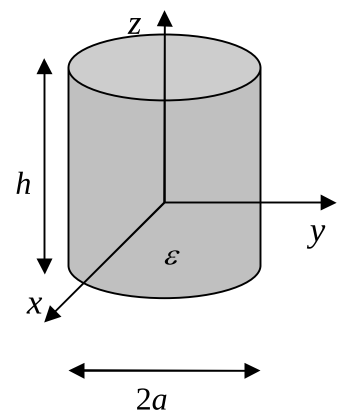
EM Scattering

Homework assignment 1

Problem 1:

An incident wave travels in free space (dielectric constant ε_0, permeability μ_0, wave number $k = \omega\sqrt{\varepsilon_0\mu_0}$) and impinges upon a dielectric cylinder with the radius a, height h, and the relative dielectric constant ε_d. The axis of the cylinder lies on the z-axis.



- Use the Born approximation to calculate the scattered electric and magnetic fields in the far-field zone in any direction $\hat{k}_s$ when the incident plane wave is moving in the +z direction ($\hat{\boldsymbol{k}}_i = \hat{\boldsymbol{z}}$) and the incident electric field is polarized along $\hat{\boldsymbol{x}}$. Take the amplitude of the incident electric field to be E_0.
- Repeat this calculation for the case where $\hat{\boldsymbol{k}}_i = \hat{\boldsymbol{x}}$ and the incident electric field is polarized along $\hat{\boldsymbol{z}}$.

Solution

In born approximation:

$$\boldsymbol{E}_s^f(\boldsymbol{r}) = \frac{\exp(-jkr)}{r}\left[\boldsymbol{E}_{i,0} - \left(\boldsymbol{E}_{i,0}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right] S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \tag{1.1}$$

$$S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) = \frac{k^2}{4\pi\epsilon_0}\int_V \exp\left[j\left(\boldsymbol{k}_s - \boldsymbol{k}_i\right)\cdot\boldsymbol{r}'\right]\delta\varepsilon_p(\boldsymbol{r}')dV' \tag{1.2}$$

In our case we use cylindrical coordinates in which

$$S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) = \frac{k^2\left(\varepsilon_d - 1\right)}{4\pi}\int_0^a\int_0^{2\pi}\int_0^h \exp\left(j\boldsymbol{k}_d\cdot\boldsymbol{r}'\right)\rho' d\rho' d\varphi' dz',\ \boldsymbol{k}_d = \boldsymbol{k}_s - \boldsymbol{k}_i \tag{1.3}$$

Here we have $\boldsymbol{r}' = \left(\rho'\cos\varphi', \rho'\sin\varphi', z'\right)$ and

$$\boldsymbol{k}_d = \left(k_{d,\rho}\cos\phi_d, k_{d,\rho}\sin\phi_d, k_{d,z}\right), k_{d,\rho} = \sqrt{k_{d,x}^2 + k_{d,y}^2} \tag{1.4}$$

so that

$$\begin{aligned}&\int_0^a\int_0^{2\pi}\int_0^h \exp\left(j\boldsymbol{k}_d\cdot\boldsymbol{r}'\right)r'dr'd\varphi'dz' = \int_0^a\int_0^{2\pi}\int_0^h \exp\left[jk_{d,\rho}\rho'\cos\left(\varphi'-\phi_d\right)+jk_{d,z}z'\right]\rho'd\rho'd\varphi'dz' \\ &= \left[\int_0^h \exp\left(jk_{d,z}z'\right)dz'\right]\int_0^a\int_0^{2\pi}\exp\left[jk_{d,\rho}\rho'\cos\left(\varphi'-\phi_d\right)\right]\rho'd\rho'd\varphi'\end{aligned} \tag{1.5}$$

Note that

$$\int_0^{2\pi}\exp\left[jk_{d,\rho}\rho'\cos\left(\varphi'-\phi_d\right)\right]d\varphi' = 2\pi J_0\left(k_{d,\rho}\rho'\right) \tag{1.6}$$

$$\begin{aligned}&\int_0^a\int_0^{2\pi}\exp\left[jk_{d,\rho}\rho'\cos\left(\varphi'-\phi_d\right)\right]\rho'd\rho'd\varphi' = 2\pi\int_0^a J_0\left(k_{d,\rho}\rho'\right)\rho'd\rho' \\ &= \frac{2\pi}{k_{d,\rho}^2}\int_0^{k_{d,\rho}a} J_0(u)u\,du = \frac{2\pi}{k_{d,\rho}^2}uJ_1(u)\Big|_0^{k_{d,\rho}a} = \frac{2\pi a}{k_{d,\rho}}J_1\left(k_{d,\rho}a\right)\end{aligned} \tag{1.7}$$

Collecting the results:

$$\begin{aligned}S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right) &= \frac{k^2\left(\varepsilon_d-1\right)}{4\pi}\int_0^a\int_0^{2\pi}\int_0^h\exp\left(j\boldsymbol{k}_d\cdot\boldsymbol{r}'\right)\rho'd\rho'd\varphi'dz' \\ &= \frac{k^2\left(\varepsilon_d-1\right)}{4\pi}\frac{\left[\exp\left(jk_{d,z}h\right)-1\right]}{jk_{d,z}}\frac{2\pi a}{k_{d,\rho}}J_1\left(k_{d,\rho}a\right) \\ &= k^2 a\left(\varepsilon_d-1\right)\frac{\sin\left(k_{d,z}h/2\right)}{k_{d,z}}\frac{J_1\left(k_{d,\rho}a\right)}{k_{d,\rho}}\exp\left(jk_{d,z}h/2\right)\end{aligned} \tag{1.8}$$

- In the first question $\boldsymbol{k}_i = k\hat{\boldsymbol{z}}$ and $\boldsymbol{k}_d = \boldsymbol{k}_s - k\hat{\boldsymbol{z}}$. Thus: $k_{d,z} = k_{s,z} - k$, and $k_{d,\rho} = k_{s,\rho} = \sqrt{k_{s,x}^2 + k_{s,y}^2}$. If one uses $k_{s,z} = k\cos\theta_s, k_{s,\rho} = k\sin\theta_s$ then

$$S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right) = a\left(\varepsilon_d-1\right)\frac{\sin\left[(kh/2)\left(\cos\theta_s-1\right)\right]}{\cos\theta_s-1}\frac{J_1\left(ka\sin\theta_s\right)}{\sin\theta_s}\exp\left[j(kh/2)\left(\cos\theta_s-1\right)\right] \tag{1.9}$$

The far field:

$$\begin{aligned}\boldsymbol{E}_s^f(\boldsymbol{r}) &= E_0 \frac{\exp(-jkr)}{r}\left[\hat{\boldsymbol{x}} - \left(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right] S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \\ &= E_0 \frac{\exp(-jkr)}{r}\left[\hat{\boldsymbol{x}} - \sin\theta_s \cos\phi_s \hat{\boldsymbol{k}}_s\right] S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)\end{aligned} \tag{1.10}$$

Here we have used $\hat{\boldsymbol{k}}_s = \left(\sin\theta_s \cos\phi_s, \sin\theta_s \sin\phi_s, \cos\theta_s\right)$.

- For the second problem $\boldsymbol{k}_d = \boldsymbol{k}_s - \boldsymbol{k}_i = \boldsymbol{k}_s - k\hat{\boldsymbol{x}}$ so that $k_{d,z} = k_{s,z} = k\cos\theta_s$ and

$$\begin{aligned}k_{d,\rho} &= \sqrt{\left(k_{s,x} - k\right)^2 + k_{s,y}^2} = k\sqrt{\left(\sin\theta_s \cos\phi_s - 1\right)^2 + \sin^2\theta_s \sin^2\phi_s} \\ &= k\sqrt{1 + \sin^2\theta_s - 2\sin\theta_s \cos\phi_s}\end{aligned}$$

As a result

$$S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) = a\left(\varepsilon_d - 1\right)\frac{\sin\left[\left(kh/2\right)\cos\theta_s\right]}{\cos\theta_s}\frac{J_1\left(ka\sqrt{1 + \sin^2\theta_s - 2\sin\theta_s\cos\phi_s}\right)}{\sqrt{1 + \sin^2\theta_s - 2\sin\theta_s\cos\phi_s}} \exp\left[j\left(kh/2\right)\cos\theta_s\right] \tag{1.11}$$

The electric field in the far zone

$$\begin{aligned}\boldsymbol{E}_s^f(\boldsymbol{r}) &= E_0 \frac{\exp(-jkr)}{r}\left[\hat{\boldsymbol{z}} - \left(\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right] S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \\ &= E_0 \frac{\exp(-jkr)}{r}\left[\hat{\boldsymbol{z}} - \cos\theta_s \hat{\boldsymbol{k}}_s\right] S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)\end{aligned} \tag{1.12}$$

Problem 2:

Using the Born approximation calculate the scattering cross section from a dielectric sphere with the dielectric constant ε_d whose radius a is much smaller than the wavelength in vacuum. Next, compute the scattering cross section by using the optical theorem and compare the results.

Solution

For a dielectric sphere we use the result derived in the power point slides:

$$S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) = \frac{k^2\left(\epsilon_r - 1\right)}{k_d^3}\left[\sin\left(k_d a\right) - k_d a\cos\left(k_d a\right)\right] \tag{1.13}$$

The far electric field in Born approximation equals:

$$\boldsymbol{E}_s^f(\boldsymbol{r}) = \frac{\exp(-jkr)}{r}\left[\boldsymbol{E}_{i,0} - \left(\boldsymbol{E}_{i,0}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right] S\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) \tag{1.14}$$

Let the incident wave travel along z and be polarized along x (this will be general enough due to the symmetry of the structure). Then $\boldsymbol{k}_d = \boldsymbol{k}_s - \boldsymbol{k}_i = \boldsymbol{k}_s - k\hat{\boldsymbol{z}}$ and

$$\boldsymbol{E}_{i,0} - \left(\boldsymbol{E}_{i,0}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s = E_0\left[\hat{\boldsymbol{x}} - \left(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right] \tag{1.15}$$

Once more, let us write $\hat{\boldsymbol{k}}_s = \left(\sin\theta_s\cos\phi_s, \sin\theta_s\sin\phi_s, \cos\theta_s\right)$. Then $k_d = 2k\sin\left(\theta_s/2\right)$ and

$$\hat{\boldsymbol{x}} - \left(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s = \hat{\boldsymbol{x}} - \sin\theta_s\cos\phi_s\hat{\boldsymbol{k}}_s \tag{1.16}$$

To calculate the scattering cross section we need to know the total power scattered. This is given by (with $\boldsymbol{r} = r\hat{\boldsymbol{k}}_s$)

$$\begin{aligned} P_s &= \frac{r^2}{2\eta}\int_0^{\pi}\int_0^{2\pi}\left|\boldsymbol{E}_s^f(\boldsymbol{r})\right|^2 \sin\theta_s d\theta_s d\phi_s \\ &= \frac{\left|E_0\right|^2}{2\eta}\int_0^{\pi}\int_0^{2\pi}\left|\hat{\boldsymbol{x}} - \sin\theta_s\cos\phi_s\hat{\boldsymbol{k}}_s\right|^2\left|S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right)\right|^2 \sin\theta_s d\theta_s d\phi_s \end{aligned} \tag{1.17}$$

Note that

$$\left|\hat{\boldsymbol{x}} - \sin\theta_s\cos\phi_s\hat{\boldsymbol{k}}_s\right|^2 = 1 - \sin^2\theta_s\cos^2\phi_s \tag{1.18}$$

Although this integral may be computed analytically, it is rather cumbersome. Therefore we use the fact that the sphere is very small compared to the wavelength so that $ka << 1$ and

$$S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right) = \frac{k^2\left(\epsilon_r - 1\right)}{k_d^3}\left[\sin\left(k_d a\right) - k_d a\cos\left(k_d a\right)\right] \sim \frac{k^2\left(\epsilon_r - 1\right)a^3}{3} \tag{1.19}$$

This allows us to write:

$$
\begin{aligned}
P_s &= \frac{r^2}{2\eta}\int_0^{\pi}\int_0^{2\pi}\left|\boldsymbol{E}_s^f(\boldsymbol{r})\right|^2 \sin\theta_s d\theta_s d\phi_s \\
&= \frac{|E_0|^2}{2\eta}\frac{k^4\left(\epsilon_r-1\right)^2 a^6}{9}\int_0^{\pi}\int_0^{2\pi}\left(1-\sin^2\theta_s\cos^2\phi_s\right)\sin\theta_s d\theta_s d\phi_s \\
&= \frac{|E_0|^2}{2\eta}\frac{\pi k^4\left(\epsilon_r-1\right)^2 a^6}{9}\int_0^{\pi}\left(1+\cos^2\theta_s\right)\sin\theta_s d\theta_s \\
&= \frac{|E_0|^2}{2\eta}\frac{\pi k^4\left(\epsilon_r-1\right)^2 a^6}{9}\frac{8}{3}
\end{aligned}
\tag{1.20}
$$

The scattering cross section:

$$
\sigma_t = \frac{8\pi}{3}\frac{\left(\epsilon_r-1\right)^2 k^4 a^6}{9} \tag{1.21}
$$

Next we use the optical theorem:

$$
\sigma_t = \frac{\eta}{\left|\boldsymbol{E}_{i,0}\right|^2}\mathrm{Re}\left[\boldsymbol{E}_{i,0}^*\cdot\boldsymbol{F}_\perp\left(\hat{\boldsymbol{z}}\right)\right] \tag{1.22}
$$

$$
\begin{aligned}
\boldsymbol{F}\left(\hat{\boldsymbol{k}}_s\right) &= \int_V \exp\left(j\boldsymbol{k}_s\cdot\boldsymbol{r}'\right)\boldsymbol{J}\left(\boldsymbol{r}'\right)dV' \\
&= j\omega\epsilon_0\left(\epsilon_d-1\right)\int_V \exp\left(j\boldsymbol{k}_s\cdot\boldsymbol{r}'\right)\boldsymbol{E}\left(\boldsymbol{r}'\right)dV' \\
&\sim j\frac{k}{\eta}\left(\epsilon_d-1\right)\boldsymbol{E}_0^i\int_V \exp\left(j\boldsymbol{k}_d\cdot\boldsymbol{r}'\right)dV'
\end{aligned}
\tag{1.23}
$$

Comparison with

$$
S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right) = \frac{k^2\left(\epsilon_d-1\right)}{4\pi}\int_V \exp\left(j\boldsymbol{k}_d\cdot\boldsymbol{r}'\right)dV' \tag{1.24}
$$

shows that

$$
\boldsymbol{F}\left(\hat{\boldsymbol{k}}_s\right) \sim j\boldsymbol{E}_{i,0}\frac{4\pi}{k}S\left(\hat{\boldsymbol{k}}_s,\hat{\boldsymbol{k}}_i\right) \sim j\boldsymbol{E}_{i,0}\frac{4\pi k\left(\epsilon_r-1\right)a^3}{3} \tag{1.25}
$$

Next consider:

$$
\boldsymbol{F}_\perp = \boldsymbol{F}-\left(\hat{\boldsymbol{k}}_s\cdot\boldsymbol{F}\right)\hat{\boldsymbol{k}}_s = j\frac{4\pi k\left(\epsilon_r-1\right)a^3}{3}\left[\boldsymbol{E}_{i,0}-\left(\hat{\boldsymbol{k}}_s\cdot\boldsymbol{E}_{i,0}\right)\hat{\boldsymbol{k}}_s\right] \tag{1.26}
$$

For evaluating $\boldsymbol{F}_\perp\left(\hat{\boldsymbol{z}}\right)$ we have to put $\hat{\boldsymbol{k}}_s = \hat{\boldsymbol{z}}$. Since the incident electric field has no component along z we have

$$\boldsymbol{F}_{\perp}(\hat{\boldsymbol{z}}) = j\frac{4\pi k(\epsilon_r - 1)a^3}{3}\boldsymbol{E}_{i,0} \tag{1.27}$$

And, therefore,

$$\sigma_t = \frac{\eta}{\left|\boldsymbol{E}_{i,0}\right|^2}\mathrm{Re}\left[\boldsymbol{E}_{i,0}^* \cdot \boldsymbol{F}_{\perp}(\hat{\boldsymbol{z}})\right] = \frac{4\pi k(\epsilon_r - 1)a^3}{3}\frac{\eta}{\left|\boldsymbol{E}_{i,0}\right|^2}\mathrm{Re}\left[j\boldsymbol{E}_{i,0}^* \cdot \boldsymbol{E}_{i,0}\right] = 0! \tag{1.28}$$

The scattering cross section which is obtained from the optical theorem is zero. This is because the Born approximation gives a result for the electric field which is correct up to the first order in $\epsilon_r - 1$. When calculating the cross section directly we use the magnitude-squared of the far electric field which is of the order $(\epsilon_r - 1)^2$. But the optical theorem directly involves $\boldsymbol{F}_{\perp}$ which, in the Born approximation, is only calculated up to the first order. Of course it cannot reproduce a quantity of the order $(\epsilon_r - 1)^2$. For that we have to go beyond the Born approximation in the optical theorem which is exact in principle.