EM Scattering

Homework assignment 1

Problem 1:

For a dielectric object with a volume V and space-dependent dielectric constant $\epsilon_p(\boldsymbol{r}')$, the scattered field in the far field zone is given by

$$\boldsymbol{E}_s(\boldsymbol{r}) = -\frac{k^2 \exp(-jkr)}{4\pi\epsilon_0 r}\hat{\boldsymbol{k}}_s \times \left[\hat{\boldsymbol{k}}_s \times \int_V \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}'\right)\left[\epsilon_p(\boldsymbol{r}') - \epsilon_0\right]\boldsymbol{E}(\boldsymbol{r}')dV'\right]$$

Here $\boldsymbol{k}_s = k\hat{\boldsymbol{k}}_s$ is the wave vector of the scattered wave along the direction of observation $\hat{\boldsymbol{k}}_s$, $\boldsymbol{E}(\boldsymbol{r}')$ is the total electric field inside the object, and $k^2 = \omega^2\epsilon_0\mu_0$ with μ_0, ϵ_0 denoting the permeability, respectively, the dielectric constant of the background medium. This result was obtained by computing the far field generated by polarization currents inside the object. It applies to materials which have no magnetic properties.

i. How should this equation be modified to include materials with magnetic properties, i.e., which are described by a permeability $\mu(\boldsymbol{r})$ different from the background permeability?

ii. Derive the scattered electric field in the Born approximation for this general case by assuming that $\mu(\boldsymbol{r}) - \mu_0$ is a small quantity. Assume that the incident wave is given by

$$\boldsymbol{E}_i(\boldsymbol{r}) = \boldsymbol{E}_i^0 \exp(-j\boldsymbol{k}_i \cdot \boldsymbol{r})$$

Solution:

(1) The general expression for the far field generated by an electric current density $\boldsymbol{J}$ is

$$\boldsymbol{E}_s(\boldsymbol{r}) = jk\eta_0 \frac{\exp(-jk_0 r)}{4\pi r}\hat{\boldsymbol{k}}_s \times \left[\hat{\boldsymbol{k}}_s \times \boldsymbol{F}\left(\hat{\boldsymbol{k}}_s\right)\right]$$

where $\eta_0 = \sqrt{\mu_0 / \epsilon_0}$ is the wave impedance, and

$$\boldsymbol{F}\left(\hat{\boldsymbol{k}}_s\right) = \int_V \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}'\right)\boldsymbol{J}(\boldsymbol{r}')dV'$$

The current can be a physical or an equivalent electric current. Instead of describing the magnetic object in terms of its permeability, one can replace it with equivalent volume and surface electric currents (both are required). The equivalent volume *electric* current density is given by $\boldsymbol{J}_e^V = \nabla \times \boldsymbol{M}$, while the equivalent surface electric current density is $\boldsymbol{J}_e^S = -\hat{\boldsymbol{n}} \times \boldsymbol{M}$ where $\hat{\boldsymbol{n}}$ is the unit normal to the surface of the object pointing outwards. Note that $\boldsymbol{M} = \boldsymbol{B}/\mu_0 - \boldsymbol{H}$ is the magnetization which, in our case, can be expressed as $\boldsymbol{M} = \delta\mu_r(\boldsymbol{r})\boldsymbol{H}$ with $\delta\mu_r(\boldsymbol{r}) = \mu(\boldsymbol{r})/\mu_0 - 1$.

Now the magnetic contribution to $\boldsymbol{F}$ is found to be

$$
\begin{aligned}
\boldsymbol{F}^m\left(\hat{\boldsymbol{k}}_s\right) &= \int_V \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}'\right)\left[\nabla' \times \boldsymbol{M}(\boldsymbol{r}')\right] dV' - \oint_S \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}'\right)\left[\hat{\boldsymbol{n}}' \times \boldsymbol{M}(\boldsymbol{r}')\right] dS' \\
&= -\int_V \nabla' \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}'\right) \times \boldsymbol{M}(\boldsymbol{r}') dV' = -j\boldsymbol{k}_s \times \int_V \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}'\right) \boldsymbol{M}(\boldsymbol{r}') dV' \\
&= -j\boldsymbol{k}_s \times \int_V \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}'\right) \boldsymbol{M}(\boldsymbol{r}') dV' = -j\boldsymbol{k}_s \times \int_V \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}'\right) \delta\mu_r(\boldsymbol{r}')\boldsymbol{H}(\boldsymbol{r}') dV'
\end{aligned}
$$

The resulting electric field is

$$
\begin{aligned}
\boldsymbol{E}_S^m(\boldsymbol{r}) &= jk_0\eta_0 \frac{\exp(-jk_0 r)}{4\pi r} \hat{\boldsymbol{k}}_s \times \left[\hat{\boldsymbol{k}}_s \times \boldsymbol{F}^m\left(\hat{\boldsymbol{k}}_s\right)\right] \\
&= -k_0^2\eta_0 \frac{\exp(-jk_0 r)}{4\pi r} \hat{\boldsymbol{k}}_s \times \int_V \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}'\right) \delta\mu_r(\boldsymbol{r}')\boldsymbol{H}(\boldsymbol{r}') dV'
\end{aligned}
$$

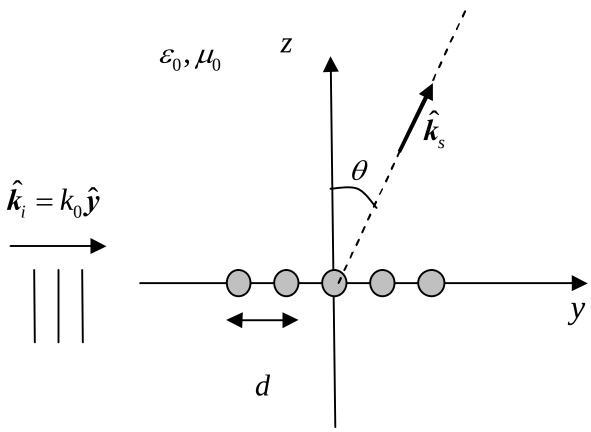
(2) In the Born approximation we approximate the magnetic field $\boldsymbol{H}(\boldsymbol{r}')$ inside the object by the incident field $\boldsymbol{H}_i(\boldsymbol{r}')$. Assuming the incident electric field to be given by $\boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right)$, the incident magnetic field becomes $(1/\eta_0)\left(\hat{\boldsymbol{k}}_i \times \boldsymbol{E}_i^0\right)\exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}\right)$ so that in the Born approximation

$$
\boldsymbol{E}_S^m(\boldsymbol{r}) = -k_0^2 \frac{\exp(-jk_0 r)}{4\pi r} \hat{\boldsymbol{k}}_s \times \left(\hat{\boldsymbol{k}}_i \times \boldsymbol{E}_i^0\right) \int_V \exp\left[j\left(\boldsymbol{k}_s - \boldsymbol{k}_i\right) \cdot \boldsymbol{r}'\right] \delta\mu_r(\boldsymbol{r}') dV'
$$

Problem 2 :

Consider an array of N small dielectric spheres in vacuum whose centers lie on the y axis. The radius and relative dielectric constant of each sphere is a and ε_r, respectively. The distance between the centers of adjacent spheres is d. An incident plane wave traveling in the y-direction hits the array. The electric field of the incident wave is along the x-axis.

Use the Born approximation and calculate the far-zone scattered electric field on the $y-z$ plane as function of the scattering angle θ. Neglect the interaction between the spheres. Approximate the integrals by assuming the spheres to be much smaller than the wavelength.



Solution

For a small sphere at $\boldsymbol{r}_0$ the Born approximation gives for the scattered field observed along $\boldsymbol{k}_s$:

$$\boldsymbol{E}_s^f(\hat{\boldsymbol{k}}_s) = \frac{\exp(-jkr)}{r}\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right)$$

$$\boldsymbol{Q}_\perp\left(\hat{\boldsymbol{k}}_s, \hat{\boldsymbol{k}}_i\right) = \frac{k^2}{4\pi\epsilon_0}\left\{\int_V \exp\left[j\left(\boldsymbol{k}_s - \boldsymbol{k}_i\right)\cdot\boldsymbol{r}'\right]\delta\varepsilon_p(\boldsymbol{r}')dV'\right\}\left[\boldsymbol{E}_{i,0} - \left(\boldsymbol{E}_{i,0}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right]$$

$$\approx \frac{k^2 V\delta\varepsilon_r}{4\pi\epsilon_0}\exp\left[j\left(\boldsymbol{k}_s - \boldsymbol{k}_i\right)\cdot\boldsymbol{r}_0\right]\left[\boldsymbol{E}_{i,0} - \left(\boldsymbol{E}_{i,0}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right]$$

$$= E_0\frac{k^2 V\delta\varepsilon_r}{4\pi\epsilon_0}\exp\left[j\left(\boldsymbol{k}_s - k\hat{\boldsymbol{y}}\right)\cdot\boldsymbol{r}_0\right]\left[\hat{\boldsymbol{x}} - \left(\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{k}}_s\right)\hat{\boldsymbol{k}}_s\right]$$

But we are just considering those directions on the y-z plane so that $\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{k}}_s = 0$ and

$$\boldsymbol{E}_s^f(\hat{\boldsymbol{k}}_s) \approx \hat{\boldsymbol{x}}E_0 \frac{\exp(-jkr)}{r}\frac{k^2a^3\delta\varepsilon_r}{3\epsilon_0}\exp\left[j\left(\boldsymbol{k}_s - k\hat{\boldsymbol{y}}\right)\cdot \boldsymbol{r}_0\right]$$

This was for a single sphere. For the array:

$$\boldsymbol{E}_s^f(\hat{\boldsymbol{k}}_s) \approx \hat{\boldsymbol{x}}E_0 \frac{\exp(-jkr)}{r}\frac{k^2a^3\delta\varepsilon_r}{3\epsilon_0}\sum_{i=1}^{N}\exp\left[j\left(\boldsymbol{k}_s - k\hat{\boldsymbol{y}}\right)\cdot \boldsymbol{r}_i\right]$$

$$\sum_{i=1}^{N}\exp\left[j\left(\boldsymbol{k}_s - k\hat{\boldsymbol{y}}\right)\cdot \boldsymbol{r}_i\right] = \sum_{n=1}^{N}\exp\left[j\left(\boldsymbol{k}_s - k\hat{\boldsymbol{y}}\right)\cdot \hat{\boldsymbol{y}}\left(nd + y_0\right)\right]$$

$$= \sum_{n=1}^{N}\exp\left[j\left(\boldsymbol{k}_s\cdot\hat{\boldsymbol{y}} - k\right)\left(nd + y_0\right)\right] = \sum_{n=1}^{N}\exp\left[jk\left(\sin\theta - 1\right)\left(nd + y_0\right)\right]$$

The latter expression is the array factor.

<u>Problem 3 :</u>

A plane wave traveling in vacuum the *x*-direction and linearly polarized along the *z*-direction is incident upon an infinitely long dielectric cylinder whose axis coincides with the z-axis. The radius and (relative) dielectric constant of the cylinder are given by a and ε_r, respectively. The amplitude of the incident wave is E_0.

i. Find the far-zone scattered field in Born approximation

ii. Is it possible to define a scattering cross section? If not which quantity must be calculated? Compute this quantity using the results of (i).

Solution

Approximate field inside the dielectric:

$$\boldsymbol{E} \sim \boldsymbol{E}_i = E_0\hat{\boldsymbol{z}}\exp\left(-jk_0x\right)$$

Equivalent (polarization) current:

$$\boldsymbol{J}_p = j\omega\left(\varepsilon - \varepsilon_0\right)\boldsymbol{E} = j\omega\varepsilon_0\delta\varepsilon_r\boldsymbol{E} = j\omega\varepsilon_0\delta\varepsilon_r E_0\hat{\boldsymbol{z}}\exp\left(-jk_0x\right)$$

2D equation for TE field generated by a current in z-direction:

$$\left(\partial_x^2+\partial_y^2\right)E_z+k_0^2E_z = j\omega\mu_0 J_z \rightarrow E_z\left(\boldsymbol{\rho}\right) = -j\omega\mu_0\int_S G_{2D}(\boldsymbol{\rho},\boldsymbol{\rho}')J_z\left(\boldsymbol{\rho}'\right)dS'$$

$$\boldsymbol{\rho}=(x,y), G_{2D}(\boldsymbol{\rho},\boldsymbol{\rho}')=\frac{1}{4j}H_0^{(2)}\left(k\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|\right)$$

Far field:

$$k\rho >> 1, \rho >> \rho'$$

$$G_{2D}(\boldsymbol{\rho},\boldsymbol{\rho}') \sim \frac{1}{4j}\sqrt{\frac{2}{\pi k_0\rho}}\exp\left(-jk\rho+\frac{j\pi}{4}\right)\exp\left(jk_0\hat{\boldsymbol{\rho}}\cdot\boldsymbol{\rho}'\right)$$

$$E_z^f \approx -\frac{\omega\mu_0}{4}\sqrt{\frac{2}{\pi k_0\rho}}\exp\left(-jk\rho+\frac{j\pi}{4}\right)F_z\left(\hat{\rho}\right)$$

$$F_z\left(\hat{\rho}\right)=\int_S \exp\left(jk_0\hat{\rho}\cdot\rho'\right)J_z\left(\boldsymbol{\rho}'\right)dS'$$

$$\hat{\rho}=(\cos\varphi,\sin\varphi)$$

In our case, thus,

$$F_z\left(\hat{\rho}\right)=F_z\left(\varphi\right)= j\omega\varepsilon_0\delta\varepsilon_r E_0\int_S \exp\left(jk_0\hat{\rho}\cdot\rho'\right)\exp\left(-jk_0x'\right)dS'$$

$$= j\omega\varepsilon_0\delta\varepsilon_r E_0\int_0^a\int_0^{2\pi}\exp\left[jk_0\left(\cos\varphi-1\right)x'+jk_0\sin\varphi y'\right]dS'$$

$$\left(\cos\varphi-1\right)x'+\sin\varphi y'=\rho'\left[\left(\cos\varphi-1\right)\cos\varphi'+\sin\varphi\sin\varphi'\right]$$

$$=2\rho'\sin\frac{\varphi}{2}\left(-\sin\frac{\varphi}{2}\cos\varphi'+\cos\frac{\varphi}{2}\sin\varphi'\right)$$

$$=2\rho'\sin\frac{\varphi}{2}\sin\left(\varphi'-\frac{\varphi}{2}\right)$$

so that

$$F_z\left(\varphi\right)= j\omega\varepsilon_0\delta\varepsilon_r E_0\int_0^a\int_0^{2\pi}\exp\left[j2k_0\rho'\sin\frac{\varphi}{2}\sin\left(\varphi'-\frac{\varphi}{2}\right)\right]\rho'd\rho'd\varphi'$$

$$= j\omega\varepsilon_0\delta\varepsilon_r E_0\int_0^a\int_0^{2\pi}\exp\left(jz\sin\varphi'\right)\rho'd\rho'd\varphi'$$

$$z=2k_0\rho'\sin\frac{\varphi}{2}$$

$$F_z\left(\varphi\right)=2\pi j\omega\varepsilon_0\delta\varepsilon_r E_0\int_0^a J_0\left(2k_0\rho'\sin\frac{\varphi}{2}\right)\rho'd\rho'$$

Use

$$\frac{d}{dz}\left[zJ_1(z)\right] = zJ_0(z)$$

$$\rightarrow \int_0^a J_0\left(2k_0\rho' \sin\frac{\varphi}{2}\right)\rho' d\rho' = \frac{1}{\left(2k_0 \sin\frac{\varphi}{2}\right)^2} \int_0^{2k_0 a \sin\frac{\varphi}{2}} J_0(z)\, z dz = \frac{a}{2k_0 \sin\frac{\varphi}{2}} J_1\left(2k_0 a \sin\frac{\varphi}{2}\right)$$

$$\rightarrow F_z(\varphi) = j\omega\varepsilon_0\delta\varepsilon_r E_0 \frac{a}{2k_0 \sin\frac{\varphi}{2}} J_1\left(2k_0 a \sin\frac{\varphi}{2}\right)$$

Far field result:

$$\begin{aligned} E_z^f &\approx -\frac{\omega\mu_0}{4}\sqrt{\frac{2}{\pi k_0 \rho}} \exp\left(-jk\rho + \frac{j\pi}{4}\right) F_z(\hat{\rho}) \\ &= -jE_0 \frac{k_0 a \delta\varepsilon_r}{8\sin\frac{\varphi}{2}} \sqrt{\frac{2}{\pi k_0 \rho}} \exp\left(-jk\rho + \frac{j\pi}{4}\right) J_1\left(2k_0 a \sin\frac{\varphi}{2}\right) \end{aligned}$$

Far field Poynting vector:

$$\boldsymbol{S} = \hat{\boldsymbol{\rho}} \frac{\left|E_s^f\right|^2}{2\eta_0} = \hat{\boldsymbol{\rho}} \frac{\left|E_0\right|^2}{2\eta_0} \left(\frac{k_0 a \delta\varepsilon_r}{8\sin\frac{\varphi}{2}}\right)^2 \frac{2}{\pi k_0 \rho} \left[J_1\left(2k_0 a \sin\frac{\varphi}{2}\right)\right]^2$$

Total power (per unit length) radiated:

$$P = \frac{\left|E_0\right|^2}{2\eta_0} \left(\frac{k_0 a \delta\varepsilon_r}{8\sin\frac{\varphi}{2}}\right)^2 \frac{2}{\pi k_0} \int_0^{2\pi} \left[J_1\left(2k_0 a \sin\frac{\varphi}{2}\right)\right]^2 d\varphi$$

scattering width:

$$\sigma_w = \left(\frac{k_0 a \delta\varepsilon_r}{8\sin\frac{\varphi}{2}}\right)^2 \frac{2}{\pi k_0} \int_0^{2\pi} \left[J_1\left(2k_0 a \sin\frac{\varphi}{2}\right)\right]^2 d\varphi$$