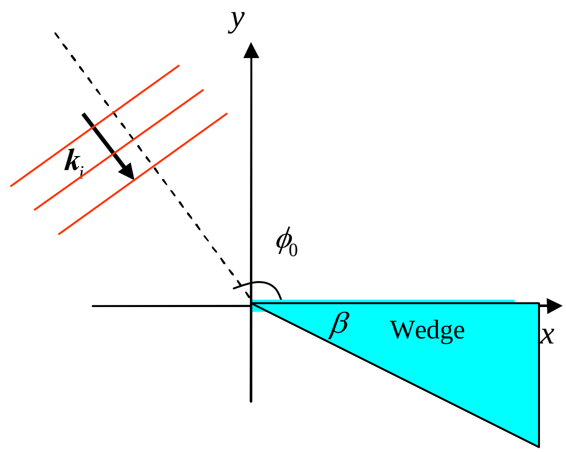
EM Scattering

Homework assignment 3

Problem 1:

A uniform TM^z plane wave (electric field along the z-direction) is normally incident on a perfectly conducting wedge with $\beta = 30\deg$ at an angle ϕ_0. The amplitude of the incident wave is E_0. Calculate the current density on the top and bottom surfaces of the half plane. Plus the current density on these two surfaces for $\phi_0 = 45, 90, 180\deg$. The dielectric constant and permeability of the surrounding medium are ε_0, μ_0.



Solution

The surface current density equals $\hat{\boldsymbol{n}} \times \boldsymbol{H}$. Note that $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{\phi}}$ on the top and bottom surfaces, respectively ($\phi = 0, 2\pi - \beta$). In the TM^z case the magnetic field has no z-component. The only component contributing to the surface current is H_ρ. Thus:

$$H_\rho^{TM}\left(\rho,\phi\right) = -\frac{1}{j\omega\mu}\frac{\partial E_z}{\rho\partial\phi} = -\frac{A_0}{j\omega\mu}\frac{4\pi}{2\pi-\beta}\sum_{m=1}^{\infty}\left(j\right)^{\nu_m}\frac{\nu_m}{\rho}\sin\left(\nu_m\phi_0\right)J_{\nu_m}(k\rho)\cos\left(\nu_m\phi\right) \quad (1.1)$$

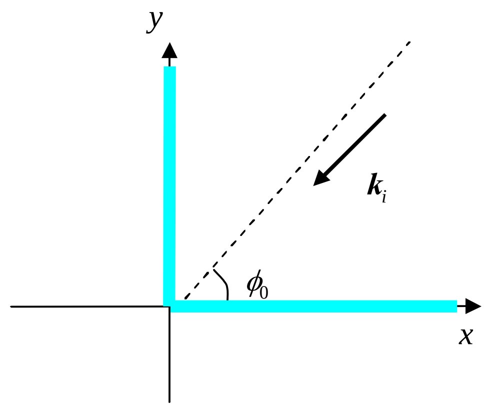
$$\phi = 0 \rightarrow \hat{\boldsymbol{n}}\times\boldsymbol{H} = -H_\rho\left(\rho,\phi\right)\hat{\boldsymbol{z}} = \frac{\hat{\boldsymbol{z}}}{\rho}\frac{A_0}{j\omega\mu}\frac{4\pi}{2\pi-\beta}\sum_{m=1}^{\infty}\left(j\right)^{\nu_m}\nu_m\sin\left(\nu_m\phi_0\right)J_{\nu_m}(k\rho) \quad (1.2)$$

$$\phi = 2\pi-\beta \rightarrow \hat{\boldsymbol{n}}\times\boldsymbol{H} = H_\rho\left(\rho,2\pi-\beta\right)\hat{\boldsymbol{z}} = -\hat{\boldsymbol{z}}\frac{A_0}{j\omega\mu}\frac{4\pi}{2\pi-\beta}\sum_{m=1}^{\infty}\left(j\right)^{\nu_m}\frac{\nu_m}{\rho}\sin\left(\nu_m\phi_0\right)J_{\nu_m}(k\rho)\cos\left(m\pi\right)$$
$$= -\frac{\hat{\boldsymbol{z}}}{\rho}\frac{A_0}{j\omega\mu}\frac{4\pi}{2\pi-\beta}\sum_{m=1}^{\infty}(-1)^m\left(j\right)^{\nu_m}\nu_m\sin\left(\nu_m\phi_0\right)J_{\nu_m}(k\rho)$$

(1.3)

Problem 2:

A uniform TM^z plane wave (electric field along the z-direction) is normally incident on a perfectly conducting right-angled corner at an angle $\phi_0 < 90\deg$. The amplitude of the incident wave is E_0. Calculate the scattered field.



Solution

The total field is

$$E_z^{TM}(\rho,\phi) = A_0 \frac{4\pi}{2\pi-\beta} \sum_{m=1}^{\infty} (j)^{\nu_m} J_{\nu_m}(k\rho) \sin(\nu_m \phi) \sin(\nu_m \phi_0) \tag{1.4}$$

where $\beta = 3\pi/2, \nu_m = 2m$. Thus

$$\begin{aligned} E_z^{TM}(\rho,\phi) &= 8A_0 \sum_{m=1}^{\infty} (-1)^m J_{2m}(k\rho) \sin(2m\phi) \sin(2m\phi_0) \\ &= 4A_0 \sum_{m=1}^{\infty} (-1)^m J_{2m}(k\rho) \left\{ \cos\left[2m(\phi-\phi_0)\right] - \cos\left[2m(\phi+\phi_0)\right] \right\} \end{aligned} \tag{1.5}$$

Now, let us consider the relationship

$$\exp(jz\cos\vartheta) = J_0(z) + 2\sum_{m=1}^{\infty} (j)^m \cos(m\vartheta) J_m(z) \tag{1.6}$$

From which is follows that

$$\exp\left[jz\cos(\pi-\vartheta)\right] = \exp\left[-jz\cos\vartheta\right] = J_0(z) + 2\sum_{m=1}^{\infty} (-j)^m \cos(m\vartheta) J_m(z) \tag{1.7}$$

Adding up the two expressions yields

$$\exp(jz\cos\vartheta) + \exp\left[jz\cos(\pi-\vartheta)\right] = 2J_0(z) + 4\sum_{m=1}^{\infty} (j)^{2m} \cos(2m\vartheta) J_{2m}(z) \tag{1.8}$$

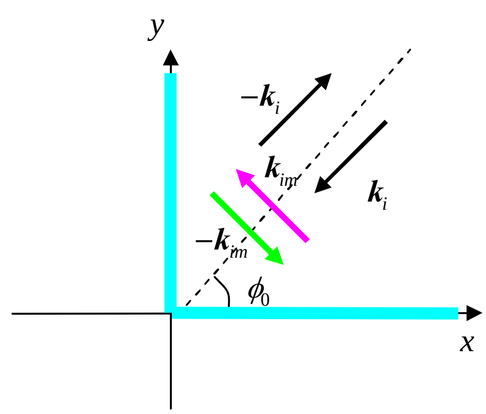
Hence:

$$4\sum_{m=1}^{\infty}(-1)^m J_{2m}(k\rho)\cos\left[2m\left(\phi\mp\phi_0\right)\right]= \\ \exp\left(jz\cos\left(\phi\mp\phi_0\right)\right)+\exp\left[jz\cos\left(\pi-\left(\phi\mp\phi_0\right)\right)\right]-2J_0(k\rho) \tag{1.9}$$

And

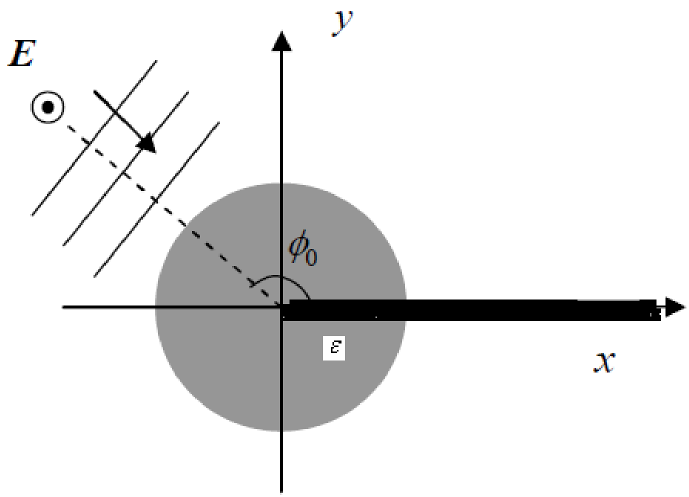
$$E_z^{TM}(\rho,\phi)=A_0\exp\left(jk\rho\cos\left(\phi-\phi_0\right)\right)+A_0\exp\left[jk\rho\cos\left(\pi-\left(\phi-\phi_0\right)\right)\right] \\ -A_0\exp\left(jk\rho\cos\left(\phi+\phi_0\right)\right)-A_0\exp\left[jk\rho\cos\left(\pi-\left(\phi+\phi_0\right)\right)\right]= \\ A_0\exp\left(-j\boldsymbol{k}_i\cdot\boldsymbol{\rho}\right)+A_0\exp\left(j\boldsymbol{k}_i\cdot\boldsymbol{\rho}\right) \\ -A_0\exp\left(-j\boldsymbol{k}_{im}\cdot\boldsymbol{\rho}\right)-A_0\exp\left(j\boldsymbol{k}_{im}\cdot\boldsymbol{\rho}\right) \tag{1.10}$$

See figure below for the definition of $\boldsymbol{k}_{im}$ **.** This is a combination of four plane waves. The same result could have been obtained using the method of images.



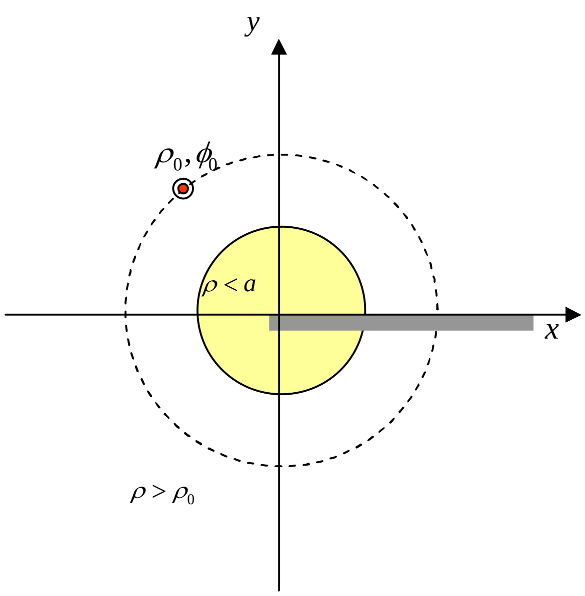
Problem 3:

Consider an infinitely long, dielectric circular cylinder whose axis coincides with the *z*-axis and has a relative dielectric constant of ε_d , and a radius of a . A perfectly conducting half infinite-plane protrudes the cylinder up to its axis as shown in the figure. A plane wave with the wave vector $k_i=(k_{ix},k_{iy},0)$ is normally incident on the cylinder. The electric field of the incident wave is polarized along the z-direction. Compute the scattered field.



<u>**Solution**</u>

We expand the solution inside and outside the cylinder and in each case take account of the boundary conditions on the half-plane. There are three regions given by $\rho < a$, $\rho_0 > \rho > a$, and $\rho > \rho_0$ with ρ_0 the distance of the source to the z-axis.



Inside the insulating cylinder we expand the electric field as

$$E_z^{TM}(\rho,\phi) = \sum_{m=1} a_m J_{\nu_m}(k_d \rho) \sin(\nu_m \phi) \tag{1.11}$$

Where $\nu_m = m/2$ and $k_d = k_0 \sqrt{\varepsilon_d}$. In the region $\rho_0 > \rho > a$ we have

$$E_z^{TM}(\rho,\phi)=\sum_{m=1}\left[c_m J_{\nu_m}(k_0\rho)+d_m Y_{\nu_m}(k_0\rho)\right]\sin(\nu_m\phi) \tag{1.12}$$

And, finally, for $\rho > \rho_0$ we have

$$E_z^{TM}(\rho,\phi)=\sum_{m=1} b_m H_{\nu_m}^{(2)}(k_0\rho)\sin(\nu_m\phi) \tag{1.13}$$

The corresponding magnetic fields are

$$H_\varphi^{TM}(\rho,\phi)=\frac{1}{j\omega\mu_0}\begin{cases} k_d\sum_{m=1} a_m J'_{\nu_m}(k_d\rho)\sin(\nu_m\phi) & \rho < a \\ k_0\sum_{m=1}\left[c_m J'_{\nu_m}(k_0\rho)+d_m Y'_{\nu_m}(k_0\rho)\right]\sin(\nu_m\phi) & a<\rho<\rho_0 \\ k_0\sum_{m=1} b_m H_{\nu_m}^{(2)\prime}(k_0\rho)\sin(\nu_m\phi) & \rho>\rho_0 \end{cases} \tag{1.14}$$

At $\rho = a$ the tangential magnetic and electric fields should be continuous:

$$\begin{aligned}
&\sum_{m=1} a_m J_{\nu_m}(k_d a)\sin(\nu_m\phi)=\sum_{m=1}\left[c_m J_{\nu_m}(k_0 a)+d_m Y_{\nu_m}(k_0 a)\right]\sin(\nu_m\phi)\\
&k_d\sum_{m=1} a_m J'_{\nu_m}(k_d a)\sin(\nu_m\phi)=k_0\sum_{m=1}\left[c_m J'_{\nu_m}(k_0 a)+d_m Y'_{\nu_m}(k_0 a)\right]\sin(\nu_m\phi)\\
&\rightarrow\\
&a_m J_{\nu_m}(k_d a)=c_m J_{\nu_m}(k_0 a)+d_m Y_{\nu_m}(k_0 a)\\
&k_d a_m J'_{\nu_m}(k_d a)=k_0\left[c_m J'_{\nu_m}(k_0 a)+d_m Y'_{\nu_m}(k_0 a)\right]
\end{aligned} \tag{1.15}$$

As a result:

$$\begin{aligned}
&a_m J_{\nu_m}(k_d a)=c_m J_{\nu_m}(k_0 a)+d_m Y_{\nu_m}(k_0 a)\\
&k_d a_m J'_{\nu_m}(k_d a)=k_0\left[c_m J'_{\nu_m}(k_0 a)+d_m Y'_{\nu_m}(k_0 a)\right]
\end{aligned} \tag{1.16}$$

From which we have

$$\frac{c_m}{d_m}=-\frac{J_{\nu_m}(k_d a)Y'_{\nu_m}(k_0 a)-\sqrt{\varepsilon_d}\,J'_{\nu_m}(k_d a)Y_{\nu_m}(k_0 a)}{J_{\nu_m}(k_d a)J'_{\nu_m}(k_0 a)-\sqrt{\varepsilon_d}\,J'_{\nu_m}(k_d a)J_{\nu_m}(k_0 a)} \tag{1.17}$$

Where we have used

$$J_{\nu_m}(z)Y'_{\nu_m}(z)-J'_{\nu_m}(z)Y_{\nu_m}(z)=\frac{2}{\pi z} \tag{1.18}$$

At the source the electric field remains continuous while the tangential magnetic field undergoes a jump:

$$\begin{aligned} E_z^{TM}\left(\rho_0^+,\phi\right) &= E_z^{TM}\left(\rho_0^-,\phi\right) \\ H_\phi^{TM}\left(\rho_0^+,\phi\right) - H_\phi^{TM}\left(\rho_0^-,\phi\right) &= \frac{I_0}{\rho_0}\delta\left(\phi-\phi_0\right) \end{aligned} \tag{1.19}$$

This yields

$$\begin{aligned} &\sum_{m=1}\left[c_m J_{\nu_m}(k_0\rho_0) + d_m Y_{\nu_m}(k_0\rho_0)\right]\sin\left(\nu_m\phi\right) = \sum_{m=1} b_m H_{\nu_m}^{(2)}(k_0\rho_0)\sin\left(\nu_m\phi\right) \\ &\sum_{m=1} b_m H_{\nu_m}^{(2)\prime}(k_0\rho_0)\sin\left(\nu_m\phi\right) = \sum_{m=1}\left[c_m J_{\nu_m}'(k_0\rho_0) + d_m Y_{\nu_m}'(k_0\rho_0)\right]\sin\left(\nu_m\phi\right) + \frac{j\omega\mu_0 I_0}{k_0\rho_0}\delta\left(\phi-\phi_0\right) \end{aligned} \tag{1.20}$$

So that

$$\begin{aligned} &c_m J_{\nu_m}(k_0\rho_0) + d_m Y_{\nu_m}(k_0\rho_0) = b_m H_{\nu_m}^{(2)}(k_0\rho_0) \\ &c_m J_{\nu_m}'(k_0\rho_0) + d_m Y_{\nu_m}'(k_0\rho_0) = b_m H_{\nu_m}^{(2)\prime}(k_0\rho_0) - \frac{j\omega\mu_0 I_0}{k_0\rho_0}\frac{2}{2\pi}\sin\left(\nu_m\phi_0\right) \end{aligned} \tag{1.21}$$

From which it follows that

$$\begin{aligned} &\left[c_m J_{\nu_m}'(k_0\rho_0) + d_m Y_{\nu_m}'(k_0\rho_0)\right] H_{\nu_m}^{(2)}(k_0\rho_0) - \\ &\left[c_m J_{\nu_m}(k_0\rho_0) + d_m Y_{\nu_m}(k_0\rho_0)\right] H_{\nu_m}^{(2)\prime}(k_0\rho_0) = -H_{\nu_m}^{(2)}(k_0\rho_0)\frac{j\omega\mu_0 I_0}{k_0\rho_0}\frac{2}{2\pi}\sin\left(\nu_m\phi_0\right) \\ &\rightarrow \frac{2j}{\pi k_0\rho_0}c_m + \frac{2}{\pi k_0\rho_0}d_m = -H_{\nu_m}^{(2)}(k_0\rho_0)\frac{j\omega\mu_0 I_0}{k_0\rho_0}\frac{2}{2\pi}\sin\left(\nu_m\phi_0\right) \\ &\rightarrow d_m + jc_m = -\frac{j\omega\mu_0 I_0}{2}H_{\nu_m}^{(2)}(k_0\rho_0)\sin\left(\nu_m\phi_0\right) \end{aligned} \tag{1.22}$$

$$\begin{aligned} &\frac{c_m}{d_m} = -\frac{J_{\nu_m}(k_d a)Y_{\nu_m}'(k_0 a) - \sqrt{\varepsilon_d}J_{\nu_m}'(k_d a)Y_{\nu_m}(k_0 a)}{J_{\nu_m}(k_d a)J_{\nu_m}'(k_0 a) - \sqrt{\varepsilon_d}J_{\nu_m}'(k_d a)J_{\nu_m}(k_0 a)} \rightarrow \\ &d_m\left[1 - j\frac{J_{\nu_m}(k_d a)Y_{\nu_m}'(k_0 a) - \sqrt{\varepsilon_d}J_{\nu_m}'(k_d a)Y_{\nu_m}(k_0 a)}{J_{\nu_m}(k_d a)J_{\nu_m}'(k_0 a) - \sqrt{\varepsilon_d}J_{\nu_m}'(k_d a)J_{\nu_m}(k_0 a)}\right] = -\frac{j\omega\mu_0 I_0}{2}H_{\nu_m}^{(2)}(k_0\rho_0)\sin\left(\nu_m\phi_0\right) \\ &\rightarrow d_m\left[\frac{J_{\nu_m}(k_d a)H_{\nu_m}^{(2)\prime}(k_0 a) - \sqrt{\varepsilon_d}J_{\nu_m}'(k_d a)H_{\nu_m}^{(2)}(k_0 a)}{J_{\nu_m}(k_d a)J_{\nu_m}'(k_0 a) - \sqrt{\varepsilon_d}J_{\nu_m}'(k_d a)J_{\nu_m}(k_0 a)}\right] = -\frac{j\omega\mu_0 I_0}{2}H_{\nu_m}^{(2)}(k_0\rho_0)\sin\left(\nu_m\phi_0\right) \quad (1.23) \\ &\rightarrow d_m = -\frac{j\omega\mu_0 I_0}{2}\frac{J_{\nu_m}(k_d a)J_{\nu_m}'(k_0 a) - \sqrt{\varepsilon_d}J_{\nu_m}'(k_d a)J_{\nu_m}(k_0 a)}{J_{\nu_m}(k_d a)H_{\nu_m}^{(2)\prime}(k_0 a) - \sqrt{\varepsilon_d}J_{\nu_m}'(k_d a)H_{\nu_m}^{(2)}(k_0 a)}H_{\nu_m}^{(2)}(k_0\rho_0)\sin\left(\nu_m\phi_0\right) \\ &c_m = \frac{j\omega\mu_0 I_0}{2}\frac{J_{\nu_m}(k_d a)Y_{\nu_m}'(k_0 a) - \sqrt{\varepsilon_d}J_{\nu_m}'(k_d a)Y_{\nu_m}(k_0 a)}{J_{\nu_m}(k_d a)H_{\nu_m}^{(2)\prime}(k_0 a) - \sqrt{\varepsilon_d}J_{\nu_m}'(k_d a)H_{\nu_m}^{(2)}(k_0 a)}H_{\nu_m}^{(2)}(k_0\rho_0)\sin\left(\nu_m\phi_0\right) \end{aligned}$$

If we let ρ_0 go to infinity but let

$$A_0 = -\frac{\omega\mu I_0}{4}\sqrt{\frac{2}{\pi k\rho_0}}\exp\left(-jk\rho_0 + j\pi/4\right) \tag{1.24}$$

Remain a constant then

$$d_m \to -\frac{j\omega\mu_0 I_0}{2}\frac{J_{\nu_m}(k_d a)J'_{\nu_m}(k_0 a) - \sqrt{\varepsilon_d}J'_{\nu_m}(k_d a)J_{\nu_m}(k_0 a)}{J_{\nu_m}(k_d a)H^{(2)\prime}_{\nu_m}(k_0 a) - \sqrt{\varepsilon_d}J'_{\nu_m}(k_d a)H^{(2)}_{\nu_m}(k_0 a)} j^{\nu_m}\sqrt{\frac{2}{\pi k\rho_0}}\exp\left[-\left(jk\rho_0 - j\pi/4\right)\right]\sin\left(\nu_m\phi_0\right)$$

$$= 2jA_0\frac{J_{\nu_m}(k_d a)J'_{\nu_m}(k_0 a) - \sqrt{\varepsilon_d}J'_{\nu_m}(k_d a)J_{\nu_m}(k_0 a)}{J_{\nu_m}(k_d a)H^{(2)\prime}_{\nu_m}(k_0 a) - \sqrt{\varepsilon_d}J'_{\nu_m}(k_d a)H^{(2)}_{\nu_m}(k_0 a)} j^{\nu_m}\sin\left(\nu_m\phi_0\right)$$

$$c_m \to -2jA_0\frac{J_{\nu_m}(k_d a)Y'_{\nu_m}(k_0 a) - \sqrt{\varepsilon_d}J'_{\nu_m}(k_d a)Y_{\nu_m}(k_0 a)}{J_{\nu_m}(k_d a)H^{(2)\prime}_{\nu_m}(k_0 a) - \sqrt{\varepsilon_d}J'_{\nu_m}(k_d a)H^{(2)}_{\nu_m}(k_0 a)} j^{\nu_m}\sin\left(\nu_m\phi_0\right)$$

(1.25)

The field outside the cylinder:

$$E_z^{TM}\left(\rho,\phi\right) = \sum_{m=1}\left[c_m J_{\nu_m}(k_0\rho) + d_m Y_{\nu_m}(k_0\rho)\right]\sin\left(\nu_m\phi\right) \tag{1.26}$$