EM Scattering

Homework assignment 2

Problem 1:

A plane wave traveling in the x direction in vacuum is scattered by a perfectly conducting, infinite cylinder with the radius of 1cm whose axis coincides with the z-axis. The frequency of the incident wave is 30GHz and its electric field is polarized along the z-axis. The amplitude of the incident electric field is E_0.

- Calculate the electric and magnetic fields in the far zone and plot their amplitude as function of angle
- Calculate the current density on the surface of the cylinder and plot it as function of angle

Solution:

Let us first examine the polarization of the incident field. The unit vectors in the plane normal to the incident wave vector $\boldsymbol{k}_i = k\hat{\boldsymbol{x}}$ are $\hat{\boldsymbol{h}}_i = \left|\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_i\right|^{-1} \hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_i = \hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{v}}_i = \hat{\boldsymbol{h}}_i \times \hat{\boldsymbol{k}}_i = -\hat{\boldsymbol{z}}$. The incident wave has only a component along $\hat{\boldsymbol{v}}_i$ so that this is a TM^z scattering problem. Thus,

$$\begin{aligned} \boldsymbol{E}_s^{TM}(\boldsymbol{r}) &= -\sum_{m=-\infty}^{\infty} v_m \frac{J_m\left(k_{i,\rho}a\right)}{H_m^{(2)}\left(k_{i,\rho}a\right)} \boldsymbol{N}_{m,k_{i,z}}^{H}(\rho,\phi,z) \\ v_m &= -\left(\boldsymbol{E}_i^0 \cdot \hat{\boldsymbol{v}}_i\right) \frac{k}{k_{i,\rho}} (-j)^m \exp\left(jm\phi_i\right) \end{aligned} \tag{1.1}$$

Note that $k_{i,\rho} = k$ as we have to deal with a normally incident plane wave ($k_{i,z} = 0$). The corresponding magnetic field is given by

$$\boldsymbol{H}_s^{TM}(\boldsymbol{r}) = -\frac{j}{\eta} \sum_{m=-\infty}^{\infty} v_m \frac{J_m\left(k_{i,\rho}a\right)}{H_m^{(2)}\left(k_{i,\rho}a\right)} \boldsymbol{M}_{m,k_{i,z}}^{H}(\rho,\phi,z) \tag{1.2}$$

Collecting the results for the scattered field (with $\phi_i = 0$):

$$\boldsymbol{E}_s^{TM}(\boldsymbol{r}) = -E_0 \sum_{m=-\infty}^{\infty} (-j)^m \frac{J_m(ka)}{H_m^{(2)}(ka)} \boldsymbol{N}_{m,0}^{H}(r,\phi,z)$$

$$\boldsymbol{H}_s^{TM}(\boldsymbol{r}) = -E_0 \frac{j}{\eta} \sum_{m=-\infty}^{\infty} (-j)^m \frac{J_m(ka)}{H_m^{(2)}(ka)} \boldsymbol{M}_{m,0}^{H}(\rho,\phi,z)$$

$$\boldsymbol{N}_{m,0}^{H}(r,\phi,z) = \frac{1}{k^2}\left[-\hat{\boldsymbol{r}}\, jk_{i,z} \frac{\partial H_m^{(2)}}{\partial r} - \hat{\boldsymbol{\phi}}\, \frac{mk_{i,z}}{r} H_m^{(2)} + \hat{\boldsymbol{z}}\, k_{i,\rho}^2 H_m^{(2)}\right] \exp\left(-jk_{i,z}z - jm\varphi\right) \quad (1.3)$$

$$= \hat{\boldsymbol{z}}\, H_m^{(2)}(kr) \exp(-jm\varphi)$$

$$\boldsymbol{M}_{m,0}^{H}(\rho,\phi,z) = \left[-\hat{\boldsymbol{r}}\, \frac{jm}{kr} H_m^{(2)} - \hat{\boldsymbol{\phi}}\, \frac{\partial H_m^{(2)}}{k\partial r}\right] \exp\left(-jk_{i,z}z - jm\varphi\right)$$

$$= \left[-\hat{\boldsymbol{r}}\, \frac{jm}{kr} H_m^{(2)}(kr) - \hat{\boldsymbol{\phi}} H_m^{(2)\prime}(kr)\right] \exp(-jm\varphi)$$

Far-zone fields:

$$\begin{aligned} \boldsymbol{E}_s^{TM}(\boldsymbol{r}) &= -\hat{\boldsymbol{z}}\, E_0 \sum_{m=-\infty}^{\infty} (-j)^m \frac{J_m(ka)}{H_m^{(2)}(ka)} H_m^{(2)}(kr) \exp(-jm\varphi) \\ &\to -\hat{\boldsymbol{z}}\, E_0 \sqrt{\frac{2}{\pi kr}} \exp\left(-jkr + j\pi/4\right) \sum_{m=-\infty}^{\infty} \frac{J_m(ka)}{H_m^{(2)}(ka)} \exp(-jm\varphi) \end{aligned} \quad (1.4)$$

$$\boldsymbol{H}_s^{TM}(\boldsymbol{r}) = -E_0 \frac{j}{\eta} \sum_{m=-\infty}^{\infty} (-j)^m \frac{J_m(ka)}{H_m^{(2)}(ka)} \left[-\hat{\boldsymbol{r}}\, \frac{jm}{kr} H_m^{(2)}(kr) - \hat{\boldsymbol{\phi}} H_m^{(2)\prime}(kr)\right] \exp(-jm\varphi)$$

$$\to \hat{\boldsymbol{\phi}} E_0 \frac{j}{\eta} \sum_{m=-\infty}^{\infty} (-j)^m \frac{J_m(ka)}{H_m^{(2)}(ka)} H_m^{(2)\prime}(kr) \exp(-jm\varphi) \quad (1.5)$$

$$\to \hat{\boldsymbol{\phi}} E_0 \frac{1}{\eta} \sqrt{\frac{2}{\pi kr}} \exp\left(-jkr + j\pi/4\right) \sum_{m=-\infty}^{\infty} \frac{J_m(ka)}{H_m^{(2)}(ka)} \exp(-jm\varphi)$$

The function we have to plot is $\sum_{m=-\infty}^{\infty} \frac{J_m(ka)}{H_m^{(2)}(ka)} \exp(-jm\varphi)$ for $ka = 2\pi$.

Part 2:

The surface current density on the cylinder is found from the exact solution for the magnetic field. We have:

$$\boldsymbol{J}_s = \hat{\boldsymbol{r}} \times \boldsymbol{H}_s^{TM}(a,\varphi) = \hat{\boldsymbol{z}} E_0 \frac{j}{\eta} \sum_{m=-\infty}^{\infty} (-j)^m \frac{J_m(ka)}{H_m^{(2)}(ka)} H_m^{(2)\prime}(ka) \exp(-jm\varphi) \quad (1.6)$$

To this we should add the contribution from the incident field. The electric field of the incident plane wave is along z. Its magnetic field is

$$\boldsymbol{H}_i = \frac{1}{\eta_0}\hat{\boldsymbol{k}}_i \times \boldsymbol{E}_i = \frac{1}{\eta_0}\left(\hat{\boldsymbol{x}}\times\hat{\boldsymbol{z}}\right)E_0\exp\left(-jkx\right) \tag{1.7}$$

$$\begin{aligned}\boldsymbol{J}_{i,s} &= \hat{\boldsymbol{r}}\times\boldsymbol{H}_i = \frac{E_0\exp\left(-jkx\right)}{\eta}\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{x}}\times\hat{\boldsymbol{z}}\right) = -\hat{\boldsymbol{z}}\frac{E_0\exp\left(-jkx\right)}{\eta}\left(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{x}}\right)\\ &= -\hat{\boldsymbol{z}}\frac{E_0\exp\left(-jkx\right)}{\eta}\cos\phi\end{aligned} \tag{1.8}$$

Next, use

$$\exp\left(-jkx\right) = \exp\left(-jka\cos\phi\right) = \sum_{m=-\infty}^{\infty}(-j)^m J_m(ka)\exp\left(-jm\phi\right) \tag{1.9}$$

$$\begin{aligned}\boldsymbol{J}_{i,s} &= -\hat{\boldsymbol{z}}\frac{E_0\exp\left(-jka\cos\phi\right)}{\eta_0}\cos\phi = \hat{\boldsymbol{z}}\frac{E_0}{j\eta k}\frac{\partial}{\partial a}\exp\left(-jka\cos\phi\right)\\ &= \hat{\boldsymbol{z}}\frac{E_0}{j\eta k}\frac{\partial}{\partial a}\sum_{m=-\infty}^{\infty}(-j)^m J_m(ka)\exp\left(-jm\phi\right) =\\ &= \hat{\boldsymbol{z}}\frac{E_0}{j\eta}\sum_{m=-\infty}^{\infty}(-j)^m J_m'(ka)\exp\left(-jm\phi\right)\end{aligned} \tag{1.10}$$

Adding the two currents:

$$\begin{aligned}&\hat{\boldsymbol{z}}\frac{E_0}{j\eta}\sum_{m=-\infty}^{\infty}(-j)^m\left[J_m'(ka) - \frac{J_m\left(ka\right)}{H_m^{(2)}\left(ka\right)}H_m^{(2)\prime}\left(ka\right)\right]\exp\left(-jm\phi\right)\\ &= \hat{\boldsymbol{z}}\frac{E_0}{j\eta}\sum_{m=-\infty}^{\infty}(-j)^m\left[\frac{J_m'(ka)H_m^{(2)}\left(ka\right) - H_m^{(2)\prime}\left(ka\right)J_m\left(ka\right)}{H_m^{(2)}\left(ka\right)}\right]\exp\left(-jm\phi\right)\\ &= \hat{\boldsymbol{z}}\frac{E_0}{j\eta}\sum_{m=-\infty}^{\infty}(-j)^m\left[\frac{2j/\pi ka}{H_m^{(2)}\left(ka\right)}\right]\exp\left(-jm\phi\right)\\ &= \hat{\boldsymbol{z}}\frac{2E_0}{k\eta\pi a}\sum_{m=-\infty}^{\infty}(-j)^m\left[\frac{\exp\left(-jm\phi\right)}{H_m^{(2)}\left(ka\right)}\right]\end{aligned} \tag{1.11}$$

Problem 2:

The problem of scattering from perfectly conducting cylinders was treated in the class by using the $\boldsymbol{M}$ and $\boldsymbol{N}$ functions. The same method may be exploited to solve the scattering problem from a dielectric cylinder. Consider a dielectric cylinder with the radius a and relative dielectric constant ε_d. The axis of the cylinder coincides with the z-axis. The incident wave travels along $\boldsymbol{k}_i$. For simplicity assume normal incidence, i.e., $k_{i,z} = 0$.

- In the region inside the cylinder write the electric field vector as a series in appropriate $\boldsymbol{M}$ and $\boldsymbol{N}$ functions.

Solution:

Inside the cylinder the vector wave equation is still valid but now with the wave number $k_d = k\sqrt{\varepsilon_d}$. Besides, we have to use ordinary Bessel functions (Why?). Therefore we write the ***total*** field as

$$\begin{aligned}
\boldsymbol{E}_t(\boldsymbol{r}) &= \sum_{m=-\infty}^{\infty} c_m \boldsymbol{M}_{m,0}^{J}(r,\phi) + \sum_{m=-\infty}^{\infty} d_m \boldsymbol{N}_{m,0}^{J}(r,\phi),\ r < a \\
\boldsymbol{H}_t(\boldsymbol{r}) &= \frac{j\sqrt{\varepsilon_d}}{\eta} \sum_{m=-\infty}^{\infty} c_m \boldsymbol{N}_{m,0}^{J}(r,\phi) + \sum_{m=-\infty}^{\infty} d_m \boldsymbol{M}_{m,0}^{J}(r,\phi) \\
\boldsymbol{M}_{m,0}^{J}(\rho,\phi) &= \left[-\hat{\boldsymbol{r}}\ \frac{jm}{k_d r} J_m\left(k_d r\right) - \hat{\boldsymbol{\phi}} J_m'\left(k_d r\right) \right] \exp\left(-jm\varphi\right) \\
\boldsymbol{N}_{m,0}^{J}(r,\phi) &= \hat{\boldsymbol{z}}\ J_m\left(k_d r\right) \exp\left(-jm\varphi\right)
\end{aligned} \tag{1.12}$$

- Outside the cylinder write the field as the incident plus the scattered field and expand each in series.

Solution:

Outside the cylinder we use the Hankel functions for the scattered field and ordinary Bessel functions for the incident field but now we use k instead of k_d:

$$\begin{aligned}
\boldsymbol{E}_s(\boldsymbol{r}) &= \sum_{m=-\infty}^{\infty} a_m \boldsymbol{M}_{m,0}^{H}(r,\phi) + \sum_{m=-\infty}^{\infty} b_m \boldsymbol{N}_{m,0}^{H}(r,\phi),\ r > a \\
\boldsymbol{H}_s(\boldsymbol{r}) &= \frac{j}{\eta} \sum_{m=-\infty}^{\infty} a_m \boldsymbol{N}_{m,0}^{H}(r,\phi) + \sum_{m=-\infty}^{\infty} b_m \boldsymbol{M}_{m,0}^{H}(r,\phi) \\
\boldsymbol{M}_{m,0}^{H}(\rho,\phi) &= \left[-\hat{\boldsymbol{r}}\ \frac{jm}{kr} H_m^{(2)}\left(kr\right) - \hat{\boldsymbol{\phi}} H_m^{(2)\prime}\left(kr\right) \right] \exp\left(-jm\varphi\right) \\
\boldsymbol{N}_{m,0}^{H}(r,\phi) &= \hat{\boldsymbol{z}} H_m^{(2)}\left(kr\right) \exp\left(-jm\varphi\right)
\end{aligned} \tag{1.13}$$

$$
\begin{aligned}
&\boldsymbol{E}_i(\boldsymbol{r}) = \sum_{m=-\infty}^{\infty} u_m \boldsymbol{M}_{m,0}^{J,0}(\rho,\phi) + \sum_{m=-\infty}^{\infty} v_m \boldsymbol{N}_{m,0}^{J,0}(\rho,\phi)\\
&\boldsymbol{H}_i(\boldsymbol{r}) = \frac{j}{\eta}\sum_{m=-\infty}^{\infty} u_m \boldsymbol{N}_{m,0}^{J,0}(\rho,\phi) + \sum_{m=-\infty}^{\infty} v_m \boldsymbol{M}_{m,0}^{J,0}(\rho,\phi)\\
&\boldsymbol{M}_{m,0}^{J,0}(\rho,\phi) = \left[-\hat{\boldsymbol{r}}\ \frac{jm}{kr} J_m\left(kr\right) - \hat{\boldsymbol{\phi}} J_m'\left(kr\right)\ \right]\exp\left(-jm\varphi\right)\\
&\boldsymbol{N}_{m,0}^{J,0}(r,\phi) = \hat{\boldsymbol{z}}\ J_m\left(kr\right)\exp\left(-jm\varphi\right)\\
&u_m = -j\left(\boldsymbol{E}_i^0\cdot\hat{\boldsymbol{h}}_i\right)\frac{k}{k_{i,\rho}}(-j)^m \exp\left(jm\phi_i\right) = -j\left(\boldsymbol{E}_i^0\cdot\hat{\boldsymbol{h}}_i\right)(-j)^m \exp\left(jm\phi_i\right)\\
&v_m = -\left(\boldsymbol{E}_i^0\cdot\hat{\boldsymbol{v}}_i\right)\frac{k}{k_{i,\rho}}(-j)^m \exp\left(jm\phi_i\right) = -\left(\boldsymbol{E}_i^0\cdot\hat{\boldsymbol{v}}_i\right)(-j)^m \exp\left(jm\phi_i\right)
\end{aligned} \tag{1.14}
$$

- Match the solutions at the surface of the cylinder and find the scattered field.

<u>**Solution:**</u>

We first match the tangential components of the electric field at $r = a$:

$$
\begin{aligned}
d_m J_m\left(k_d a\right) &= b_m H_m^{(2)}\left(ka\right) + v_m J_m\left(ka\right)\\
c_m J_m'\left(k_d a\right) &= a_m H_m^{(2)\prime}\left(ka\right) + u_m J_m'\left(ka\right)
\end{aligned} \tag{1.15}
$$

We next match the tangential components of the magnetic field at $r = a$:

$$
\begin{aligned}
\sqrt{\varepsilon_d}\, c_m J_m\left(k_d a\right) &= a_m H_m^{(2)}\left(ka\right) + u_m J_m\left(ka\right)\\
\sqrt{\varepsilon_d}\, d_m J_m'\left(k_d a\right) &= b_m H_m^{(2)\prime}\left(ka\right) + v_m J_m'\left(ka\right)
\end{aligned} \tag{1.16}
$$

From these equations is follows that

$$
\begin{aligned}
&c_m J_m'\left(k_d a\right) = a_m H_m^{(2)\prime}\left(ka\right) + u_m J_m'\left(ka\right)\\
&\sqrt{\varepsilon_d}\, c_m J_m\left(k_d a\right) = a_m H_m^{(2)}\left(ka\right) + u_m J_m\left(ka\right)\\
&\rightarrow a_m = -\frac{J_m'\left(k_d a\right) H_m^{(2)}\left(ka\right) - \sqrt{\varepsilon_d}\, J_m\left(k_d a\right) H_m^{(2)\prime}\left(ka\right)}{J_m'\left(k_d a\right) J_m\left(ka\right) - \sqrt{\varepsilon_d}\, J_m\left(k_d a\right) J_m'\left(ka\right)} u_m
\end{aligned} \tag{1.17}
$$

$$
\begin{aligned}
&d_m J_m\left(k_d a\right) = b_m H_m^{(2)}\left(ka\right) + v_m J_m\left(ka\right)\\
&\sqrt{\varepsilon_d}\, d_m J_m'\left(k_d a\right) = b_m H_m^{(2)\prime}\left(ka\right) + v_m J_m'\left(ka\right)\\
&\rightarrow b_m = -\frac{J_m\left(k_d a\right) H_m^{(2)\prime}\left(ka\right) - \sqrt{\varepsilon_d}\, J_m'\left(k_d a\right) H_m^{(2)}\left(ka\right)}{J_m\left(k_d a\right) J_m'\left(ka\right) - \sqrt{\varepsilon_d}\, J_m'\left(k_d a\right) J_m\left(ka\right)} v_m
\end{aligned} \tag{1.18}
$$

These coefficients, when substituted in (1.13) are used to calculate the scattered field.