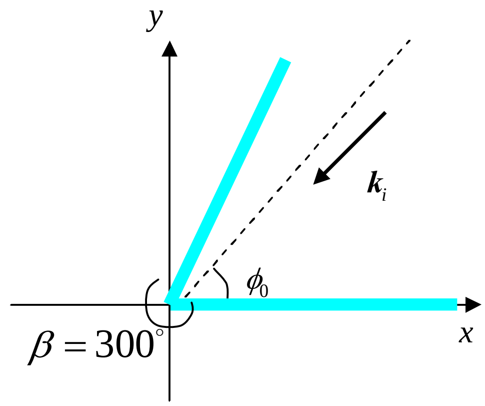
EM Scattering

Homework assignment 3

Problem 1:



A uniform TM^z plane wave (electric field along the z-direction) is normally incident on a perfectly conducting wedge with the angle $\beta = 300^\circ$ at an angle $\phi_0 < 60\,\deg$. The amplitude of the incident wave is E_0. Calculate the scattered field and show that it corresponds to the solution found from the method of images.

Solution

The total field in the far zone is

$$E_z^{TM}(\rho,\phi) = A_0 \frac{4\pi}{2\pi-\beta}\sum_{m=1}^{\infty}(j)^{3m} J_{\nu_m}(k\rho)\sin(\nu_m\phi)\sin(\nu_m\phi_0) \tag{1.1}$$

where $\beta = 5\pi/3, \nu_m = 3m$. Thus

$$\begin{aligned} E_z^{TM}(\rho,\phi) &= 6A_0\sum_{m=1}^{\infty}(j)^{3m} J_{3m}(k\rho)\sin(3m\phi)\sin(3m\phi_0) \\ &= 3A_0\sum_{m=1}^{\infty}(j)^{3m} J_{3m}(k\rho)\left\{\cos\left[3m(\phi-\phi_0)\right]-\cos\left[3m(\phi+\phi_0)\right]\right\} \end{aligned} \tag{1.2}$$

Now, let us consider the relationship

$$\exp(jz\cos\vartheta) = J_0(z) + 2\sum_{m=1}^{\infty}(j)^{m}\cos(m\vartheta)J_m(z) \tag{1.3}$$

From which is follows that

$$\exp\left(jz\cos\vartheta\right)=J_0(z)+2\sum_{m=1}^{\infty}\left(j\right)^m\cos\left(m\vartheta\right)J_m(z)$$

$$=J_0(z)+2\sum_{m=1}^{\infty}\left(j\right)^{3m}\cos\left(3m\vartheta\right)J_{3m}(z)+2\sum_{m=1}^{\infty}\left(j\right)^{3m-1}\cos\left[\left(3m-1\right)\vartheta\right]J_{3m-1}(z) \quad (1.4)$$

$$+2\sum_{m=1}^{\infty}\left(j\right)^{3m-2}\cos\left[\left(3m-2\right)\vartheta\right]J_{3m-2}(z)$$

$$\exp\left[jz\cos\left(\vartheta+\frac{2\pi}{3}\right)\right]=J_0(z)+2\sum_{m=1}^{\infty}\left(-j\right)^m\cos\left(m\vartheta+\frac{2\pi m}{3}\right)J_m(z)$$

$$=J_0(z)+2\sum_{m=1}^{\infty}\left(-j\right)^{3m}\cos\left(3m\vartheta\right)J_{3m}(z)+2\sum_{m=1}^{\infty}\left(-j\right)^{3m-1}\cos\left[\left(3m-1\right)\vartheta-\frac{2\pi}{3}\right]J_{3m-1}(z) \quad (1.5)$$

$$2\sum_{m=1}^{\infty}\left(-j\right)^{3m-1}\cos\left[\left(3m-2\right)\vartheta-\frac{4\pi}{3}\right]J_{3m-2}(z)$$

$$\exp\left[jz\cos\left(\vartheta-\frac{2\pi}{3}\right)\right]=J_0(z)+2\sum_{m=1}^{\infty}\left(-j\right)^m\cos\left(m\vartheta-\frac{2\pi m}{3}\right)J_m(z)$$

$$=J_0(z)+2\sum_{m=1}^{\infty}\left(-j\right)^{3m}\cos\left(3m\vartheta\right)J_{3m}(z)+2\sum_{m=1}^{\infty}\left(-j\right)^{3m-1}\cos\left[\left(3m-1\right)\vartheta+\frac{2\pi}{3}\right]J_{3m-1}(z) \quad (1.6)$$

$$2\sum_{m=1}^{\infty}\left(-j\right)^{3m-2}\cos\left[\left(3m-2\right)\vartheta+\frac{4\pi}{3}\right]J_{3m-2}(z)$$

Adding up the last two expressions yields

$$\exp\left[jz\cos\left(\vartheta+\frac{2\pi}{3}\right)\right]+\exp\left[jz\cos\left(\vartheta-\frac{2\pi}{3}\right)\right]$$

$$=2J_0(z)+4\sum_{m=1}^{\infty}\left(-j\right)^{3m}\cos\left(3m\vartheta\right)J_{3m}(z)+4\cos\frac{2\pi}{3}\sum_{m=1}^{\infty}\left(-j\right)^{3m-1}\cos\left[\left(3m-1\right)\vartheta\right]J_{3m-1}(z) \quad (1.7)$$

$$+4\cos\frac{2\pi}{3}\sum_{m=1}^{\infty}\left(-j\right)^{3m-2}\cos\left[\left(3m-2\right)\vartheta\right]J_{3m-2}(z)$$

so that

$$-\exp\left[jz\cos\left(\vartheta+\frac{2\pi}{3}\right)\right]-\exp\left[jz\cos\left(\vartheta-\frac{2\pi}{3}\right)\right]+2J_0(z)+4\sum_{m=1}^{\infty}\left(-j\right)^{3m}\cos\left(3m\vartheta\right)J_{3m}(z)$$

$$=2\sum_{m=1}^{\infty}\left(-j\right)^{3m-1}\cos\left[\left(3m-1\right)\vartheta\right]J_{3m-1}(z)+2\sum_{m=1}^{\infty}\left(-j\right)^{3m-2}\cos\left[\left(3m-2\right)\vartheta\right]J_{3m-2}(z) \quad (1.8)$$

we thus have

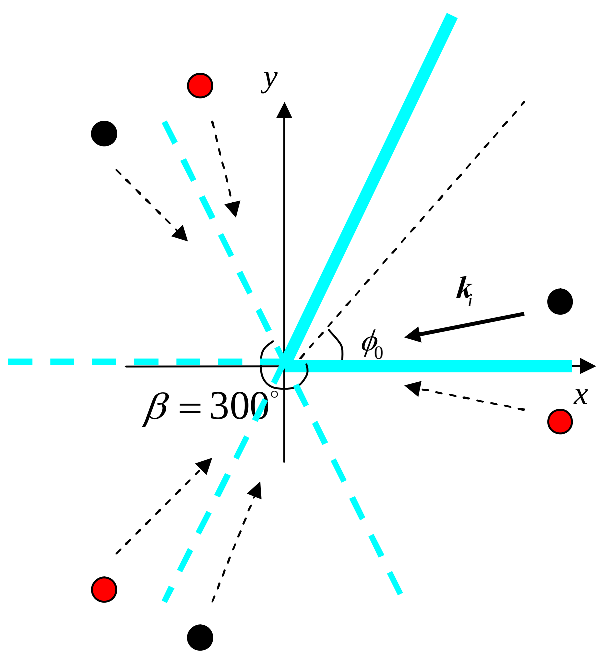
$$
\begin{aligned}
&J_0(z)+2\sum_{m=1}^{\infty}(j)^{3m}\cos(3m\vartheta)J_{3m}(z)=\exp(jz\cos\vartheta)-2\sum_{m=1}^{\infty}(j)^{3m-1}\cos\left[(3m-1)\vartheta\right]J_{3m-1}(z)\\
&-2\sum_{m=1}^{\infty}(j)^{3m-2}\cos\left[(3m-2)\vartheta\right]J_{3m-2}(z)\\
&=\exp(jz\cos\vartheta)-2J_0(z)+\exp\left[jz\cos\left(\vartheta+\frac{2\pi}{3}\right)\right]+\exp\left[jz\cos\left(\vartheta-\frac{2\pi}{3}\right)\right]\\
&-4\sum_{m=1}^{\infty}(-j)^{3m}\cos(3m\vartheta)J_{3m}(z)
\end{aligned}
\tag{1.9}
$$

and

$$
\begin{aligned}
&J_0(z)+2\sum_{m=1}^{\infty}(j)^{3m}\cos(3m\vartheta)J_{3m}(z)=\\
&=\frac{1}{3}\exp(jz\cos\vartheta)+\frac{1}{3}\exp\left[jz\cos\left(\vartheta+\frac{2\pi}{3}\right)\right]+\frac{1}{3}\exp\left[jz\cos\left(\vartheta-\frac{2\pi}{3}\right)\right]
\end{aligned}
\tag{1.10}
$$

As a result

$$
\begin{aligned}
&E_z^{TM}(\rho,\phi)=3A_0\sum_{m=1}^{\infty}(j)^{3m}J_{3m}(k\rho)\left\{\cos\left[3m(\phi-\phi_0)\right]-\cos\left[3m(\phi+\phi_0)\right]\right\}\\
&=A_0\exp\left[jk\rho\cos(\phi-\phi_0)\right]+A_0\exp\left[jk\rho\cos\left(\phi-\phi_0+\frac{2\pi}{3}\right)\right]+A_0\exp\left[jk\rho\cos\left(\phi-\phi_0-\frac{2\pi}{3}\right)\right]\\
&-A_0\exp\left[jk\rho\cos(\phi+\phi_0)\right]-A_0\exp\left[jk\rho\cos\left(\phi+\phi_0+\frac{2\pi}{3}\right)\right]-A_0\exp\left[jk\rho\cos\left(\phi+\phi_0-\frac{2\pi}{3}\right)\right]
\end{aligned}
$$

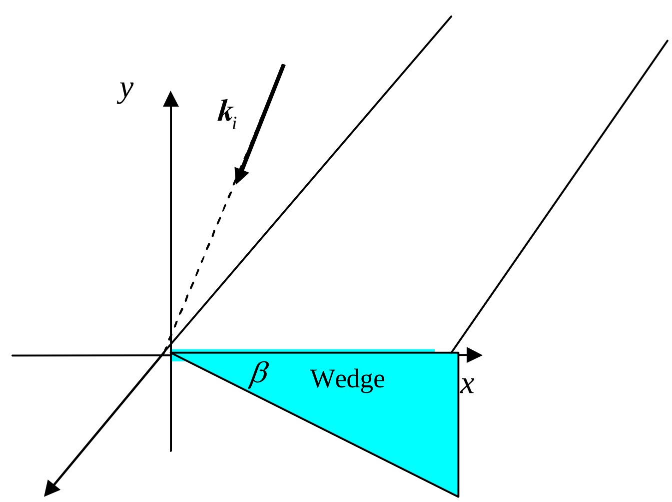


(1.11)

This is a combination of six plane waves which may be considered to be generated by a main source at infinity plus five images. Each image generates a wave whose wavevector represents an element of the solution. In reality, these waves are caused by multiple reflections from the wedge surface.

Problem 2:

By using a line of electric current $I = I_0 \exp(-jk_z z), k_z < k_0$, as the source, solve the problem of oblique TM^z scattering from a perfectly conducting wedge where the incident magnetic field lies in the x-y plane and the incident wave vector is not normal to the wedge but has a component along z, i.e., $\boldsymbol{k}_i = (k_{ix}, k_{iy}, k_{iz})$.



Solution

Due to the uniformity of the structure in the z-direction, we expect a source current of the type shown generates a solution whose dependence on z is also of the type $\exp(-jk_z z)$. We thus return to the Maxwell equations in cylindrical coordinates and assume (like in solving the modes of a waveguide):

$$\boldsymbol{E} = \tilde{\boldsymbol{E}}(\rho,\varphi)\exp(-jk_z z), \boldsymbol{H} = \tilde{\boldsymbol{H}}(\rho,\varphi)\exp(-jk_z z) \tag{1.12}$$

$$\begin{aligned}
&\frac{1}{\rho}\frac{\partial E_z}{\partial\varphi}-\frac{\partial E_\varphi}{\partial z}=-j\omega\mu_0 H_\rho && \frac{1}{\rho}\frac{\partial \tilde{E}_z}{\partial\varphi}+jk_z\tilde{E}_\varphi=-j\omega\mu_0\tilde{H}_\rho\\
&\frac{\partial E_\rho}{\partial z}-\frac{\partial E_z}{\partial\rho}=-j\omega\mu_0 H_\varphi \qquad \text{------>>} && -jk_z\tilde{E}_\rho-\frac{\partial \tilde{E}_z}{\partial\rho}=-j\omega\mu_0\tilde{H}_\varphi\\
&\frac{1}{\rho}\frac{\partial\left(\rho E_\varphi\right)}{\partial\rho}-\frac{1}{\rho}\frac{\partial E_\rho}{\partial\varphi}=-j\omega\mu_0 H_z && \frac{1}{\rho}\frac{\partial\left(\rho \tilde{E}_\varphi\right)}{\partial\rho}-\frac{1}{\rho}\frac{\partial \tilde{E}_\rho}{\partial\varphi}=-j\omega\mu_0\tilde{H}_z
\end{aligned} \tag{1.13}$$

$$\begin{aligned}
&\frac{1}{\rho}\frac{\partial H_z}{\partial\varphi}-\frac{\partial H_\varphi}{\partial z}=j\omega\varepsilon_0 E_\rho && \frac{1}{\rho}\frac{\partial \tilde{H}_z}{\partial\varphi}+jk_z\tilde{H}_\varphi=j\omega\varepsilon_0\tilde{E}_\rho\\
&\frac{\partial H_\rho}{\partial z}-\frac{\partial H_z}{\partial\rho}=j\omega\varepsilon_0 E_\varphi \qquad \text{-------->>} && -jk_z\tilde{H}_\rho-\frac{\partial \tilde{H}_z}{\partial\rho}=j\omega\varepsilon_0\tilde{E}_\varphi\\
&\frac{1}{\rho}\frac{\partial\left(\rho H_\varphi\right)}{\partial\rho}-\frac{1}{\rho}\frac{\partial H_\rho}{\partial\varphi}=j\omega\varepsilon_0 E_z+J_z && \frac{1}{\rho}\frac{\partial\left(\rho \tilde{H}_\varphi\right)}{\partial\rho}-\frac{1}{\rho}\frac{\partial \tilde{H}_\rho}{\partial\varphi}=j\omega\varepsilon_0\tilde{E}_z+\tilde{J}_z
\end{aligned} \tag{1.14}$$

Like in a waveguide there will be TM and TE modes. We choose the TM problem since our aim is to solve the related scattering problem (also the type of source used enforces this choice). As a result

$$\begin{aligned}
&\tilde{H}_\varphi=\frac{\omega\varepsilon_0}{k_z}\tilde{E}_\rho,\tilde{H}_\rho=-\frac{\omega\varepsilon_0}{k_z}\tilde{E}_\varphi\\
&\tilde{E}_\varphi=\frac{-jk_z}{k_0^2-k_z^2}\frac{1}{\rho}\frac{\partial\tilde{E}_z}{\partial\varphi},\tilde{E}_\rho=\frac{-jk_z}{k_0^2-k_z^2}\frac{\partial\tilde{E}_z}{\partial\rho}
\end{aligned} \tag{1.15}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\tilde{E}_z}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\tilde{E}_z}{\partial\varphi^2}+k_\rho^2\tilde{E}_z=-\frac{k_\rho^2}{j\omega\varepsilon_0}\tilde{J}_z \tag{1.16}$$

$$k_\rho^2=k_0^2-k_z^2 \tag{1.17}$$

The rest is similar to the method used for the 2D problem. Assuming the line source to be located at ρ_0,ϕ_0:

$$\tilde{J}_z=\frac{I_0}{\rho_0}\delta\left(\rho-\rho_0\right)\delta\left(\varphi-\varphi_0\right) \tag{1.18}$$

we separately consider the regions $\rho<\rho_0$ and $\rho>\rho_0$. Given the boundary condition on the wedge surface we expand the field as

$$\tilde{E}_z = \begin{cases} \sum_{m=1}^{\infty} a_m \sin(\nu_m \varphi) J_{\nu_m}(k_\rho \rho) & \rho < \rho_0 \\ \sum_{m=1}^{\infty} b_m \sin(\nu_m \varphi) H_{\nu_m}^{(2)}(k_\rho \rho) & \rho > \rho_0 \end{cases} \tag{1.19}$$

Matching condition at the source: the electric field along z must be continuous while

$$\left.\frac{\partial \tilde{E}_z}{\partial \rho}\right|_{\rho_0^+} - \left.\frac{\partial \tilde{E}_z}{\partial \rho}\right|_{\rho_0^-} = -\frac{k_\rho^2}{j\omega\varepsilon_0}\frac{I_0}{\rho_0}\delta(\varphi-\varphi_0) = \frac{k_\rho^2}{k_0^2}\frac{j\omega\mu_0 I_0}{\rho_0}\delta(\varphi-\varphi_0) \tag{1.20}$$

Final result for the field inside the source circle:

$$\begin{aligned} E_z^{TM}(\rho,\phi) &= -\exp(-jk_z z)\frac{k_\rho^2}{k_0^2}\frac{\pi\omega\mu I_0}{2\pi-\beta}\sum_{m=1}^{\infty} H_{\nu_m}^{(2)}(k_\rho\rho_0)\sin(\nu_m\phi_0) J_{\nu_m}(k_\rho\rho)\sin(\nu_m\phi) \\ E_\varphi &= \frac{jk_z\exp(-jk_z z)}{k_0^2-k_z^2}\frac{k_\rho^2}{k_0^2\rho}\frac{\pi\omega\mu I_0}{2\pi-\beta}\sum_{m=1}^{\infty} H_{\nu_m}^{(2)}(k_\rho\rho_0)\sin(\nu_m\phi_0) J_{\nu_m}(k_\rho\rho)\nu_m\cos(\nu_m\phi) \\ E_\rho &= \frac{jk_z\exp(-jk_z z)}{k_0^2-k_z^2}\frac{k_\rho^3}{k_0^2}\frac{\pi\omega\mu I_0}{2\pi-\beta}\sum_{m=1}^{\infty} H_{\nu_m}^{(2)}(k_\rho\rho_0)\sin(\nu_m\phi_0) J'_{\nu_m}(k_\rho\rho)\sin(\nu_m\phi) \end{aligned} \tag{1.21}$$

To solve the scattering problem we must move the source to infinity. In the absence of the wedge, the induced field is the solution of

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\tilde{E}_z}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\tilde{E}_z}{\partial\varphi^2} + k_\rho^2\tilde{E}_z = \frac{k_\rho^2}{k^2}\frac{j\omega\mu_0 I_0}{\rho_0}\delta(\rho-\rho_0)\delta(\varphi-\varphi_0) \tag{1.22}$$

so that

$$\begin{aligned} \tilde{E}_z^i &= -\frac{k_\rho^2}{k^2} j\omega\mu_0 I_0 \frac{1}{4j} H_0^{(2)}\left(k_\rho\left|\boldsymbol{\rho}-\boldsymbol{\rho_0}\right|\right) \rightarrow \\ E_z^i &= -\frac{k_\rho^2}{k^2}\frac{\omega\mu_0 I_0}{4} H_0^{(2)}\left(k_\rho\left|\boldsymbol{\rho}-\boldsymbol{\rho_0}\right|\right)\exp(-jk_z z) \end{aligned} \tag{1.23}$$

As $\rho_0 \to \infty$ this function approaches

$$\begin{aligned} E_z(\rho,\phi) &\sim A_0 \exp\left[jk\rho\cos(\phi-\phi_0) - jk_z z\right] \\ A_0 &= -\frac{k_\rho^2}{k^2}\frac{\omega\mu I_0}{4}\sqrt{\frac{2}{\pi k\rho_0}}\exp(-jk\rho_0 + j\pi/4) \end{aligned} \tag{1.24}$$

Letting $\rho_0 \to \infty$ in (1.21) and using the above result leads to:

$$E_z^{TM}\left(\rho,\phi\right)=\exp\left(-jk_z z\right)\frac{4\pi A_0}{2\pi-\beta}\sum_{m=1}^{\infty} j^{\nu_m}\sin\left(\nu_m\phi_0\right)J_{\nu_m}(k_\rho\rho)\sin\left(\nu_m\phi\right) \tag{1.25}$$