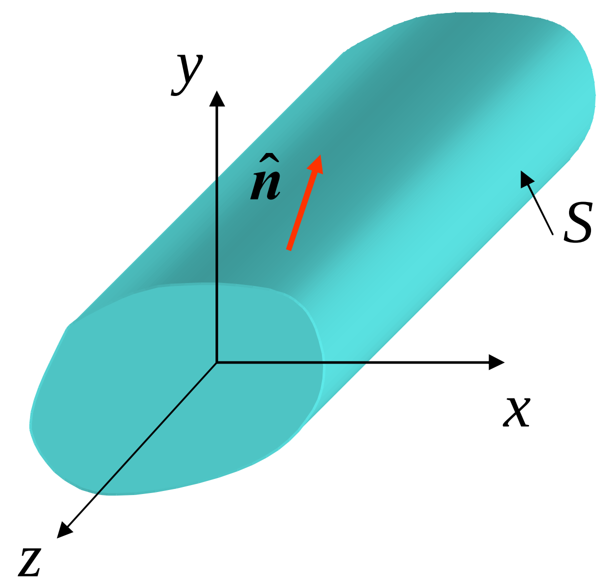
2D Surface integral equations for conductors

- To keep the analysis simple, we again consider the 2D case where the object is cylindrical and infinitely long, while the incident wave vector has no z-component

- TE: H has only a z-component and E lies on the x-y plane

- TM: E has only a z-component and H lies on the x-y plane

$$\boldsymbol{\rho} = (x, y)$$



2D Surface integral equations for conductors

- We can use the result obtained for dielectrics by letting the tangential electric field to be zero. For the TM case we use the 2D EFIE (MFIE is also applicable, but this is easier)

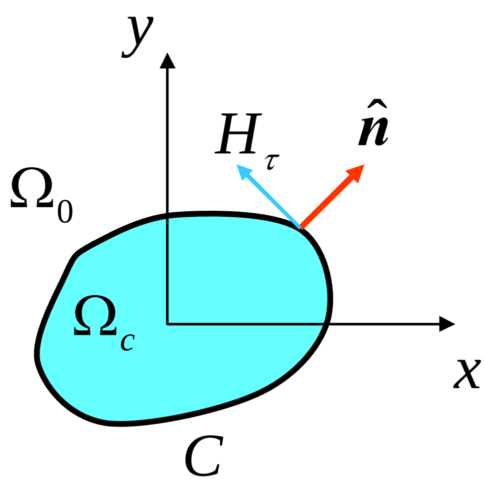
$$E_{i,z}(\boldsymbol{\rho}') = j\omega\mu_0 \oint_C G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) H_\tau(\boldsymbol{\rho}) dl$$

$\boldsymbol{\rho}' \to C$ from Ω_c

- Note that

$$\boldsymbol{J}_s(\boldsymbol{\rho}) = J_{s,z}(\boldsymbol{\rho})\hat{\boldsymbol{z}}$$

$$J_{s,z}(\boldsymbol{\rho}) = H_\tau(\boldsymbol{\rho})$$

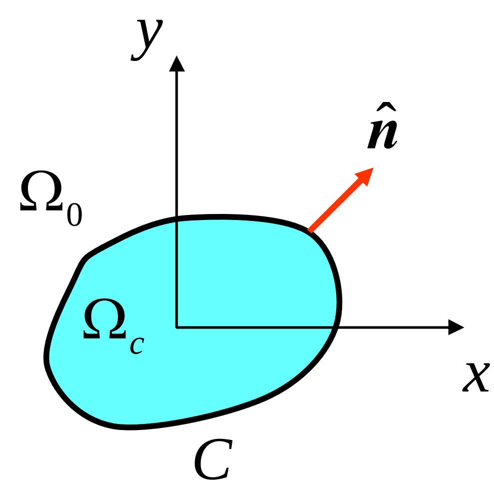


2D Surface integral equations for conductors

- In more detail:

$$E_{i,z}(\boldsymbol{\rho}') = \frac{\omega\mu_0}{4} \oint_C H_0^{(2)}\left(k_0 \left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right) J_{s,z}(\boldsymbol{\rho}) dl$$

- This is a Fredholm integral equation of the 1st kind which has to be solved over the boundary curve C

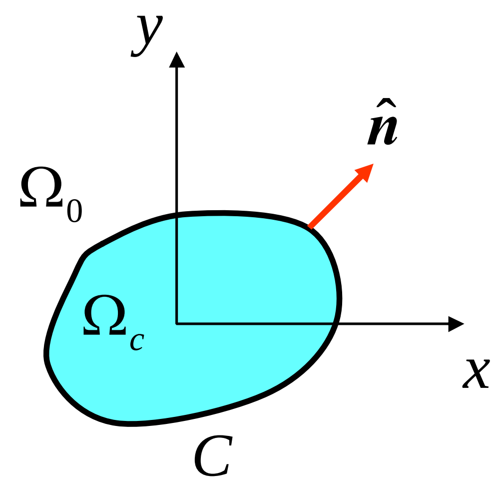


2D Surface integral equations for conductors

- For the TE case we use 2D MFIE (EFIE also possible)

$$H_{i,z}(\boldsymbol{\rho}') = \oint_C \partial_n^- G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) H_z(\boldsymbol{\rho}) dl$$

$\boldsymbol{\rho}' \rightarrow C \text{ from } \Omega_c$



- Note that

$$\boldsymbol{J}_s(\boldsymbol{\rho}) = H_z(\boldsymbol{\rho})\left(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{z}}\right)$$

- In more detail:

$$H_{i,z}(\boldsymbol{\rho}') = \frac{1}{4j} \oint_C \partial_n^- H_0^{(2)}\left(k_0 \left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right) H_z(\boldsymbol{\rho}) dl$$

2D Surface integral equations for conductors

$$\partial_n^- H_0^{(2)}\left(k_0\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|\right) = -\lim_{\zeta\to 0} \nabla H_0^{(2)}\left(k_0\left|\boldsymbol{\rho}'-\zeta\hat{\boldsymbol{n}}'-\boldsymbol{\rho}\right|\right)\cdot\hat{\boldsymbol{n}}$$

- Now consider

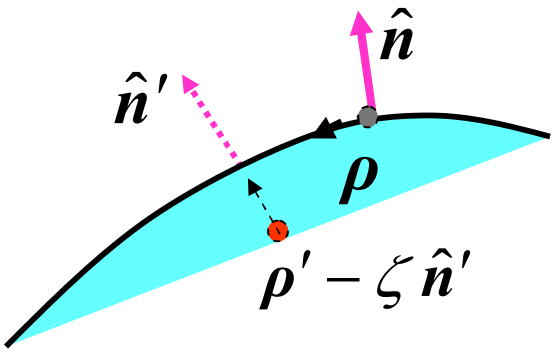
$$\nabla H_0^{(2)}\left(k_0\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|\right) = -k_0 H_0^{(2)\prime}\left(k_0\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|\right)\frac{\boldsymbol{\rho}'-\boldsymbol{\rho}}{\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|}$$

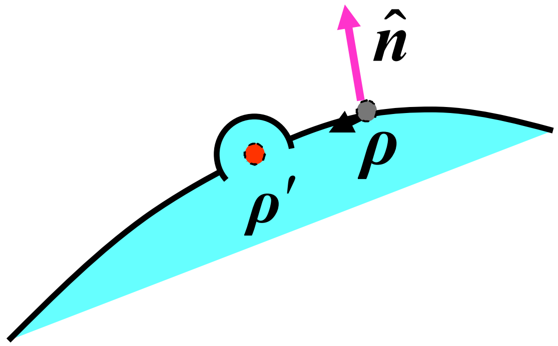
$$\partial_n^- H_0^{(2)}\left(k_0\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|\right) =$$

$$k_0 \lim_{\zeta\to 0} H_0^{(2)\prime}\left(k_0\left|\boldsymbol{\rho}'-\zeta\hat{\boldsymbol{n}}'-\boldsymbol{\rho}\right|\right)\frac{\boldsymbol{\rho}'-\zeta\hat{\boldsymbol{n}}'-\boldsymbol{\rho}}{\left|\boldsymbol{\rho}'-\zeta\hat{\boldsymbol{n}}'-\boldsymbol{\rho}\right|}\cdot\hat{\boldsymbol{n}}$$

2D Surface integral equations for conductors

- Consider a section of the boundary curve containing $\boldsymbol{\rho}'$
- Instead of taking the limit this way, we consider a different procedure
- We consider a slightly modified curve containing a small semi-circle surrounding $\boldsymbol{\rho}'$ and let the circle radius to zero





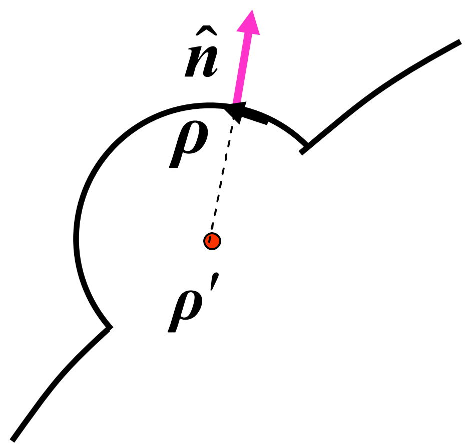
2D Surface integral equations for conductors

- The contribution of the small semi-circle (radius ε) is found by using the following approximation on this circle:

$$\partial_n^- H_0^{(2)}\left(k_0\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|\right) \sim \frac{2}{\pi j}\frac{\boldsymbol{\rho}'-\boldsymbol{\rho}}{\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|^2}\cdot\hat{\boldsymbol{n}}$$

- Note that on any point $\boldsymbol{\rho}$ on this semi-circle we have

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{\rho}-\boldsymbol{\rho}'}{\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|}$$



2D Surface integral equations for conductors

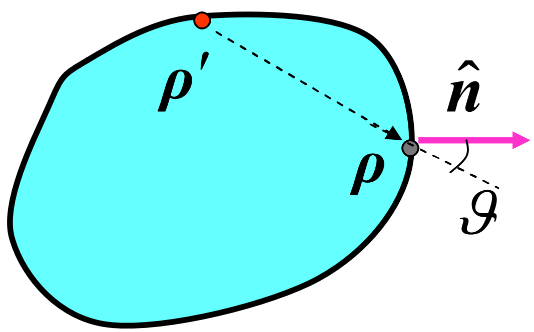
- As a result:

$$\frac{1}{4j}\oint_{semi-circle}\partial_n^- H_0^{(2)}\left(k_0\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|\right)H_z(\boldsymbol{\rho})dl=\frac{1}{2}H_z(\boldsymbol{\rho}')$$

- The contribution of the rest of the curve is written as

$$-\frac{k_0}{4j}\oint_C H_1^{(2)}\left(k_0\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|\right)\left(\frac{\boldsymbol{\rho}'-\boldsymbol{\rho}}{\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|}\cdot\hat{\boldsymbol{n}}\right)H_z(\boldsymbol{\rho})dl$$

$$=\frac{k_0}{4j}\oint_C H_1^{(2)}\left(k_0\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|\right)\cos\left[\vartheta\left(\boldsymbol{\rho},\boldsymbol{\rho}'\right)\right]H_z(\boldsymbol{\rho})dl$$



2D Surface integral equations for conductors

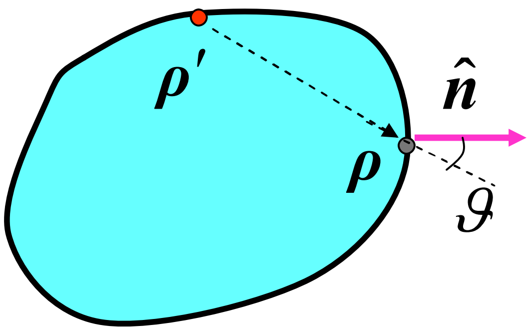
- We thus obtain the integral equation

$$H_{i,z}(\boldsymbol{\rho}') = \frac{1}{2} H_z(\boldsymbol{\rho}') + \frac{jk_0}{4} \oint_{\bar{C}} H_1^{(2)}\left(k_0 \left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right) \left(\frac{\boldsymbol{\rho}' - \boldsymbol{\rho}}{\left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|} \cdot \hat{\boldsymbol{n}} \right) H_z(\boldsymbol{\rho}) dl$$

- Or:

$$H_{i,z}(\boldsymbol{\rho}') = \frac{1}{2} H_z(\boldsymbol{\rho}') - \frac{jk_0}{4} \oint_{C} H_1^{(2)}\left(k_0 \left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right) \cos\left[\vartheta\left(\boldsymbol{\rho}, \boldsymbol{\rho}'\right)\right] H_z(\boldsymbol{\rho}) dl$$

- The integral is a principal value integral which excludes the interval 2ε around the point $\boldsymbol{\rho}'$



2D Surface integral equations for conductors

- In both TE and TM cases, after solving the surface magnetic field, the scattered field is found from the original equations
- The TE case we have for any $\boldsymbol{\rho}'$ outside the conductor

$$
\begin{aligned}
E_{s,z}(\boldsymbol{\rho}') &= -j\omega\mu_0 \oint_C G_0(\boldsymbol{\rho}', \boldsymbol{\rho}) J_{s,z}(\boldsymbol{\rho}) dl \\
&= -\frac{\omega\mu_0}{4} \oint_C H_0^{(2)}\left(k_0 \left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right) J_{s,z}(\boldsymbol{\rho}) dl \\
&= -\frac{\omega\mu_0}{4} \oint_C H_0^{(2)}\left(k_0 \left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right) H_{\tau}(\boldsymbol{\rho}) dl
\end{aligned}
$$

2D Surface integral equations for conductors

- But, we are interested in the far field:

$$H_0^{(2)}\left(k_0\left|\boldsymbol{\rho}'-\boldsymbol{\rho}\right|\right)\sim\sqrt{\frac{2}{\pi k_0\rho'}}\exp\left(-jk_0\rho'+j\pi/4\right)\exp\left(j\boldsymbol{k}_s\cdot\boldsymbol{\rho}\right)$$

$$E_{s,z}(\boldsymbol{\rho}'\to\infty)\to-\frac{\omega\mu_0}{4}\sqrt{\frac{2}{\pi k_0\rho'}}\exp\left(-jk_0\rho'+j\pi/4\right)\oint_C\exp\left(j\boldsymbol{k}_s\cdot\boldsymbol{\rho}\right)J_{s,z}(\boldsymbol{\rho})dl$$

2D Surface integral equations for conductors

- For the TE scattering we have for any $\boldsymbol{\rho}'$ outside the conductor

$$H_{s,z}(\boldsymbol{\rho}') = \nabla' \times \oint_C G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) \boldsymbol{J}_s(\boldsymbol{\rho}) dl$$

$$= -\oint_C \partial_n^- G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) H_z(\boldsymbol{\rho}) dl$$

$$= \frac{1}{4j} \oint_C \left[\hat{\boldsymbol{n}} \cdot \nabla H_0^{(2)} \left(k_0 \left| \boldsymbol{\rho}' - \boldsymbol{\rho} \right| \right) \right] H_z(\boldsymbol{\rho}) dl$$

2D Surface integral equations for conductors

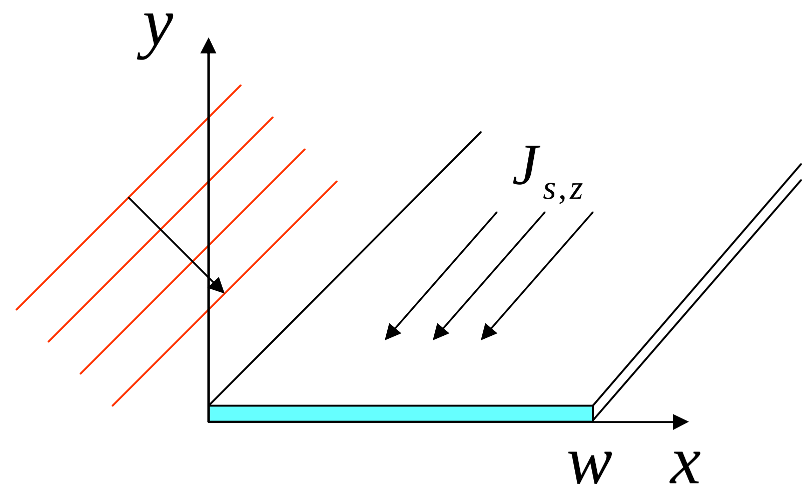
- Far field:

$$H_{s,z}(\boldsymbol{\rho}' \to \infty) \to \frac{1}{4}\sqrt{\frac{2}{\pi k_0 \rho'}} \exp\left(-jk_0\rho' + j\pi/4\right)$$

$$\oint_C \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{\rho}\right) H_z(\boldsymbol{\rho}) \left(\boldsymbol{k}_s \cdot \hat{\boldsymbol{n}}\right) dl$$

Infinitely long, finite width strip

- Consider the 2D scattering problem of a conducting strip of width w
- Consider the TM case first. The incident electric field is then

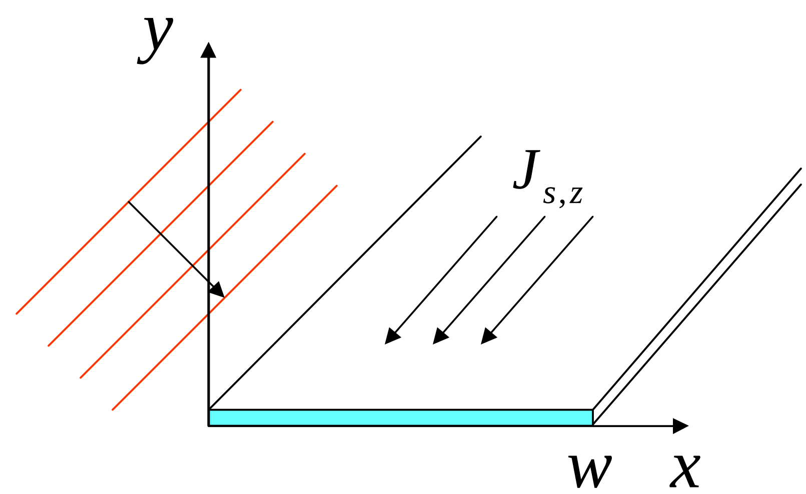


$$E_{i,z}(\boldsymbol{\rho}') = E_0 \exp\left(-jk_{i,x}x - jk_{i,y}y\right)$$

- We then have to solve

$$E_{i,z}(\boldsymbol{\rho}') = \frac{\omega\mu_0}{4} \oint_C H_0^{(2)}\left(k_0 \left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right) J_{s,z}(\boldsymbol{\rho}) dl$$

Infinitely long, finite width strip



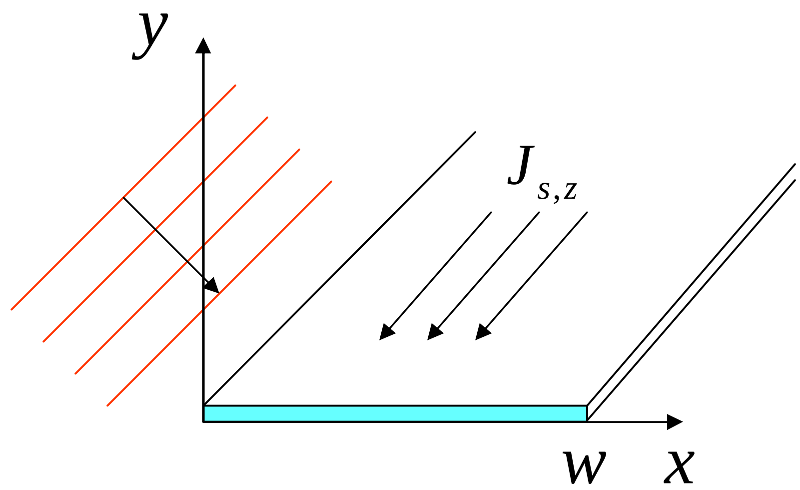
- We take the strip to be very thin
- Then we just have to consider the total surface current (top plus bottom surfaces). As a result we obtain the equation

$$E_0 \exp\left(-jk_{i,x}x'\right) = \frac{\omega\mu_0}{4}\int_0^w H_0^{(2)}\left(k_0\left|x'-x\right|\right) J_{s,z}(x)$$

- Can be solved by numerical methods

Infinitely long, finite width strip

- If the wire width is much smaller than the wavelength



$$k_0 w \ll 1 \rightarrow k_0 |x' - x| \ll 1$$

$$H_0^{(2)}\left(k_0 |x' - x|\right) \approx \frac{2}{\pi j} \ln\left(k_0 |x' - x|\right)$$

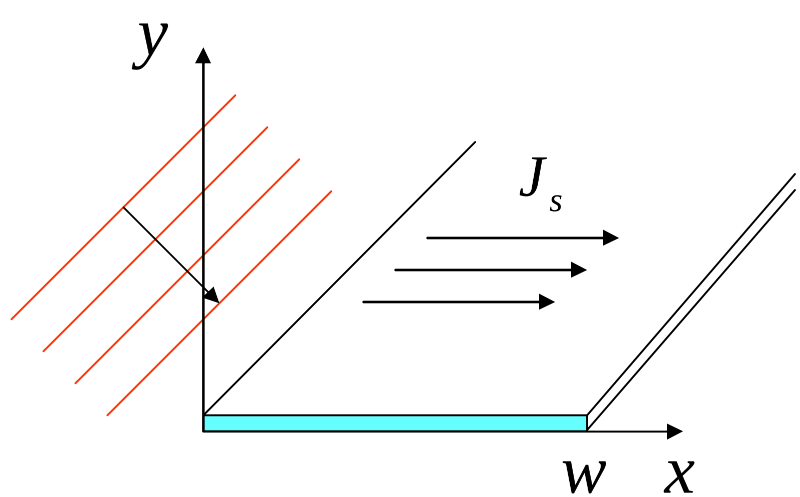
$$E_0 \exp\left(-jk_{i,x} x'\right) = \frac{\omega \mu_0}{2\pi j} \int_0^w \ln\left(k_0 |x' - x|\right) J_{s,z}(x)$$

Infinitely long, finite width strip

- Now consider the TE case where we use the MFIE formulation:

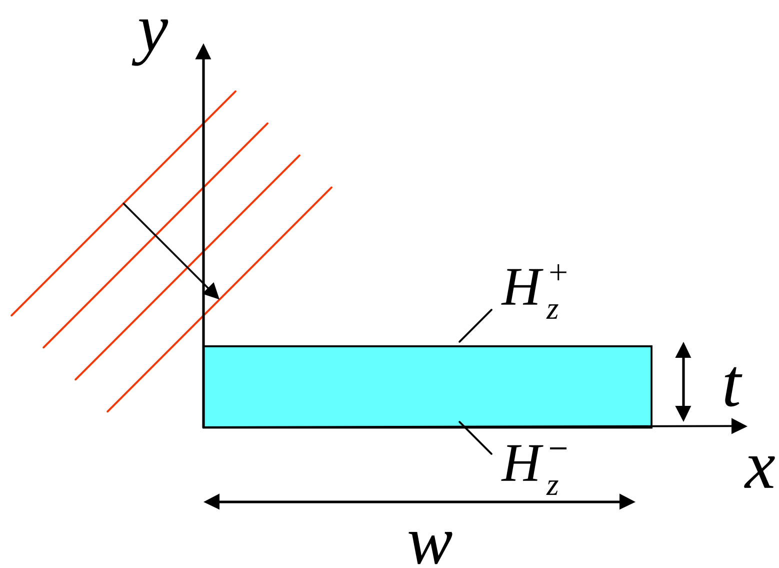
$$H_{i,z}(\boldsymbol{\rho}') = \frac{1}{2} H_z(\boldsymbol{\rho}') + \frac{jk_0}{4} \oint_{\bar{C}} H_1^{(2)}\left(k_0 \left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right) \left(\frac{\boldsymbol{\rho}' - \boldsymbol{\rho}}{\left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|} \cdot \hat{\boldsymbol{n}} \right) H_z(\boldsymbol{\rho}) dl$$

- But we should be careful with letting the thickness of the conductor approach zero
- The correct limiting process should be considered



Infinitely long, finite width strip

- Denote magnetic fields on the top and bottom surface by $H_z^{\pm}(x)$ and the incident fields on the top and bottom by $H_{i,z}^{\pm}(x)$



- Neglect the sidewall contributions. For the field on the top surface we have

$$H_{i,z}^{+}(x') = \frac{1}{2} H_z^{+}(x') - \frac{jk_0 t}{4} \int_0^w \frac{H_1^{(2)}\left[k_0\sqrt{(x-x')^2+t^2}\right]}{\sqrt{(x-x')^2+t^2}} H_z^{-}(x)\,dx$$

Infinitely long, finite width strip

- On the bottom surface

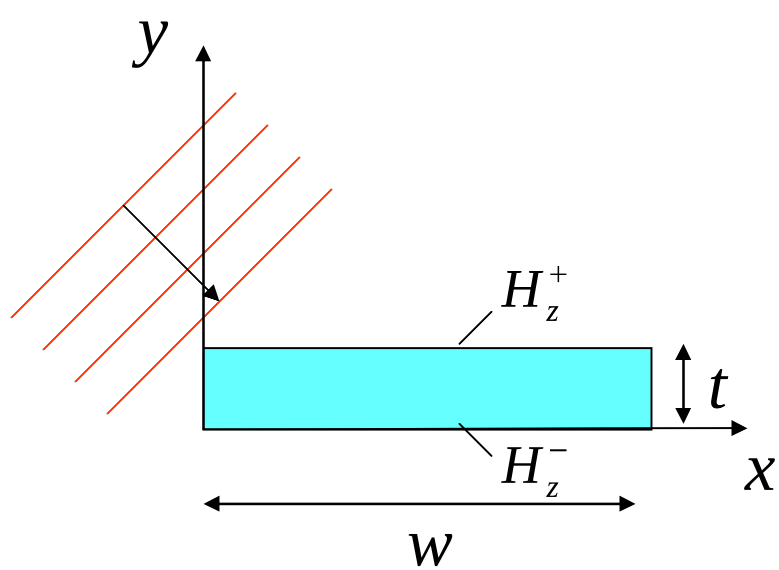
$$H_{i,z}^{-}(x') = \frac{1}{2}H_z^{-}(x') - \frac{jk_0 t}{4}\int_0^w \frac{H_1^{(2)}\left[k_0\sqrt{(x-x')^2+t^2}\right]}{\sqrt{(x-x')^2+t^2}} H_z^{+}(x)dx$$

- Note that on the top surface

$$\boldsymbol{J}_s = J_x^{+}\hat{\boldsymbol{x}}, \quad J_x^{+}(x') = H_z^{+}(x')$$

- Whereas on the bottom surface

$$\boldsymbol{J}_s = J_x^{-}\hat{\boldsymbol{x}}, \quad J_x^{-}(x') = -H_z^{-}(x')$$



Infinitely long, finite width strip

- Subtracting equations leads to

$$H_{i,z}^{+}(x') - H_{i,z}^{+}(x') = \frac{1}{2} J_x(x') + \frac{jk_0 t}{4} \int_0^w \frac{H_1^{(2)}\left[k_0\sqrt{(x-x')^2 + t^2}\right]}{\sqrt{(x-x')^2 + t^2}} J_x(x)dx$$

$$J_x(x) = J_x^{+}(x) + J_x^{-}(x)$$ ← Total current

- Bear in mind that

$$\frac{H_1^{(2)}(z)}{z} = \frac{1}{2}\left[H_0^{(2)}(z) + H_2^{(2)}(z)\right]$$

Infinitely long, finite width strip

- Keep in mind that if the thickness is much smaller than the wavelength then the left hand side is

$$H_{i,z}^{+}(x') - H_{i,z}^{+}(x') \approx t\frac{\partial H_{i,z}}{\partial y} = \left(j\omega\epsilon_0 t\right)E_{i,z}$$

- Also note that if the width of the strip is much smaller than the wavelength, then the right hand side can be approximated by

$$\frac{1}{2}J_x(x') - \frac{t}{2\pi}\int_0^w \frac{J_x(x)}{\left(x - x'\right)^2 + t^2}dx$$