Problem 1:

A uniform TM\(^2\) plane wave (electric field along the z-direction) is normally incident on a perfectly conducting wedge with the angle \(\beta = 300^\circ\) at an angle \(\phi_0 < 60\,\text{deg.}\). The amplitude of the incident wave is \(E_0\). Calculate the scattered field and show that it corresponds to the solution found from the method of images.

Solution

The total field in the far zone is

\[
E_{z}^{TM}(\rho, \phi) = A_0 \frac{4\pi}{2\pi - \beta} \sum_{m=1}^{\infty} (j)^{3m} J_{3m}(k\rho) \sin(v_m\phi) \sin(v_m\phi_0) \tag{1.1}
\]

where \(\beta = 5\pi / 3, v_m = 3m\). Thus

\[
E_{z}^{TM}(\rho, \phi) = 6A_0 \sum_{m=1}^{\infty} (j)^{3m} J_{3m}(k\rho) \sin(3m\phi) \sin(3m\phi_0) \tag{1.2}
\]

\[
= 3A_0 \sum_{m=1}^{\infty} (j)^{3m} J_{3m}(k\rho) \left\{ \cos[3m(\phi - \phi_0)] - \cos[3m(\phi + \phi_0)] \right\}
\]

Now, let us consider the relationship

\[
\exp(jz \cos \theta) = J_0(z) + 2 \sum_{m=1}^{\infty} (j)^m \cos(m\theta) J_m(z) \tag{1.3}
\]

From which is follows that
\[
\exp(jz \cos \vartheta) = J_0(z) + 2 \sum_{m=1}^{\infty} (j)^m \cos(m \vartheta) J_m(z)
\]
\[
= J_0(z) + 2 \sum_{m=1}^{\infty} (j)^{3m} \cos(3m \vartheta) J_{3m}(z) + 2 \sum_{m=1}^{\infty} (j)^{3m-1} \cos((3m-1) \vartheta) J_{3m-1}(z) \quad (1.4)
\]
\[
+ 2 \sum_{m=1}^{\infty} (j)^{3m-2} \cos((3m-2) \vartheta) J_{3m-2}(z)
\]
\[
\exp\left[jz \cos\left(\vartheta + \frac{2\pi}{3}\right)\right] = J_0(z) + 2 \sum_{m=1}^{\infty} (-j)^m \cos\left(m \vartheta + \frac{2\pi m}{3}\right) J_m(z)
\]
\[
= J_0(z) + 2 \sum_{m=1}^{\infty} (-j)^{3m} \cos(3m \vartheta) J_{3m}(z) + 2 \sum_{m=1}^{\infty} (-j)^{3m-1} \cos((3m-1) \vartheta - \frac{2\pi}{3}) J_{3m-1}(z) \quad (1.5)
\]
\[
2 \sum_{m=1}^{\infty} (-j)^{3m-1} \cos((3m-2) \vartheta - \frac{4\pi}{3}) J_{3m-2}(z)
\]
\[
\exp\left[jz \cos\left(\vartheta - \frac{2\pi}{3}\right)\right] = J_0(z) + 2 \sum_{m=1}^{\infty} (-j)^m \cos\left(m \vartheta - \frac{2\pi m}{3}\right) J_m(z)
\]
\[
= J_0(z) + 2 \sum_{m=1}^{\infty} (-j)^{3m} \cos(3m \vartheta) J_{3m}(z) + 2 \sum_{m=1}^{\infty} (-j)^{3m-1} \cos((3m-1) \vartheta + \frac{2\pi}{3}) J_{3m-1}(z) \quad (1.6)
\]
\[
2 \sum_{m=1}^{\infty} (-j)^{3m-2} \cos((3m-2) \vartheta + \frac{4\pi}{3}) J_{3m-2}(z)
\]

Adding up the last two expressions yields
\[
\exp\left[jz \cos\left(\vartheta + \frac{2\pi}{3}\right)\right] + \exp\left[jz \cos\left(\vartheta - \frac{2\pi}{3}\right)\right]
\]
\[
= 2J_0(z) + 4 \sum_{m=1}^{\infty} (-j)^{3m} \cos(3m \vartheta) J_{3m}(z) + 4 \cos\frac{2\pi}{3} \sum_{m=1}^{\infty} (-j)^{3m-1} \cos((3m-1) \vartheta) J_{3m-1}(z) \quad (1.7)
\]
\[
+ 4 \cos\frac{2\pi}{3} \sum_{m=1}^{\infty} (-j)^{3m-2} \cos((3m-2) \vartheta) J_{3m-2}(z)
\]

so that
\[
-\exp\left[jz \cos\left(\vartheta + \frac{2\pi}{3}\right)\right] - \exp\left[jz \cos\left(\vartheta - \frac{2\pi}{3}\right)\right] + 2J_0(z) + 4 \sum_{m=1}^{\infty} (-j)^{3m} \cos(3m \vartheta) J_{3m}(z)
\]
\[
= 2 \sum_{m=1}^{\infty} (-j)^{3m-1} \cos((3m-1) \vartheta) J_{3m-1}(z) + 2 \sum_{m=1}^{\infty} (-j)^{3m-2} \cos((3m-2) \vartheta) J_{3m-2}(z) \quad (1.8)
\]

we thus have
\[ J_0(z) + 2 \sum_{m=1}^{\infty} (j)^{3m} \cos(3m\theta) J_{3m}(z) = \exp(jz \cos \theta) - 2 \sum_{m=1}^{\infty} (j)^{3m-1} \cos[(3m-1)\theta] J_{3m-1}(z) \]
\[ -2 \sum_{m=1}^{\infty} (j)^{3m-2} \cos[(3m-2)\theta] J_{3m-2}(z) \]
\[ = \exp(jz \cos \theta) - 2J_0(z) + \exp[jz \cos \left(\theta + \frac{2\pi}{3}\right)] + \exp[jz \cos \left(\theta - \frac{2\pi}{3}\right)] \] 
\[ -4 \sum_{m=1}^{\infty} (-j)^{3m} \cos(3m\theta) J_{3m}(z) \]

and
\[ J_0(z) + 2 \sum_{m=1}^{\infty} (j)^{3m} \cos(3m\theta) J_{3m}(z) = \]
\[ = \frac{1}{3} \exp(jz \cos \theta) + \frac{1}{3} \exp[jz \cos \left(\theta + \frac{2\pi}{3}\right)] + \frac{1}{3} \exp[jz \cos \left(\theta - \frac{2\pi}{3}\right)] \] 

As a result
\[ E_z^{TM} (\rho, \phi) = 3A_0 \sum_{m=1}^{\infty} (j)^{3m} J_{3m}(k \rho) \{ \cos[3m(\phi - \phi_0)] - \cos[3m(\phi + \phi_0)] \} \]
\[ = A_0 \exp[jk \rho \cos(\phi - \phi_0)] + A_0 \exp[jk \rho \cos(\phi - \phi_0 + \frac{2\pi}{3})] + A_0 \exp[jk \rho \cos(\phi - \phi_0 - \frac{2\pi}{3})] \]
\[ - A_0 \exp[jk \rho \cos(\phi + \phi_0)] - A_0 \exp[jk \rho \cos(\phi + \phi_0 + \frac{2\pi}{3})] - A_0 \exp[jk \rho \cos(\phi + \phi_0 - \frac{2\pi}{3})] \]
This is a combination of six plane waves which may be considered to be generated by a main source at infinity plus five images. Each image generates a wave whose wavevector represents an element of the solution. In reality, these waves are caused by multiple reflections from the wedge surface.

**Problem 2:**

By using a line of electric current $I = I_0 \exp(-j k_z z), k_z < k_0$, as the source, solve the problem of oblique TM$^z$ scattering from a perfectly conducting wedge where the incident magnetic field lies in the x-y plane and the incident wave vector is not normal to the wedge but has a component along $z$, i.e., $k_i = (k_{ix}, k_{iy}, k_{iz})$.

**Solution**

Due to the uniformity of the structure in the $z$-direction, we expect a source current of the type shown generates a solution whose dependence on $z$ is also of the type $\exp(-j k_z z)$. We thus return to the Maxwell equations in cylindrical coordinates and assume (like in solving the modes of a waveguide):

$$E = \hat{E}(\rho, \varphi) \exp(-j k_z z), \quad H = \hat{H}(\rho, \varphi) \exp(-j k_z z) \quad (1.12)$$
Like in a waveguide there will be TM and TE modes. We choose the TM problem since our aim is to solve the related scattering problem (also the type of source used enforces this choice). As a result

\[
\frac{1}{\rho} \frac{\partial \tilde{E}_\rho}{\partial \varphi} - \frac{\partial \tilde{E}_\varphi}{\partial \rho} = -j \omega \mu_0 H_\rho,
\]
\[
\frac{1}{\rho} \frac{\partial \tilde{E}_\varphi}{\partial \varphi} = -j \omega \mu_0 H_\varphi.
\]

\[
\frac{1}{\rho} \frac{\partial \tilde{H}_\rho}{\partial \varphi} - \frac{\partial \tilde{H}_\varphi}{\partial \rho} = -j k_z \tilde{E}_\rho - \frac{\partial \tilde{E}_\rho}{\partial \varphi} = -j \omega \mu_0 H_\rho,
\]
\[
\frac{1}{\rho} \frac{\partial \tilde{H}_\varphi}{\partial \varphi} - \frac{\partial \tilde{E}_\rho}{\partial \varphi} = -j k_z \tilde{H}_\rho - \frac{\partial \tilde{E}_\rho}{\partial \varphi} = -j \omega \mu_0 H_\varphi
\]

(1.13)

\[
\frac{1}{\rho} \frac{\partial \rho \tilde{E}_\rho}{\partial \varphi} - \frac{1}{\rho} \frac{\partial \rho \tilde{E}_\varphi}{\partial \varphi} = -j \omega \mu_0 H_\rho,
\]
\[
\frac{1}{\rho} \frac{\partial \rho \tilde{H}_\rho}{\partial \varphi} - \frac{1}{\rho} \frac{\partial \rho \tilde{H}_\varphi}{\partial \varphi} = -j \omega \mu_0 H_\varphi
\]

(1.14)

The rest is similar to the method used for the 2D problem. Assuming the line source to be located at \( \rho_0, \phi_0 \):

\[
\tilde{J}_\rho = \frac{I_0}{\rho_0} \delta (\rho - \rho_0) \delta (\varphi - \phi_0)
\]

(1.18)

we separately consider the regions \( \rho < \rho_0 \) and \( \rho > \rho_0 \). Given the boundary condition on the wedge surface we expand the field as
\[ \vec{E}_z = \begin{cases} \sum_{m=1}^{\infty} a_m \sin(v_m \phi) J_{v_m}(k_\rho \rho) & \rho < \rho_0 \\ \sum_{m=1}^{\infty} b_m \sin(v_m \phi) H^{(2)}_{v_m}(k_\rho \rho) & \rho > \rho_0 \end{cases} \tag{1.19} \]

Matching condition at the source: the electric field along z must be continuous while

\[ \left. \frac{\partial \vec{E}_z}{\partial \rho} \right|_{\rho_0} - \frac{\partial \vec{E}_z}{\partial \rho} \right|_{\rho_0} = -\frac{k_\rho^2}{j\omega \rho_0} I_0 \delta(\phi - \phi_0) = \frac{k_\rho^2 j\omega \mu I_0}{\rho_0} \delta(\phi - \phi_0) \tag{1.20} \]

Final result for the field inside the source circle:

\[ E_z^{TM}(\rho, \phi) = -\exp(-jk_z z) \frac{k_\rho^2 \pi \omega \mu I_0}{k_0^2 2\pi - \beta} \sum_{m=1}^{\infty} H^{(2)}_{v_m}(k_\rho \rho_0) \sin(v_m \phi_0) J_{v_m}(k_\rho \rho) \sin(v_m \phi) \]

\[ E_\phi = \frac{jk_z}{k_0^2 - k_z^2} \frac{k_\rho^2 \pi \omega \mu I_0}{k_0^2 2\pi - \beta} \sum_{m=1}^{\infty} H^{(2)}_{v_m}(k_\rho \rho_0) \sin(v_m \phi_0) J_{v_m}(k_\rho \rho) v_m \cos(v_m \phi) \tag{1.21} \]

\[ E_\rho = \frac{jk_z}{k_0^2 - k_z^2} \frac{k_\rho^2 \pi \omega \mu I_0}{k_0^2 2\pi - \beta} \sum_{m=1}^{\infty} H^{(2)}_{v_m}(k_\rho \rho_0) \sin(v_m \phi_0) J_{v_m}(k_\rho \rho) \sin(v_m \phi) \]

To solve the scattering problem we must move the source to infinity. In the absence of the wedge, the induced field is the solution of

\[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \vec{E}_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \vec{E}_z}{\partial \phi^2} + k_\rho^2 \vec{E}_z = \frac{k_\rho^2 j\omega \mu I_0}{\rho_0} \delta(\rho - \rho_0) \delta(\phi - \phi_0) \tag{1.22} \]

so that

\[ \vec{E}_z = -\frac{k_\rho^2 j\omega \mu I_0}{4j} H^{(2)}_0(k_\rho |\rho - \rho_0|) \rightarrow \]

\[ \vec{E}_z = -\frac{k_\rho^2 \omega \mu I_0}{4j} H^{(2)}_0(k_\rho |\rho - \rho_0|) \exp(-jk_z z) \tag{1.23} \]

As \( \rho_0 \to \infty \) this function approaches

\[ E_z(\rho, \phi) \sim A_0 \exp\left[-jk_\rho \cos(\phi - \phi_0) - jk_z z\right] \]

\[ A_0 = -\frac{k_\rho^2 \omega \mu I_0}{4 \sqrt{\pi k_\rho \rho_0}} \exp(-jk_\rho \rho_0 + j \pi / 4) \tag{1.24} \]

Letting \( \rho_0 \to \infty \) in (1.21) and using the above result leads to:
\[ E^\text{TM}_z (\rho, \phi) = \exp(-j k \zeta z) \frac{4 \pi A_0}{2 \pi - \beta} \sum_{m=1}^{\infty} j^m \sin (\nu_m \phi_0) J_{\nu_m} (k \rho) \sin (\nu_m \phi) \] (1.25)