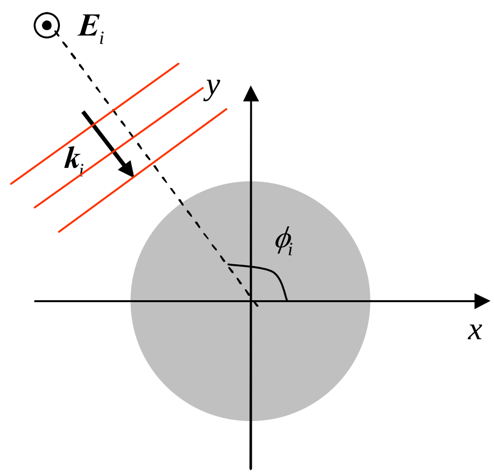
EM Scattering

Homework assignment 3

Problem 1:



Consider the problem of normal scattering from an infinitely long, perfectly conducting cylinder of radius a whose axis is along the z-axis. Assume the electric field of the incident wave to be polarized along z. Find the current on the surface of the cylinder by using the surface integral equation method. Calculate the scattered field using this current. The dielectric constant and permeability of the surrounding medium are ε_0, μ_0.

Solution

We use the electric field integral equation formulation for this problem in which:

$$E_{i,z}(\boldsymbol{\rho}') = j\omega\mu_0 \oint_C G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) J_{s,z}(\boldsymbol{\rho}) dl \tag{1.1}$$

Here $\boldsymbol{\rho}'$ approaches the surface from within the cylinder (actually this does not affect the solution). The Green's function is

$$G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) = \frac{1}{4j} H_0^{(2)}\left(k_0 \left|\boldsymbol{\rho}' - \boldsymbol{\rho}\right|\right) \tag{1.2}$$

Using the addition theorem of Bessel functions ($\rho' < \rho$):

$$\begin{aligned} &H_0^{(2)}\left(\sqrt{u^2 + v^2 - 2uv\cos\beta}\right) = \sum_{m=-\infty}^{\infty} H_m^{(2)}(u) J_m(v) \exp\left(-jm\beta\right) \qquad u \ge v \\ &\rightarrow G_0^{(2D)}(\boldsymbol{\rho}', \boldsymbol{\rho}) = \frac{1}{4j} \sum_{m=-\infty}^{\infty} H_m^{(2)}\left(k_0\rho\right) J_m\left(k_0\rho'\right) \exp\left[jm\left(\phi - \phi'\right)\right] \end{aligned} \tag{1.3}$$

Now, we have

$$
\begin{aligned}
E_{i,z}(\rho',\phi') &= j\omega\mu_0 a \int_0^{2\pi} G_0^{(2D)}(\rho',\phi';a,\phi) J_{s,z}(\phi) d\phi \\
&= \frac{\omega\mu_0 a}{4} \sum_{m=-\infty}^{\infty} H_m^{(2)}\left(k_0 a\right) J_m\left(k_0\rho'\right) \exp\left(-jm\phi'\right) \int_0^{2\pi} \exp\left(jm\phi\right) J_{s,z}(\phi) d\phi
\end{aligned} \tag{1.4}
$$

Let us consider the Fourier expansion of $J_{s,z}(\phi)$

$$
J_{s,z}(\phi) = \sum_{n=-\infty}^{\infty} b_n \exp\left(-jn\phi\right), b_n = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(jn\phi\right) J_{s,z}(\phi) d\phi \tag{1.5}
$$

Hence from (1.4):

$$
E_{i,z}(\rho',\phi') = \frac{\omega\mu_0 \pi a}{2} \sum_{m=-\infty}^{\infty} H_m^{(2)}\left(k_0 a\right) J_m\left(k_0\rho'\right) \exp\left(-jm\phi'\right) b_m \tag{1.6}
$$

On the other hand:

$$
E_{i,z} = E_0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{\rho}'\right) = E_0 \exp\left[-jk\rho' \cos\left(\phi' - \phi_i\right)\right] \tag{1.7}
$$

We also have the relationship

$$
\exp\left(-jz\cos\vartheta\right) = \sum_{m=-\infty}^{\infty} (-j)^m J_m(z) \exp\left(-jm\vartheta\right) \tag{1.8}
$$

So that

$$
E_{i,z} = E_0 \exp\left[-jk\rho' \cos\left(\phi' - \phi_i\right)\right] = E_0 \sum_{m=-\infty}^{\infty} (-j)^m J_m(k\rho') \exp\left[-jm\left(\phi' - \phi_i\right)\right] \tag{1.9}
$$

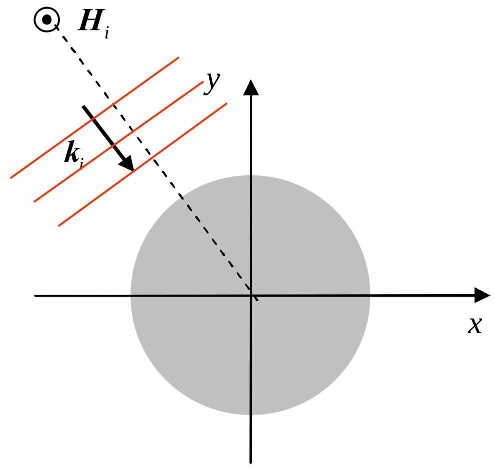
Comparison with (1.6) shows that

$$
\begin{aligned}
b_m &= E_0 \frac{2}{\omega\mu_0\pi a} \frac{(-j)^m}{H_m^{(2)}\left(k_0 a\right)} \exp\left(jm\phi_i\right) \\
J_{s,z}(\phi) &= \sum_{m=-\infty}^{\infty} b_m \exp\left(-jm\phi\right) = \frac{2E_0}{\omega\mu_0\pi a} \sum_{n=-\infty}^{\infty} \frac{(-j)^m \exp\left(jm\left(\phi_i - \phi\right)\right)}{H_m^{(2)}\left(k_0 a\right)}
\end{aligned} \tag{1.10}
$$

This is exactly equal to the total current found from the problem of homework 2 where M and N functions where used.

Problem 2:

Repeat the same problem, but now assume that it is the magnetic field of the incident wave which is polarized along z.



Solution

Now, use the magnetic field integral equation

$$H_{i,z}(\boldsymbol{\rho}') = \oint_C \partial_n^- G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho}) H_z(\boldsymbol{\rho}) dl = -\oint_C H_z(\boldsymbol{\rho}) \hat{\boldsymbol{n}} \cdot \nabla G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho}) dl \tag{1.11}$$

Here $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\rho}}$ so that

$$-\hat{\boldsymbol{n}} \cdot \nabla G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho}) = -\frac{\partial G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho})}{\partial \rho} \tag{1.12}$$

Again $\boldsymbol{\rho}'$ approaches the surface from within the cylinder. Note that in this case

$$-\frac{\partial G_0^{(2D)}(\boldsymbol{\rho}',\boldsymbol{\rho})}{\partial \rho} = -\frac{k_0}{4j} \sum_{m=-\infty}^{\infty} H_m^{(2)\prime}(k_0 \rho) J_m(k_0 \rho') \exp\left[-jm(\phi' - \phi)\right] \tag{1.13}$$

Substitution in (1.11) yields (with $\rho = a$):

$$H_{i,z}(\rho',\phi') = -\frac{k_0 a}{4j} \sum_{m=-\infty}^{\infty} H_m^{(2)\prime}(k_0 a) J_m(k_0 \rho') \exp[-jm\phi'] \int_0^{2\pi} \exp[jm\phi] H_z(\phi) d\phi \tag{1.14}$$

Fourier series for surface magnetic field:

$$H_z(\phi) = \sum_{n=-\infty}^{\infty} c_n \exp(-jn\phi), c_n = \frac{1}{2\pi} \int_0^{2\pi} \exp(jn\phi) H_z(\phi) d\phi \tag{1.15}$$

So that

$$H_{i,z}(\rho',\phi') = -\frac{\pi k_0 a}{2j} \sum_{m=-\infty}^{\infty} H_m^{(2)\prime}(k_0 a) J_m(k_0 \rho') \exp[-jm\phi'] c_n \tag{1.16}$$

Let now

$$H_{i,z} = H_0 \exp\left[-jk_0 \rho' \cos(\phi' - \phi_i)\right] = H_0 \sum_{m=-\infty}^{\infty} (-j)^m J_m(k_0 \rho') \exp\left[-jm(\phi' - \phi_i)\right] \tag{1.17}$$

The resulting problem is very similar to the previous one. We find

$$c_m = -\frac{2jH_0}{\pi k_0 a} \frac{(-j)^m \exp(jm\phi_i)}{H_m^{(2)\prime}(k_0 a)} \tag{1.18}$$

In neither derivation taking the limit $\rho' \to a$ was actually needed!!!