EM Scattering

Homework assignment 2

Problem 1:

A plane wave traveling in the x direction in vacuum is scattered by a perfectly conducting, infinite cylinder with the radius of 1cm whose axis coincides with the z-axis. The frequency of the incident wave is 20GHz. The amplitude of the incident electric field is E_0. For **both** cases where the electric field is polarized along the y- and z-axes:

- Calculate the electric field in the far zone and plot its amplitude as function of angle
- Calculate the current density on the surface of the cylinder and plot it as function of angle

(You need routines for Hankel and Bessel functions. Matlab would be OK.)

Solution

Note that $\hat{\boldsymbol{k}}_i = \hat{\boldsymbol{x}}, k_{iz} = 0$

$$\hat{\boldsymbol{h}}_i = \frac{\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_i}{\left|\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_i\right|} = \hat{\boldsymbol{y}}, \quad \hat{\boldsymbol{v}}_i = \hat{\boldsymbol{h}}_i \times \hat{\boldsymbol{k}}_i = -\hat{\boldsymbol{z}}$$

In the two cases considered only TE_z, respectively, TM_z scattered fields are generated.

Resulting far field for polarization along y

$$\boldsymbol{E}_s^{TE,f} = -\hat{\boldsymbol{\phi}} \left(\boldsymbol{E}_i^0 \cdot \hat{\boldsymbol{h}}_i\right) \sqrt{\frac{2}{\pi k_{i,\rho}\rho}} \exp\left(-jk_{i,\rho}\rho - jk_{i,z}z + \frac{j\pi}{4}\right) \sum_{m=-\infty}^{\infty} \frac{J_m'\left(k_{i,\rho}a\right)}{H_m^{(2)\prime}\left(k_{i,\rho}a\right)} \exp\left[jm\left(\phi_i - \phi\right)\right]$$

$$\left(\boldsymbol{E}_i^0 \cdot \hat{\boldsymbol{h}}_i\right) = E_0, k_{i,\rho} = k, \phi_i = 0 \rightarrow$$

$$\boldsymbol{E}_s^{TE,f} = -\hat{\boldsymbol{\phi}}\, E_0 \sqrt{\frac{2}{\pi k\rho}} \exp\left(-jk\rho + \frac{j\pi}{4}\right) \sum_{m=-\infty}^{\infty} \frac{J_m'\left(ka\right)}{H_m^{(2)\prime}\left(ka\right)} \exp\left[-jm\phi\right]$$

And for z-polarization

$$\boldsymbol{E}_s^{TM,f} = \left[-\frac{k_{i,z}}{k}\hat{\boldsymbol{\rho}} + \frac{k_{i,\rho}}{k}\hat{\boldsymbol{z}}\right]\left(\boldsymbol{E}_i^0 \cdot \hat{\boldsymbol{v}}_i\right)\sqrt{\frac{2}{\pi k_{i,\rho}\rho}}\exp\left(-jk_{i,\rho}\rho - jk_{i,z}z + \frac{j\pi}{4}\right)$$

$$\sum_{m=-\infty}^{\infty}\frac{J_m\left(k_{i,\rho}a\right)}{H_m^{(2)}\left(k_{i,\rho}a\right)}\exp\left[jm\left(\phi_i - \phi\right)\right]$$

$$\rightarrow \boldsymbol{E}_s^{TM,f} = -\hat{\boldsymbol{z}}E_0\sqrt{\frac{2}{\pi k\rho}}\exp\left(-jk\rho + \frac{j\pi}{4}\right)\sum_{m=-\infty}^{\infty}\frac{J_m\left(ka\right)}{H_m^{(2)}\left(ka\right)}\exp\left[-jm\phi\right]$$

To calculate the current note that the surface current density on the cylinder is found from the exact solution for the magnetic field. We have:

$$\boldsymbol{J}_s = \hat{\boldsymbol{r}} \times \left[\boldsymbol{H}_s(a,\varphi) + \boldsymbol{H}_i(a,\varphi)\right] \qquad (0.1)$$

For the y-polarized field

$$\boldsymbol{H}_s^{TE}(\boldsymbol{r}) = -\frac{j}{\eta}\sum_{m=-\infty}^{\infty}\frac{u_m J_m'\left(k_{i,\rho}a\right)}{H_m^{(2)\prime}\left(k_{i,\rho}a\right)}\boldsymbol{N}_{m,k_{i,z}}^{H}(\rho,\phi,z)$$

$$u_m = -j\left(\boldsymbol{E}_i^0 \cdot \hat{\boldsymbol{h}}_i\right)\frac{k}{k_{i,\rho}}(-j)^m \exp\left(jm\phi_i\right)$$

$$\rightarrow \boldsymbol{H}_s^{TE}(\boldsymbol{r}) = -\frac{E_0}{\eta}\sum_{m=-\infty}^{\infty}\frac{(-j)^m J_m'\left(ka\right)}{H_m^{(2)\prime}\left(ka\right)}\boldsymbol{N}_{m,0}^{H}(\rho,\phi,z)$$

$$\boldsymbol{N}_{m,0}^{H}(\boldsymbol{r}) = \exp\left(-jm\phi\right)\hat{\boldsymbol{z}}\, H_m^{(2)}\left(k\rho\right)$$

The incident magnetic field

$$\boldsymbol{H}_i = \hat{\boldsymbol{z}}\frac{E_0}{\eta}\exp\left(-jkx\right)$$

Current density

$$\boldsymbol{J}_s^{TE} = \hat{\boldsymbol{\rho}} \times \left[\boldsymbol{H}_s^{TE}(a,\varphi) + \boldsymbol{H}_i(a,\varphi)\right] =$$

$$-\hat{\boldsymbol{\varphi}}\,\frac{E_0}{\eta}\left[\exp\left(-jka\cos\phi\right) - \sum_{m=-\infty}^{\infty}\frac{(-j)^m J_m'\left(ka\right)H_m^{(2)}\left(ka\right)}{H_m^{(2)\prime}\left(ka\right)}\exp\left(-jm\phi\right)\right]$$

$$\exp\left(-jz\cos\vartheta\right) = \sum_{m=-\infty}^{\infty}(-j)^m J_m(z)\exp\left(-jm\vartheta\right)$$

$$\rightarrow \exp\left(-jka\cos\phi\right) = \sum_{m=-\infty}^{\infty}(-j)^m J_m(ka)\exp\left(-jm\phi\right)$$

$$\boldsymbol{J}_s^{TE} = -\hat{\boldsymbol{\varphi}}\ \frac{E_0}{\eta}\left[\sum_{m=-\infty}^{\infty}(-j)^m\left\{J_m(ka) - \frac{J'_m(ka)H_m^{(2)}(ka)}{H_m^{(2)\prime}(ka)}\right\}\exp(-jm\phi)\right]$$

$$= \frac{2j}{\pi ka}\hat{\boldsymbol{\varphi}}\ \frac{E_0}{\eta}\left[\sum_{m=-\infty}^{\infty}(-j)^m\frac{\exp(-jm\phi)}{H_m^{(2)\prime}(ka)}\right]$$

For the z-polarized field:

$$\boldsymbol{H}_s^{TM}(\boldsymbol{r}) = -\frac{j}{\eta}\sum_{m=-\infty}^{\infty}v_m\frac{J_m(k_{i,\rho}a)}{H_m^{(2)}(k_{i,\rho}a)}\boldsymbol{M}_{m,k_{i,z}}^H(\rho,\phi,\mathrm{z})$$

$$v_m = -\left(\boldsymbol{E}_i^0\cdot\hat{\boldsymbol{v}}_i\right)\frac{k}{k_{i,\rho}}(-j)^m\exp(jm\phi_i)$$

$$\rightarrow \boldsymbol{H}_s^{TM}(\boldsymbol{r}) = -\frac{jE_0}{\eta}\sum_{m=-\infty}^{\infty}(-j)^m\frac{J_m(ka)}{H_m^{(2)}(ka)}\boldsymbol{M}_{m,0}^H(\rho,\phi,\mathrm{z})$$

$$\boldsymbol{M}_{m,0}^H(\rho,\phi,\mathrm{z}) = -\exp(-jm\phi)\left[\hat{\rho}\ \frac{jm}{k\rho}H_m^{(2)}(k\rho) + \hat{\boldsymbol{\phi}}H_m^{(2)\prime}(k\rho)\right]$$

Incident magnetic field

$$\boldsymbol{H}_i = -\hat{\boldsymbol{y}}\frac{E_0}{\eta}\exp(-jkx)$$

$$\boldsymbol{J}_s^{TM} = \hat{\boldsymbol{\rho}}\times\left[\boldsymbol{H}_s^{TM}(a,\varphi) + \boldsymbol{H}_i(a,\varphi)\right] =$$

$$\frac{jE_0}{\eta}\hat{\boldsymbol{\rho}}\times\hat{\boldsymbol{\phi}}\sum_{m=-\infty}^{\infty}(-j)^m\frac{J_m(ka)H_m^{(2)\prime}(ka)}{H_m^{(2)}(ka)}\exp(-jm\phi) - \hat{\boldsymbol{\rho}}\times\hat{\boldsymbol{y}}\frac{E_0}{\eta}\exp(-jkx) =$$

$$\frac{jE_0}{\eta}\hat{\boldsymbol{z}}\sum_{m=-\infty}^{\infty}(-j)^m\frac{J_m(ka)H_m^{(2)\prime}(ka)}{H_m^{(2)}(ka)}\exp(-jm\phi) - \hat{\boldsymbol{z}}\cos\varphi\frac{E_0}{\eta}\exp(-jka\cos\varphi)$$

$$\cos\varphi\exp(-jka\cos\varphi) = -\frac{1}{jk}\frac{d}{da}\exp(-jka\cos\varphi)$$

$$= -\frac{1}{jk}\frac{d}{da}\sum_{m=-\infty}^{\infty}(-j)^m J_m(ka)\exp(-jm\phi) = j\sum_{m=-\infty}^{\infty}(-j)^m J'_m(ka)\exp(-jm\phi)$$

$$\rightarrow \boldsymbol{J}_s^{TM} = \frac{jE_0}{\eta}\hat{\boldsymbol{z}}\sum_{m=-\infty}^{\infty}(-j)^m\left[\frac{J_m(ka)H_m^{(2)\prime}(ka)}{H_m^{(2)}(ka)} - J'_m(ka)\right]\exp(-jm\phi)$$

$$= -\frac{2j}{\pi ka}\frac{jE_0}{\eta}\hat{\boldsymbol{z}}\sum_{m=-\infty}^{\infty}(-j)^m\frac{\exp(-jm\phi)}{H_m^{(2)}(ka)}$$

Problem 2:

Consider a region in vacuum confined by two infinite, perfectly conducting plates at $z = 0$ and $z = b$. We would like to use cylindrical coordinates (ρ, ϕ, z) to analyze a scattering problem in this configuration.

(i) Find the general solution of the vector-wave equation for the electric field in terms of the $\boldsymbol{M}$ and $\boldsymbol{N}$ functions in cylindrical coordinates. Use $\hat{\boldsymbol{z}}$ for the constant vector field needed to define these functions. Explicitly express $\boldsymbol{M}$ and $\boldsymbol{N}$ in terms of ρ, ϕ, z.

Consider now a perfectly conducting cylinder of radius a and height b put in between the two plates. The axis of the cylinder coincides with the z-axis. An incident wave whose fields are given by

$$\boldsymbol{E}_i = \hat{\boldsymbol{y}} E_0 \sin\left(\frac{\pi z}{b}\right) \exp(-j\beta x), \ \beta = \sqrt{k_0^2 - \left(\frac{\pi}{b}\right)^2},$$

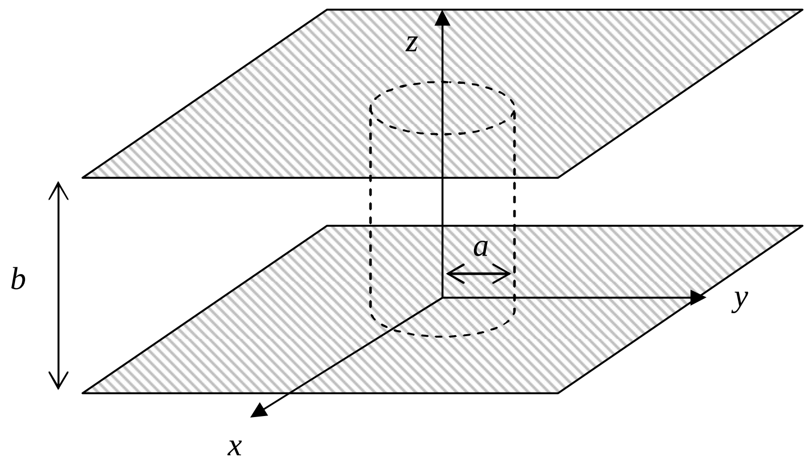
$$\boldsymbol{H}_i = \frac{E_0}{j\omega\mu_0}\left[\hat{\boldsymbol{x}}\frac{\pi}{b}\cos\left(\frac{\pi z}{b}\right) + \hat{\boldsymbol{z}} j\beta \sin\left(\frac{\pi z}{b}\right)\right]\exp(-j\beta x),$$

travels along x with the wave number β and hits the cylinder ($k_0 > \pi / b$).

(ii) Write down the scattered electric field as a series with unknown coefficients and find these coefficients by imposing appropriate boundary conditions on the surface of the cylinder.

Hint: if needed you may use

$$\exp(-j\beta x) = \sum_{m=-\infty}^{\infty} (-j)^m J_m(\beta\rho) \exp(-jm\phi)$$



Solution

Consider a region in vacuum confined by two infinite, perfectly conducting plates at $z=0$ and $z=b$. We would like to use cylindrical coordinates (ρ,ϕ,z) to analyze a scattering problem in this configuration.

(i) Find the general solution of the vector-wave equation for the electric field in terms of the $\boldsymbol{M}$ and $\boldsymbol{N}$ functions in cylindrical coordinates. Use $\hat{z}$ for the constant vector field needed to define these functions. Explicitly express $\boldsymbol{M}$ and $\boldsymbol{N}$ in terms of ρ,ϕ,z.

Here we would like to find the general solutions of the vector wave equation which satisfy the boundary conditions on the plates. First consider the scalar wave equation in cylindrical coordinates

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0$$

The general solution of this equation may be written as

$$\psi(\rho,\phi,z)=f\left(\rho\right)g\left(z\right)\exp\left(-jm\phi\right)$$
$$f\left(\rho\right)=AJ_m\left(k_\rho\rho\right)+BY_m\left(k_\rho\rho\right)=\bar{A}H_m^{(1)}\left(k_\rho\rho\right)+\bar{B}H_m^{(2)}\left(k_\rho\rho\right)$$
$$g\left(z\right)=C\sin\left(k_z z\right)+D\cos\left(k_z z\right)$$
$$k_\rho^2+k_z^2=k^2$$

The corresponding $\boldsymbol{M}$ and $\boldsymbol{N}$ functions are

$$\boldsymbol{M} = \frac{1}{k}\nabla\psi\times\hat{\boldsymbol{z}} = -\hat{\boldsymbol{\rho}}\ \frac{jm}{k\rho}\psi - \hat{\boldsymbol{\phi}}\ \frac{\partial\psi}{k\partial\rho}$$

$$\boldsymbol{N} = \frac{1}{k^2}\nabla\times\left[\nabla\psi\times\hat{\boldsymbol{z}}\right] = \frac{1}{k^2}\left[k^2\psi\hat{\boldsymbol{z}} + \frac{\partial}{\partial z}\nabla\psi\right] =$$

$$\left[\psi\hat{\boldsymbol{z}} + \frac{1}{k^2}\frac{\partial}{\partial z}\left(\hat{\boldsymbol{\rho}}\frac{\partial\psi}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{\partial\psi}{\rho\partial\phi} + \hat{\boldsymbol{z}}\frac{\partial\psi}{\partial z}\right)\right] =$$

$$\left[\left(\psi + \frac{1}{k^2}\frac{\partial^2\psi}{\partial z^2}\right)\hat{\boldsymbol{z}} + \frac{1}{k^2}\left(\frac{\partial}{\partial z}\frac{\partial\psi}{\partial\rho}\right)\hat{\boldsymbol{\rho}} + \frac{1}{k^2\rho}\left(\frac{\partial}{\partial z}\frac{\partial\psi}{\partial\phi}\right)\hat{\boldsymbol{\phi}}\right]$$

Since the tangential components of $\boldsymbol{M}$ and $\boldsymbol{N}$ should vanish on the plates (in order to be a good solution for the electric field) we must have for $\boldsymbol{M}$:

$$\psi(\rho,\phi,0) = \psi(\rho,\phi,b) = 0 \rightarrow g\left(0\right) = g\left(b\right) = 0 \rightarrow$$

$$k_z = \frac{n\pi}{b}, g\left(z\right) = C\sin\left(\frac{n\pi z}{b}\right), n = 1, 2, \cdots$$

For $\boldsymbol{N}$, however,

$$\frac{\partial}{\partial z}\psi(\rho,\phi,0) = \frac{\partial}{\partial z}\psi(\rho,\phi,b) = 0 \rightarrow g'\left(0\right) = g'\left(b\right) = 0 \rightarrow$$

$$k_z = \frac{n\pi}{b}, g\left(z\right) = D\cos\left(\frac{n\pi z}{b}\right), n = 0, 1, 2, \ldots$$

The choice of $f\left(\rho\right)$ depends on the problem.

Consider now a perfectly conducting cylinder of radius a and height b put in between the two plates.

The axis of the cylinder coincides with the z-axis. An incident wave whose fields are given by

$$\boldsymbol{E}_i = \hat{\boldsymbol{y}}E_0\sin\left(\frac{\pi z}{b}\right)\exp\left(-j\beta x\right),\ \beta = \sqrt{k_0^2 - \left(\frac{\pi}{b}\right)^2},$$

$$\boldsymbol{H}_i = \frac{E_0}{j\omega\mu_0}\left[\hat{\boldsymbol{x}}\frac{\pi}{b}\cos\left(\frac{\pi z}{b}\right) + \hat{\boldsymbol{z}}j\beta\sin\left(\frac{\pi z}{b}\right)\right]\exp\left(-j\beta x\right),$$

travels along x with the wave number β and hits the cylinder ($k_0 > \pi / b$).

(ii) Write down the scattered electric field as a series with unknown coefficients and find these coefficients by imposing appropriate boundary conditions on the surface of the cylinder.

The scattered electric field should be composed of series of $\boldsymbol{M}$ and $\boldsymbol{N}$ functions but with $f(\rho) = H_m^{(2)}(k_\rho \rho)$ to represent outgoing waves between the plates. Hence we have

$$\boldsymbol{M}_{m,n} = -\hat{\boldsymbol{\rho}}\ \frac{jm}{k\rho}\psi - \hat{\boldsymbol{\phi}}\ \frac{\partial \psi}{k\partial \rho} =$$

$$-\hat{\boldsymbol{\rho}}\ \frac{jm}{k\rho}\sin\left(\frac{n\pi z}{b}\right)H_m^{(2)}(k_{\rho n}\rho)\exp(-jm\phi)$$

$$-\hat{\boldsymbol{\phi}}\ \frac{k_{\rho n}}{k}\sin\left(\frac{n\pi z}{b}\right)H_m^{(2)\prime}(k_{\rho n}\rho)\exp(-jm\phi)$$

$$k_{\rho n}^2 = k^2 - \left(\frac{n\pi}{b}\right)^2$$

$$\boldsymbol{N}_{m,n} = \left[\left(\psi + \frac{1}{k^2}\frac{\partial^2 \psi}{\partial z^2}\right)\hat{\boldsymbol{z}} + \frac{1}{k^2}\left(\frac{\partial}{\partial z}\frac{\partial \psi}{\partial \rho}\right)\hat{\boldsymbol{\rho}} + \frac{1}{k^2 \rho}\left(\frac{\partial}{\partial z}\frac{\partial \psi}{\partial \phi}\right)\hat{\boldsymbol{\phi}}\right]$$

$$= -\left(1 - \left(\frac{n\pi}{kb}\right)^2\right)\cos\left(\frac{n\pi z}{b}\right)H_m^{(2)}(k_{\rho n}\rho)\exp(-jm\phi)\,\hat{\boldsymbol{z}}$$

$$-\frac{n\pi}{b}\frac{k_{\rho n}}{k^2}\sin\left(\frac{n\pi z}{b}\right)H_m^{(2)\prime}(k_{\rho n}\rho)\exp(-jm\phi)\,\hat{\boldsymbol{\rho}}$$

$$+\frac{jm}{k^2\rho}\frac{n\pi}{b}\sin\left(\frac{n\pi z}{b}\right)H_m^{(2)}(k_{\rho n}\rho)\exp(-jm\phi)\,\hat{\boldsymbol{\phi}}$$

The scattered field may now be written as

$$\boldsymbol{E}_s(\rho,\phi,z) = \sum_{m,n} a_{m,n}\boldsymbol{M}_{m,n}(\rho,\phi,z) + b_{m,n}\boldsymbol{N}_{m,n}(\rho,\phi,z)$$

On the surface of the cylinder ($\rho = a$) one must have

$$(\boldsymbol{E}_i + \boldsymbol{E}_s)\cdot\hat{\boldsymbol{\varphi}} = (\boldsymbol{E}_i + \boldsymbol{E}_s)\cdot\hat{\boldsymbol{z}} = 0$$

But the incident field has no z-component. Therefore, $\boldsymbol{N}_{m,n}$ does not play a role and $b_{m,n}$=0. Hence we have to consider the equation

$$\left[\boldsymbol{E}_s\left(a,\phi,z\right)+\boldsymbol{E}_i\left(a,\phi,z\right)\right]\cdot\hat{\boldsymbol{\varphi}}=0\rightarrow\sum_{m,n}a_{m,n}\boldsymbol{M}_{m,n}\left(\rho,\phi,z\right)\cdot\hat{\boldsymbol{\varphi}}=-\boldsymbol{E}_i\left(a,\phi,z\right)\cdot\hat{\boldsymbol{\varphi}}$$

Since the functions $\sin\left(\frac{n\pi z}{b}\right)$ form a complete orthogonal set on $0<z<b$, from the expression for the incident field it follows that we only need $n=1$. The equation is now reduced to

$$\sum_{m=-\infty}^{\infty}a_{m,1}\boldsymbol{M}_{m,1}\left(\rho,\phi,z\right)\cdot\hat{\boldsymbol{\varphi}}=-\boldsymbol{E}_i\left(a,\phi,z\right)\cdot\hat{\boldsymbol{\varphi}}\rightarrow$$
$$\sum_{m=-\infty}^{\infty}a_{m,1}\frac{k_{\rho n}}{k}H_m^{(2)\prime}\left(k_{\rho n}a\right)\exp\left(-jm\phi\right)=\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{\varphi}}E_0\exp\left(-j\beta x\right)$$

Take note that $\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{\varphi}}=\cos\phi$. Using the expansion of $\exp\left(-j\beta x\right)$:

$$\sum_{m=-\infty}^{\infty}a_{m,1}\frac{k_{\rho n}}{k}H_m^{(2)\prime}\left(k_{\rho n}a\right)\exp\left(-jm\phi\right)=\cos\phi E_0\sum_{m=-\infty}^{\infty}(-j)^m J_m(\beta a)\exp\left(-jm\phi\right)\rightarrow$$
$$a_{m,1}=\frac{kE_0}{k_{\rho n}H_m^{(2)\prime}\left(k_{\rho n}a\right)}\frac{1}{2\pi}\int_0^{2\pi}\exp\left(jm\phi\right)\sum_{l=-\infty}^{\infty}(-j)^l J_l(\beta a)\cos\phi\exp\left(-jl\phi\right)$$
$$=\frac{kE_0}{k_{\rho n}H_m^{(2)\prime}\left(k_{\rho n}a\right)}\frac{1}{2}\left[(-j)^{m+1}J_{m+1}(\beta a)+(-j)^{m-1}J_{m-1}(\beta a)\right]$$