Electromagnetic scattering

Graduate Course

Electrical Engineering (Communications)

1st Semester, 1388-1389

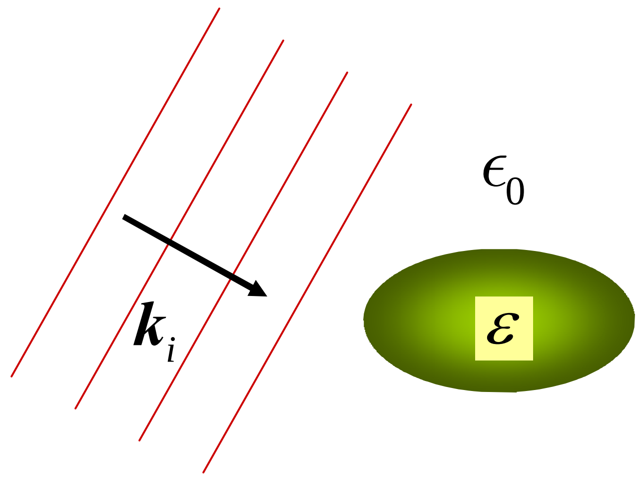
Sharif University of Technology

Contents of lecture 8

- **Contents of lecture 8:**
  - Scattering from small dielectric objects (Rayleigh scattering)
  - Volume integral equation for a dielectric particle
  - Small particle (uniform field) approximation
  - The depolarization field and factors for ellipsoids
  - Solution of the problem
  - The scattered field
  - Examples

Introduction

- We consider the problem of scattering by a dielectric object with a constant permittivity, but assume the object to have dimensions much smaller than the wavelength (inside the object)



- To express the actual field distribution inside the particle, let us return to the volume integral equation

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_i(\boldsymbol{r}) + k_0^2 \int_V \overline{\overline{\boldsymbol{G}}}_0(\boldsymbol{r}, \boldsymbol{r}') \cdot \delta\varepsilon \boldsymbol{E}(\boldsymbol{r}') dV'$$

Total field

Incident field

$$\delta\varepsilon = \varepsilon - \epsilon_0$$

Introduction

- Remember that

$$\delta\varepsilon \boldsymbol{E}(\boldsymbol{r}') = \frac{1}{\epsilon_0}(\varepsilon - \epsilon_0)\boldsymbol{E}(\boldsymbol{r}') = \frac{1}{\epsilon_0}\boldsymbol{P}(\boldsymbol{r}')$$

← Electric polarization

- Consider now the field generated by this polarization inside the dielectric particle

$$\frac{k_0^2}{\epsilon_0}\int_V \overline{\overline{\boldsymbol{G}}}_0(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{P}(\boldsymbol{r}')dV' =$$

$$\frac{1}{\epsilon_0}\int_V \left[k_0^2 G_0(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{P}(\boldsymbol{r}') - \nabla\nabla' G_0(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{P}(\boldsymbol{r}')\right]dV'$$

Introduction

- To gain more insight, we compare the ratio of the two terms inside the integral. We have

$$\frac{\nabla\nabla' G_0(\boldsymbol{r},\boldsymbol{r}')}{k_0^2 G_0(\boldsymbol{r},\boldsymbol{r}')} = \left\{ \bar{\bar{\boldsymbol{I}}}\left[1 + jk_0R\right] + \frac{\boldsymbol{RR}}{R^2}\left[\left(k_0R\right)^2 - 3j\left(k_0R\right) - 3\right] \right\} \frac{1}{\left(k_0R\right)^2}$$

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}',\ R = \left|\boldsymbol{R}\right|$$

- If the particle is small compared to wavelength, then $k_0R \ll 1$ and this ratio can be huge. Therefore, the first term in the integral is negligible.

The electrostatic approximation

- Now, consider the 2nd term

$$\nabla\nabla' G_0(\boldsymbol{r},\boldsymbol{r}') =$$

$$= \left\{ \bar{\bar{\boldsymbol{I}}} \left[1 + jk_0 R\right] + \frac{\boldsymbol{RR}}{R^2}\left[\left(k_0 R\right)^2 - 3j\left(k_0 R\right) - 3\right]\right\} \frac{\exp\left(-jk_0 R\right)}{4\pi R^3}$$

$$\approx \left\{ \bar{\bar{\boldsymbol{I}}} - 3\frac{\boldsymbol{RR}}{R^2}\right\} \frac{1}{4\pi R^3} = \nabla\nabla'\left(\frac{1}{4\pi R}\right)$$

This is the electrostatic approximation

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_i(\boldsymbol{r}) - \int_V \nabla\nabla'\left(\frac{1}{4\pi\epsilon_0 R}\right)\cdot \boldsymbol{P}(\boldsymbol{r}') dV'$$

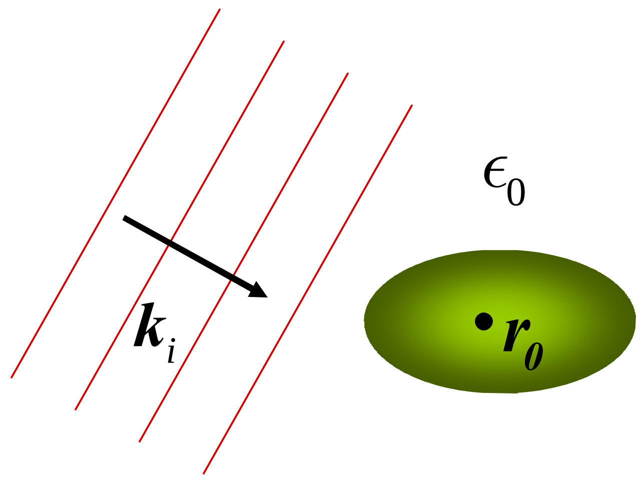
The electrostatic approximation

- Let us rewrite this field as

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_i(\boldsymbol{r}) - \nabla \oint_S \frac{\boldsymbol{P}(\boldsymbol{r}')\cdot\hat{\boldsymbol{n}}}{4\pi\epsilon_0 R} dS' - \nabla \int_V \frac{\left[-\nabla'\cdot\boldsymbol{P}(\boldsymbol{r}')\right]}{4\pi\epsilon_0 R} dV'$$

- Now, inside a small particle the incident field is almost a constant:

$$\boldsymbol{E}_i(\boldsymbol{r}) = \boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i\cdot\boldsymbol{r}\right)$$
$$\approx \boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i\cdot\boldsymbol{r}_0\right)$$



The electrostatic approximation

- We have to solve the electrostatic problem of a dielectric in a uniform, externally applied electric field
- For a dielectric particle of ellipsoid shape, it is known that the solution is a uniform (constant) polarization vector
- It generates a uniform depolarization field

$$\boldsymbol{E}_{dep} = -\nabla \oint_S \frac{\boldsymbol{P}\cdot\hat{\boldsymbol{n}}}{4\pi\epsilon_0 R}\,dS' = -\frac{1}{\epsilon_0}\,\overline{\overline{\boldsymbol{N}}}\cdot\boldsymbol{P}$$

Depolarization tensor

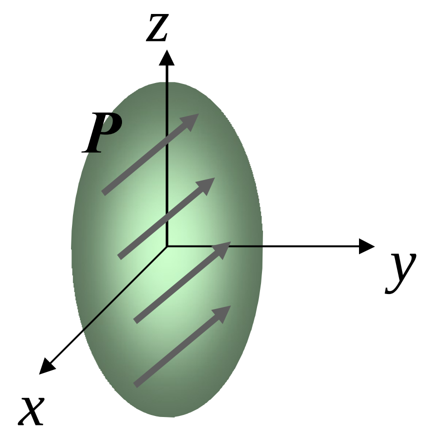
The field inside an ellipsoid

- To show this, consider a 3D ellipsoid with its principal axes chosen along the x,y, and z axes of the coordinate system
- Let us decompose the constant polarization vector also into its x,y, and z components. The total depolarization field with be the superposition of the fields generated by each component

$$\boldsymbol{E}_{dep} = \boldsymbol{E}_{dep}^{x} + \boldsymbol{E}_{dep}^{y} + \boldsymbol{E}_{dep}^{z}$$

$$\boldsymbol{E}_{dep}^{\alpha} = -P_{\alpha} \nabla \oint_{S} \frac{\hat{\boldsymbol{v}}_{\alpha} \cdot \hat{\boldsymbol{n}}}{4\pi\epsilon_0 R} dS', \quad \alpha = x, y, z$$

$$\hat{\boldsymbol{v}}_x \equiv \hat{\boldsymbol{x}}, \hat{\boldsymbol{v}}_y \equiv \hat{\boldsymbol{y}}, \hat{\boldsymbol{v}}_z \equiv \hat{\boldsymbol{z}}$$

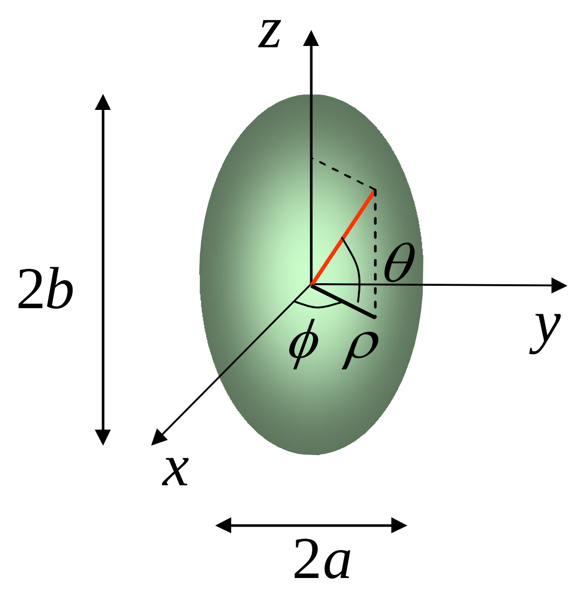


The field inside an ellipsoid

- The case of a general ellipsoid is too complicated. Thus, we analyze the problem for the particular case of an *ellipsoid of revolution*. To obtain such an object, one can draw an ellipse on the x-z plane and rotate it around the z-axis.

- We 1st use cylindrical coordinates in which the surface of the ellipsoid is given by the equation

$$\frac{\rho^2}{a^2}+\frac{z^2}{b^2}=1$$



The field inside an ellipsoid

- The problem we are trying to solve is the electrostatic problem of the electric field induced by the surface charge

$$\rho_s = P_\alpha \hat{\boldsymbol{v}}_\alpha \cdot \hat{\boldsymbol{n}}$$

- So we may solve the Laplace equation $\nabla^2 \Phi = 0$ inside and outside the ellipsoid, and match the solutions on the surface
- In cylindrical coordinates the Laplace equation reads

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

The field inside an ellipsoid

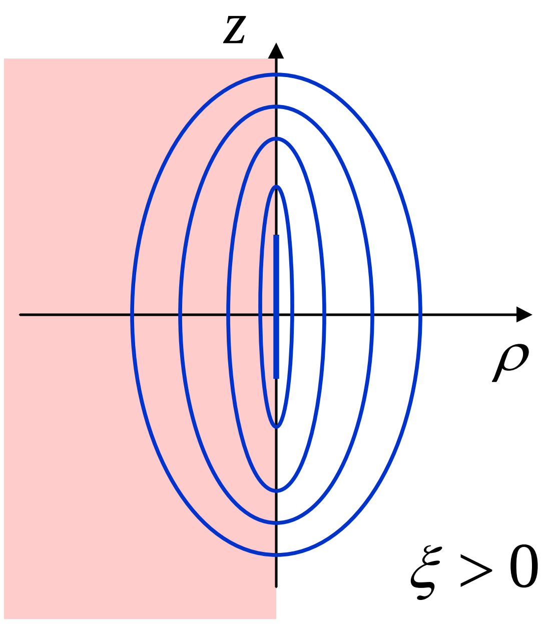
- But this coordinate system is inappropriate to solve the problem. Instead consider the parametrization (γ>0)

$$\rho = \gamma \sinh \xi \;\; \sin \theta, \;\; z = \gamma \cosh \xi \;\; \cos \theta \qquad 0 < \xi < \infty, 0 < \theta < \pi$$

- Obviously, coordinate surfaces corresponding to $\xi = \xi_0$=constant are half ellipsoids with axes length's

$\gamma \cosh \xi_0 \qquad \gamma \sinh \xi_0$

$$\left(\frac{\rho}{\gamma \sinh \xi_0} \right)^2 + \left(\frac{z}{\gamma \cosh \xi_0} \right)^2 = 1$$



The field inside an ellipsoid

- To be able to simplify the problem, the surface of our ellipsoid should coincide with one of these surfaces:

$$a = \gamma \sinh \xi_0 \qquad b = \gamma \cosh \xi_0$$

- This is possible if $\gamma^2 = b^2 - a^2$. Then the corresponding ξ is

$$\xi_e = \operatorname{arctanh}\left(\frac{a}{b}\right)$$

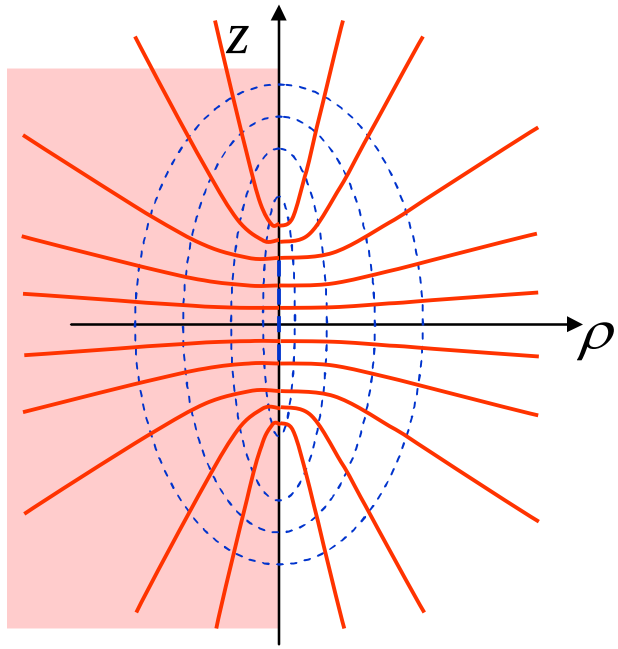
- Note that we have assumed $b > a$. If that is not the case we interchange sinh and cosh in the formulation.

The field inside an ellipsoid

- The surfaces corresponding to θ=constant are hyperboloids

$$\theta = \theta_0 \rightarrow \left(\frac{z}{\gamma \cos\theta_0}\right)^2 - \left(\frac{\rho}{\gamma \sin\theta_0}\right)^2 = 1$$

- For correct parametrization we demanded that $\xi > 0$ and $0 < \theta < \pi$
- In this way all the space with ρ > 0 will be covered



The field inside an ellipsoid

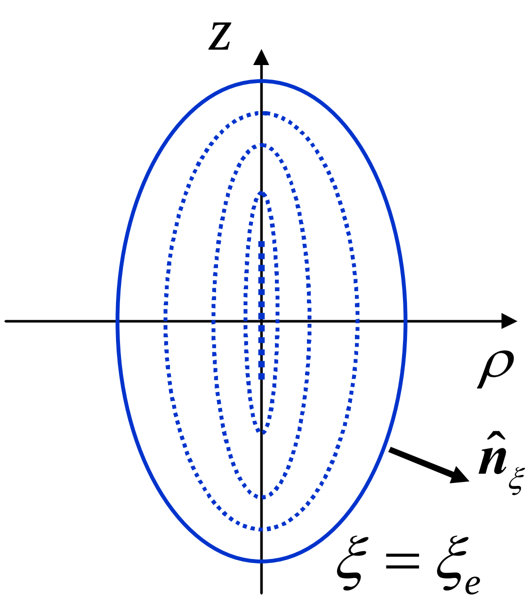
- The Laplace equation in this new coordinate system is

$$\frac{1}{\sinh\xi}\frac{\partial}{\partial\xi}\left(\sinh\xi\frac{\partial\Phi}{\partial\xi}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{\sinh^2\xi+\sin^2\theta}{\sinh^2\xi\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

- The outward normal derivative on the surface of the ellipsoid is given by

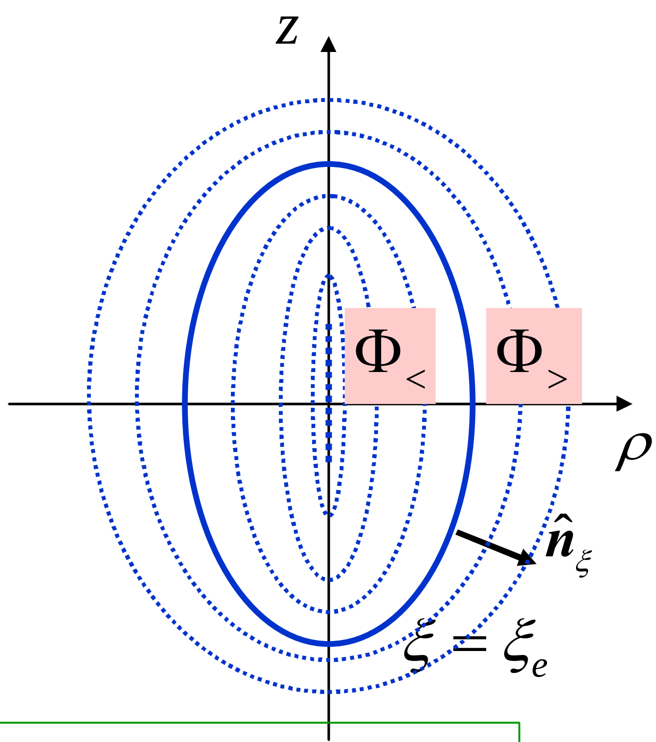
$$\hat{\boldsymbol{n}}_\xi\cdot\nabla\Phi=\frac{1}{\gamma\sqrt{\sinh^2\xi_e+\sin^2\theta}}\left.\frac{\partial\Phi}{\partial\xi}\right|_{\xi=\xi_e}$$

$$\hat{\boldsymbol{n}}_\xi=\frac{\hat{\boldsymbol{\rho}}\cosh\xi_e\sin\theta+\hat{\boldsymbol{z}}\sinh\xi_e\cos\theta}{\sqrt{\sinh^2\xi_e+\sin^2\theta}}$$



The field inside an ellipsoid

- Now we solve the Laplace equation both inside and outside the ellipsoid and match the solutions using



$$\Phi_>(\xi_e,\theta,\phi)=\Phi_<(\xi_e,\theta,\phi)$$

$$\hat{\boldsymbol{n}}_\xi\cdot\nabla\Phi_> - \hat{\boldsymbol{n}}_\xi\cdot\nabla\Phi_< = -\frac{\rho_s}{\epsilon_0}$$

$$\rho_s = P_\alpha \hat{\boldsymbol{v}}_\alpha\cdot\hat{\boldsymbol{n}}_\xi$$

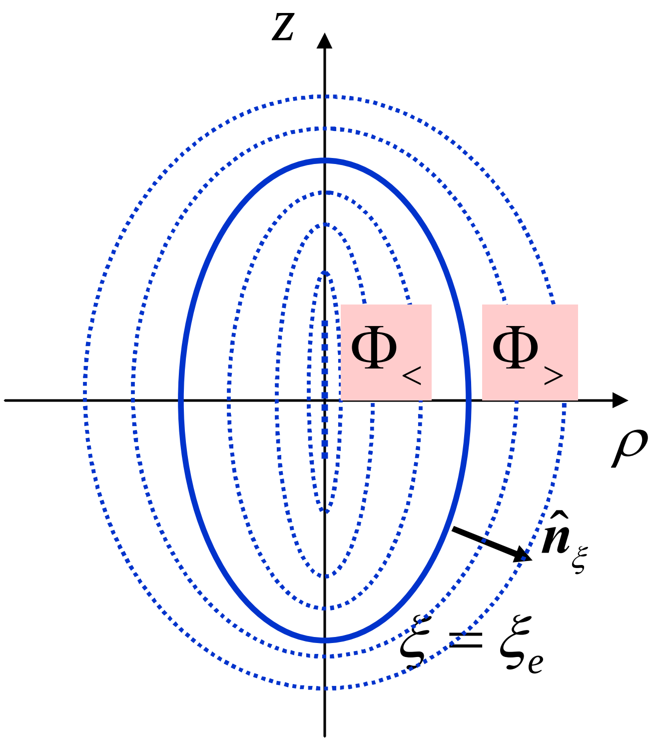
$$\left.\frac{\partial\Phi_>}{\partial\xi}\right|_{\xi=\xi_e} - \left.\frac{\partial\Phi_<}{\partial\xi}\right|_{\xi=\xi_e} = -\gamma P_\alpha \hat{\boldsymbol{v}}_\alpha\cdot\left(\hat{\boldsymbol{\rho}}\cosh\xi_e\sin\theta + \hat{\boldsymbol{z}}\sinh\xi_e\cos\theta\right)$$

The field inside an ellipsoid

- 1^{st} case: polarization along z-axis

$$\Phi_>\left(\xi_e,\theta,\phi\right)=\Phi_<\left(\xi_e,\theta,\phi\right)$$

$$\left.\frac{\partial\Phi_>}{\partial\xi}\right|_{\xi=\xi_e}-\left.\frac{\partial\Phi_<}{\partial\xi}\right|_{\xi=\xi_e}=-\gamma\frac{P_z}{\epsilon_0}\sinh\xi_e\cos\theta$$



- In both regions try solution of the type

$$\Phi=\cos\theta\, f\left(\xi\right)\rightarrow\frac{1}{\sinh\xi}\frac{d}{d\xi}\left(\sinh\xi\frac{df}{d\xi}\right)-2f=0$$

The field inside an ellipsoid

- Solving the equation:

$$s = \cosh\xi \rightarrow \left(s^2 - 1\right)\frac{d^2 f}{ds^2} + 2s\frac{df}{ds} - 2f = 0$$

- General solution:

$$f(s) = As + B\left[1 + \frac{s}{2}\ln\left(\frac{s-1}{s+1}\right)\right]$$

$$f(\xi) = A\cosh\xi + B\left[1 + \frac{\cosh\xi}{2}\ln\left(\frac{\cosh\xi - 1}{\cosh\xi + 1}\right)\right]$$

The field inside an ellipsoid

- Inside the ellipsoid (to ensure finiteness at ξ=0)

$$f_<(\xi) = A\cosh\xi$$

- Outside the ellipsoid (to ensure finiteness at ξ=∞)

$$f_>(\xi) = B\left[1+\frac{\cosh\xi}{2}\ln\left(\frac{\cosh\xi-1}{\cosh\xi+1}\right)\right]$$

- Matching:

$$A\cosh\xi_e = B\left[1+\frac{\cosh\xi_e}{2}\ln\left(\frac{\cosh\xi_e-1}{\cosh\xi_e+1}\right)\right]$$

$$B\left[\frac{1}{2}\ln\left(\frac{\cosh\xi_e-1}{\cosh\xi_e+1}\right)+\frac{\cosh\xi_e}{\sinh^2\xi_e}\right]-A=-\gamma\frac{P_z}{\epsilon_0}$$

The field inside an ellipsoid

$$A = -\gamma \frac{P_z}{\epsilon_0} \sinh^2 \xi_e \left[1 + \frac{\cosh \xi_e}{2} \ln \left(\frac{\cosh \xi_e - 1}{\cosh \xi_e + 1} \right) \right]$$

$$B = -\gamma \frac{P_z}{\epsilon_0} \sinh^2 \xi_e \cosh \xi_e$$

- Inside the ellipsoid

$$\Phi_< = \cos\theta \, f_<(\xi) = N_z \frac{P_z}{\epsilon_0} \gamma \cos\theta \cosh \xi = N_z \frac{P_z}{\epsilon_0} z$$

$$N_z = -\sinh^2 \xi_e \left[1 + \frac{\cosh \xi_e}{2} \ln \left(\frac{\cosh \xi_e - 1}{\cosh \xi_e + 1} \right) \right]$$

$$\xi_e = \operatorname{arctanh}\left(\frac{a}{b} \right)$$

$$\boldsymbol{E}^z_{dep} = -\nabla \Phi_< = -N_z P_z \hat{\boldsymbol{z}}$$

The field inside an ellipsoid

- 2^{nd} case: polarization along x-axis, the boundary conditions are

$$\Phi_{>}\left(\xi_e,\theta,\phi\right)=\Phi_{<}\left(\xi_e,\theta,\phi\right)$$

$$\left.\frac{\partial\Phi_{>}}{\partial\xi}\right|_{\xi=\xi_e}-\left.\frac{\partial\Phi_{<}}{\partial\xi}\right|_{\xi=\xi_e}=-\gamma\frac{P_x}{\epsilon_0}\cosh\xi_e\sin\theta\cos\phi$$

- In both regions try solution of the type

$$\Phi=\sin\theta\,\cos\phi\, f\left(\xi\right)$$

The field inside an ellipsoid

- From the Laplace equation we get

$$\frac{1}{\sinh\xi}\frac{\partial}{\partial\xi}\left(\sinh\xi\frac{\partial\Phi}{\partial\xi}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{\sinh^2\xi+\sin^2\theta}{\sinh^2\xi\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

$$\Rightarrow \quad \frac{1}{\sinh\xi}\frac{d}{d\xi}\left(\sinh\xi\frac{df}{d\xi}\right)-\left(2+\frac{1}{\sinh^2\xi}\right)f=0$$

$$s=\cosh\xi\rightarrow\left(s^2-1\right)\frac{d^2f}{ds^2}+2s\frac{df}{ds}-\left(\frac{2s^2-1}{s^2-1}\right)f=0$$

$$\Rightarrow \quad f=A\sqrt{s^2-1}+B\left\{\frac{s}{\sqrt{s^2-1}}+\frac{1}{2}\sqrt{s^2-1}\ln\left(\frac{s-1}{s+1}\right)\right\}$$

Depolarization factors

- Inside the ellipsoid only the first term is allowed. It then follows that the electric field is constant, and is in the x-direction.
- Summarizing the results, it follows that the electric field inside a uniformly polarized ellipsoid is given by the constant field

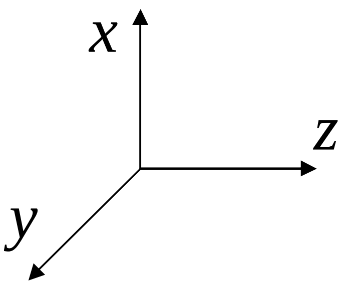
$$\boldsymbol{E}_{dep} = -\frac{1}{\epsilon_0}\bar{\bar{\boldsymbol{N}}} \cdot \boldsymbol{P} \qquad \bar{\bar{\boldsymbol{N}}} = \begin{bmatrix} N_x & 0 & 0 \\ 0 & N_y & 0 \\ 0 & 0 & N_z \end{bmatrix} \qquad N_x + N_y + N_z = 1$$

- In case of an ellipsoid of revolution we have due to symmetry:

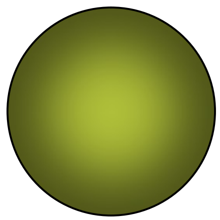
$$N_x = N_y = \frac{1}{2}(1 - N_z)$$

Depolarization factors

- Examples: some limiting cases of ellipsoids

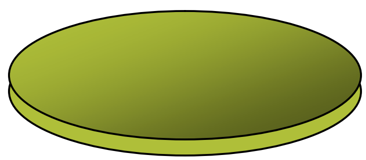


sphere



$$N_x = N_y = N_z = \frac{1}{3}$$

Infinitely thin disk



$$N_x = 1 \quad N_y = N_z = 0$$

Infinitely long rod

$$N_x = N_y = \frac{1}{2} \quad N_z = 0$$

The field inside an ellipsoid

- We can now easily find the (uniform) electric field inside the dielectric ellipsoid:

$$\boldsymbol{E} = \boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}_0\right) - \frac{1}{\epsilon_0}\overline{\overline{\boldsymbol{N}}} \cdot \boldsymbol{P} \qquad \boldsymbol{P} = \left(\varepsilon - \epsilon_0\right)\boldsymbol{E}$$

$$\boldsymbol{E} = \left[\overline{\overline{\boldsymbol{I}}} + \delta\varepsilon_r \overline{\overline{\boldsymbol{N}}}\right]^{-1} \cdot \boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}_0\right) \qquad \delta\varepsilon_r = \frac{\varepsilon}{\epsilon_0} - 1$$

The field inside an ellipsoid

- The scattered field

$$\boldsymbol{E}_s(\boldsymbol{r}) = -\frac{k^2 \exp(-jkr)}{4\pi r}\hat{\boldsymbol{k}}_s \times \left[\hat{\boldsymbol{k}}_s \times \delta\varepsilon_r \int_V \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}'\right)\boldsymbol{E}(\boldsymbol{r}')dV'\right]$$

- Using previous results, & assuming the particle to be small:

$$\boldsymbol{E}_s(\boldsymbol{r}) = -Vk^2\delta\varepsilon_r \frac{\exp\left[j\left(\boldsymbol{k}_s - \boldsymbol{k}_i\right)\cdot \boldsymbol{r}_0 - jkr\right]}{4\pi r}$$

$$\hat{\boldsymbol{k}}_s \times \hat{\boldsymbol{k}}_s \times \left[\overline{\overline{\boldsymbol{I}}} + \delta\varepsilon_r \overline{\overline{\boldsymbol{N}}}\right]^{-1} \cdot \boldsymbol{E}_i^0$$