EM Scattering

Homework assignment 4

Problem 1:

Use the Rayleigh approximation (small particle approximation) and calculate the scattering cross section of a small dielectric sphere with the dielectric constant ε_d and radius a (a is much smaller than the wavelength in vacuum). Where possible, approximate the resulting integrals.

Solution

Assume the center of the sphere to coincide with the center of the coordinates. Then from the small particle approximation the electric field inside the particle is a constant given by

$$\boldsymbol{E} = \left[\overline{\overline{\boldsymbol{I}}} + \delta\varepsilon_r \overline{\overline{\boldsymbol{N}}}\right]^{-1} \cdot \boldsymbol{E}_i^0 \exp\left(-j\boldsymbol{k}_i \cdot \boldsymbol{r}_0\right)$$

$$\boldsymbol{r}_0 = 0, \delta\varepsilon_r = \frac{\varepsilon_d}{\varepsilon_0} - 1, \overline{\overline{\boldsymbol{N}}} = \frac{1}{3}\overline{\overline{\boldsymbol{I}}} \rightarrow \boldsymbol{E} = \frac{3\varepsilon_0}{2\varepsilon_0 + \varepsilon_d}\boldsymbol{E}_i^0$$

Scattered field observed along $\hat{\boldsymbol{k}}_s$:

$$\boldsymbol{E}_s(\boldsymbol{r}, \hat{\boldsymbol{k}}_s) = -\frac{k^2 \exp\left(-jkr\right)}{4\pi r}\hat{\boldsymbol{k}}_s \times \left[\hat{\boldsymbol{k}}_s \times \delta\varepsilon_r \int_V \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{r}'\right) \boldsymbol{E}(\boldsymbol{r}') dV'\right]$$

$$\approx -\frac{k^2 V \delta\varepsilon_r \exp\left(-jkr\right)}{4\pi r}\hat{\boldsymbol{k}}_s \times \left(\hat{\boldsymbol{k}}_s \times \boldsymbol{E}\right) = -\frac{k^2 V \exp\left(-jkr\right)}{4\pi r}\frac{3\left(\varepsilon_d - \varepsilon_0\right)}{2\varepsilon_0 + \varepsilon_d}\hat{\boldsymbol{k}}_s \times \left(\hat{\boldsymbol{k}}_s \times \boldsymbol{E}_i^0\right)$$

Scattering cross section: total scattered power divided by the magnitude of the incident Poynting vector:

$$P = \oint_{r\to\infty} \frac{\left|E_s\right|^2}{2\eta} dS = \frac{1}{2\eta}\frac{k^4 V^2}{16\pi^2}\frac{9\left(\varepsilon_d - \varepsilon_0\right)^2}{\left(2\varepsilon_0 + \varepsilon_d\right)^2}\oint_{r\to\infty}\left|\hat{\boldsymbol{k}}_s \times \left(\hat{\boldsymbol{k}}_s \times \boldsymbol{E}_i^0\right)\right|^2 \sin\theta_s d\theta_s d\phi_s$$

Since we are dealing with a sphere, without any loss of generality, we may take the polarization of the incident wave to be along z so that

$$\hat{\boldsymbol{k}}_s \times \left(\hat{\boldsymbol{k}}_s \times \boldsymbol{E}_i^0 \right) = E_0 \hat{\boldsymbol{k}}_s \times \left(\hat{\boldsymbol{k}}_s \times \hat{\boldsymbol{z}} \right) = \hat{\boldsymbol{\theta}}_s E_0 \sin\theta_s$$

$$\oint_{r\to\infty} \left| \hat{\boldsymbol{k}}_s \times \left(\hat{\boldsymbol{k}}_s \times \boldsymbol{E}_i^0 \right) \right|^2 \sin\theta_s d\theta_s d\phi_s = \left| E_0 \right|^2 2\pi \int_0^{\pi} \sin^3\theta_s d\theta_s = \frac{8\pi}{3} \left| E_0 \right|^2$$

$$P = \frac{1}{2\eta} \frac{k^4 V^2}{16\pi^2} \frac{9\left(\varepsilon_d - \varepsilon_0 \right)^2}{\left(2\varepsilon_0 + \varepsilon_d \right)^2} \frac{8\pi}{3} \left| E_0 \right|^2 = \frac{\left| E_0 \right|^2}{2\eta} \frac{k^4 a^6}{9} \frac{9\left(\varepsilon_d - \varepsilon_0 \right)^2}{\left(2\varepsilon_0 + \varepsilon_d \right)^2} \frac{8\pi}{3}$$

$$\to \sigma = \frac{k^4 a^6}{9} \frac{9\left(\varepsilon_d - \varepsilon_0 \right)^2}{\left(2\varepsilon_0 + \varepsilon_d \right)^2} \frac{8\pi}{3}$$

The same result could have been obtained by using the result obtained for the Born approximation where $\boldsymbol{E} \approx \boldsymbol{E}_i^0$ and just normalize the end result by $\left(\frac{3\varepsilon_0}{2\varepsilon_0 + \varepsilon_d} \right)^2$.

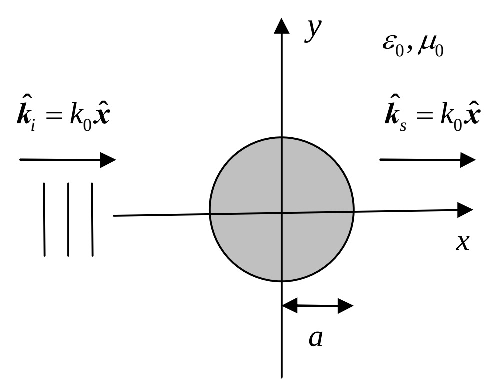
Problem 2:

A plane wave which is propagating in vacuum along the x-axis (wave number k_0) is normally incident on an infinitely long, perfectly conducting cylinder of radius a whose axis is along z. The electric field of the incident wave is linearly polarized along the z-axis and has an amplitude of E_0. We would like to use the approximation of physical optics to analyze this problem.

(i) Find the electric surface current induced on the cylinder using this approximation.

(ii) Find the amplitude of the far-zone electric field scattered in the x direction (the forward scattered wave).

Hint: use the expression for the far-zone field of 2D objects discussed in lecture 7 (integral equations).



Solution

First we evaluate the incident magnetic field:

$$\boldsymbol{E}_i = E_0 \hat{\boldsymbol{z}} \exp(-jkx) \to \boldsymbol{H}_i = -\hat{\boldsymbol{y}} \frac{E_0}{\eta} \exp(-jkx)$$

Tangential component of the incident magnetic field on the surface of the cylinder

$$\boldsymbol{H}_{i,t} = \left(\boldsymbol{H}_i \cdot \hat{\boldsymbol{\varphi}}\right)\hat{\boldsymbol{\varphi}} = -\hat{\boldsymbol{\varphi}}\left(\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\varphi}}\right)\frac{E_0}{\eta}\exp\left(-jkx\right) = -\hat{\boldsymbol{\varphi}}\frac{E_0}{\eta}\cos\phi\exp\left(-jkx\right)$$
$$= -\hat{\boldsymbol{\varphi}}\frac{E_0}{\eta}\cos\phi\exp\left(-jka\cos\phi\right)$$

Induced current in the approximation of physical optics

$$3\pi/2 > \phi > \pi/2: \ \boldsymbol{J}_s = 2\hat{\boldsymbol{n}} \times \boldsymbol{H}_{i,t} = 2\hat{\boldsymbol{\rho}} \times \boldsymbol{H}_{i,t} = -2\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\varphi}}\frac{E_0}{\eta}\cos\phi\exp\left(-jka\cos\phi\right)$$
$$= -\hat{\boldsymbol{z}}\frac{2E_0}{\eta}\cos\phi\exp\left(-jka\cos\phi\right)$$
$$3\pi/2 < \phi < 2\pi: \ \boldsymbol{J}_s = 0$$

(ii) Find the amplitude of the far-zone electric field scattered in the x direction (the forward scattered wave).

$$E_{s,z} = -\frac{\omega\mu_0}{4}\sqrt{\frac{2}{\pi k_0 \rho'}}\exp\left(-jk_0\rho' + j\pi/4\right)\oint_C \exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{\rho}\right) J_{s,z}(\boldsymbol{\rho})dl$$
$$= \frac{2E_0}{\eta}\frac{\omega\mu_0 a}{4}\sqrt{\frac{2}{\pi k_0 \rho'}}\exp\left(-jk_0\rho' + j\pi/4\right)\int_{\pi/2}^{3\pi/2}\exp\left(j\boldsymbol{k}_s \cdot \boldsymbol{\rho}\right)\cos\phi\exp\left(-jka\cos\phi\right)d\phi$$

Forward scattering: $\boldsymbol{k}_s = k\hat{\boldsymbol{x}}$ so that

$$E_{s,z} = \frac{2E_0}{\eta}\frac{\omega\mu_0 a}{4}\sqrt{\frac{2}{\pi k_0 \rho'}}\exp\left(-jk_0\rho' + j\pi/4\right)\int_{\pi/2}^{3\pi/2}\exp\left(jkx\right)\cos\phi\exp\left(-jka\cos\phi\right)d\phi$$
$$= \frac{2E_0}{\eta}\frac{\omega\mu_0 a}{4}\sqrt{\frac{2}{\pi k_0 \rho'}}\exp\left(-jk_0\rho' + j\pi/4\right)\int_{\pi/2}^{3\pi/2}\cos\phi d\phi = -\frac{4E_0}{\eta}\frac{\omega\mu_0 a}{4}\sqrt{\frac{2}{\pi k_0 \rho'}}\exp\left(-jk_0\rho' + j\pi/4\right)$$