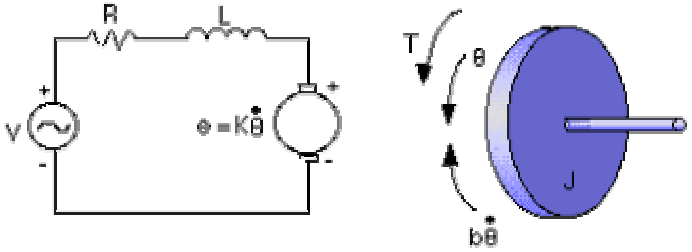
Example: A State-Space Controller for DC Motor Position Control

The electric circuit of the armature and the free body diagram of the rotor are shown in the following figure:



For this example, we will assume the following values for the physical parameters. These values were derived by experiment from an actual motor in Carnegie Mellon's undergraduate controls lab.

* moment of inertia of the rotor (J) = 3.2284E-6 kg.m^2/s^2
* damping ratio of the mechanical system (b) = 3.5077E-6 Nms
* electromotive force constant (K=Ke=Kt) = 0.0274 Nm/Amp
* electric resistance (R) = 4 ohm
* electric inductance (L) = 2.75E-6 H
* input (V): Source Voltage
* output (theta): position of shaft
* The rotor and shaft are assumed to be rigid

System Equations

The motor torque, **T**, is related to the armature current, **i**, by a constant factor **Kt**. The back emf, **e**, is related to the rotational velocity by the following equations:

$$T = K_t i$$

$$e = K_e \dot{\theta}$$

In SI units (which we will use), **Kt** (armature constant) is equal to **Ke** (motor constant).

From the figure above we can write the following equations based on Newton's law combined with Kirchhoff's law:

$$J\ \ddot{\theta} + b\ \dot{\theta} = Ki$$

$$L\frac{di}{dt} + Ri = V - K\ \dot{\theta}$$

1. Transfer Function

Using Laplace Transforms the above equations can be expressed in terms of s.

$$s(Js + b)\Theta(s) = KI(s)$$

$$(Ls + R)I(s) = V - Ks\Theta(s)$$

By eliminating I(s) we can get the following transfer function, where the rotating speed is the output and the voltage is an input.

$$\frac{\dot{\theta}}{V} = \frac{K}{(Js + b)(Ls + R) + K^2}$$

However during this example we will be looking at the position, as being the output. We can obtain the position by integrating Theta Dot, therefore we just need to divide the transfer function by s.

$$\frac{\theta}{V} = \frac{K}{s((Js + b)(Ls + R) + K^2)}$$

From the main problem, the dynamic equations in state-space form are the following:

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{K}{J} \\ 0 & -\frac{K}{L} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} V$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix}$$

With a 1 rad reference added to the system, the design criteria are:

- Settling time less than 0.04 seconds
- Overshoot less than 16%
- Zero steady-state error to a step input

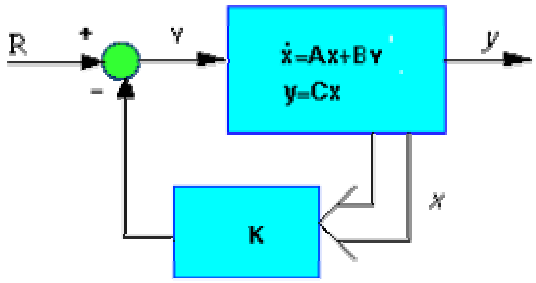
- Zero steady-state error to a step disturbance

Create a new m-file and type in the following commands (refer to the main problem for the details of getting those commands).

```
J=3.2284E-6;
b=3.5077E-6;
K=0.0274;
R=4;
L=2.75E-6;

A=[0 1 0
   0 -b/J K/J
   0 -K/L -R/L];
B=[0 ; 0 ; 1/L];
C=[1 0 0];
D=[0];
```

Designing the full-state feedback controller

Since all of the state variables in our problem are very easy to measure (simply add an ammeter for current, a tachometer for speed, and a potentiometer for position), we can design a full-state feedback controller for the system without worrying about having to add an observer. The schematic for a full-state feedback system is:



Recall that the characteristic polynomial for this closed-loop system is the determinant of **(sI-(A-BKc))** where **s** is the Laplace variable. Since the matrices **A** and **B*Kc** are both 3x3 matrices, there should be 3 poles for the system. By designing a full-state feedback controller, we can move these three poles anywhere we want them. We shall first try to place them at -100 + 100i and -100-100i (note that this corresponds to a zeta = 0.5 which gives 0.16% overshoot and a sigma = 100 which leads to a .04 sec settling time). Once we come up with the poles we want, Matlab will find the controller matrix, **Kc**, for us. Simply add the following code to the end of your m-file :

```
p1=-100+100i;
```

```
p2=-100-100i;
p3=-200;
Kc=place(A,B,[p1,p2,p3]);
```

Now look at the schematic above again. We see that after adding the K matrix into the system, the state-space equations become:

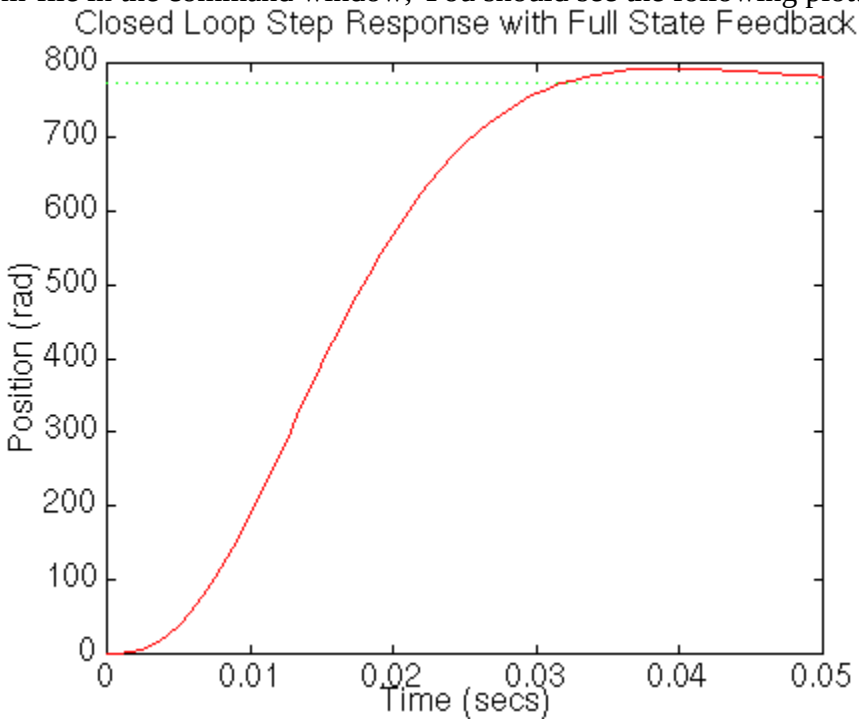
$$\dot{X} = (A - BK)X + Bu$$

$$Y = CX$$

We can see the closed-loop response by simply adding the following line to the end of your m-file:

```
t=0:0.001:.05;
step (A-B*Kc,B,C,D,1,t)
```

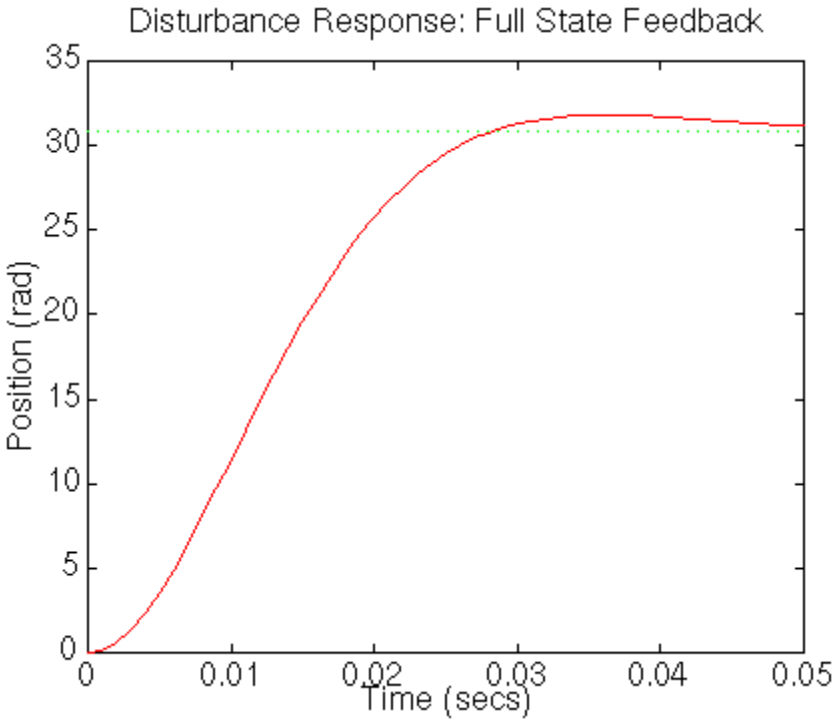
Run your m-file in the command window, You should see the following plot:



Disturbance Response

In order to get the disturbance response, we must provide the proper input to the system. Physically, a disturbance is a torque which acts on the inertia of the motor. A torque acts as an additive term in the second state equation (which gets divided by J, as do all the other terms in this equation). We can simulate this simply by modifying our closed loop input matrix, B, to have a 1/J in the second row. Add the following line to your m-file and re-run.

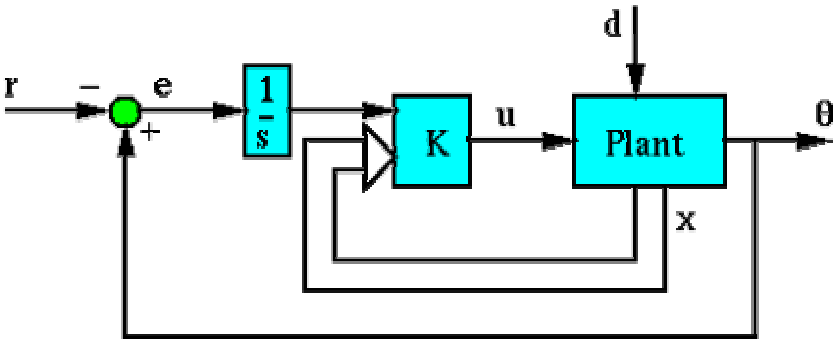
```
step(A-B*Kc,[0;1/J;0],C,D,1,t)
```



This is not a zero steady-state error to a disturbance, and we will have to compensate for this.

Adding Integral Action

We know that if we put an extra integrator in series with the plant it can remove steady-state error to an input. If the integrator comes before the injection of the disturbance, it will cancel the disturbance in steady state. This changes our control structure so it now resembles the following:



We can model the integrator by augmenting our state equations with an extra state which is the integral of the output. This adds an extra equation which states that the derivative of the integral of theta is theta. This equation will be placed at the top of our matrices. The input, r, now enters the system before the integrator, so it appears in the newly added top equation. The output of the system remains the same.

$$\frac{d}{dt}\begin{bmatrix} \int\theta \\ \theta \\ \dot{\theta} \\ i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{b}{J} & \frac{K}{J} \\ 0 & 0 & -\frac{K}{L} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} \int\theta \\ \theta \\ \dot{\theta} \\ i \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \int\theta \\ \theta \\ \dot{\theta} \\ i \end{bmatrix}$$

These equations represent the dynamics of the system before the loop is closed. We will refer to the matrices in this equation as A_a, B_a, C_a, and D_a. We will refer to the state vector of the augmented system as x_a. Note that the reference, r, does not affect the states (except the integrator state) or the output of the plant - this is expected, since there is no path from the reference to the plant input, u, without implementing the feedback matrix, Kc.

In order to find the closed loop equations, we have to look at how the input, u, affects the plant. In this case, it is exactly the same as in the unaugmented equations. Therefore, there is a vector, call it B_{au}, which replaces B_a when we are treating u as the input. This is just our old B vector with an extra zero added as a first row. Since $u=Kc*x_a$ is the input to the plant for the closed loop, but r is the input to the closed loop system, the closed loop equations will depend on both B_{au} and B_a. The closed loop equations will become:

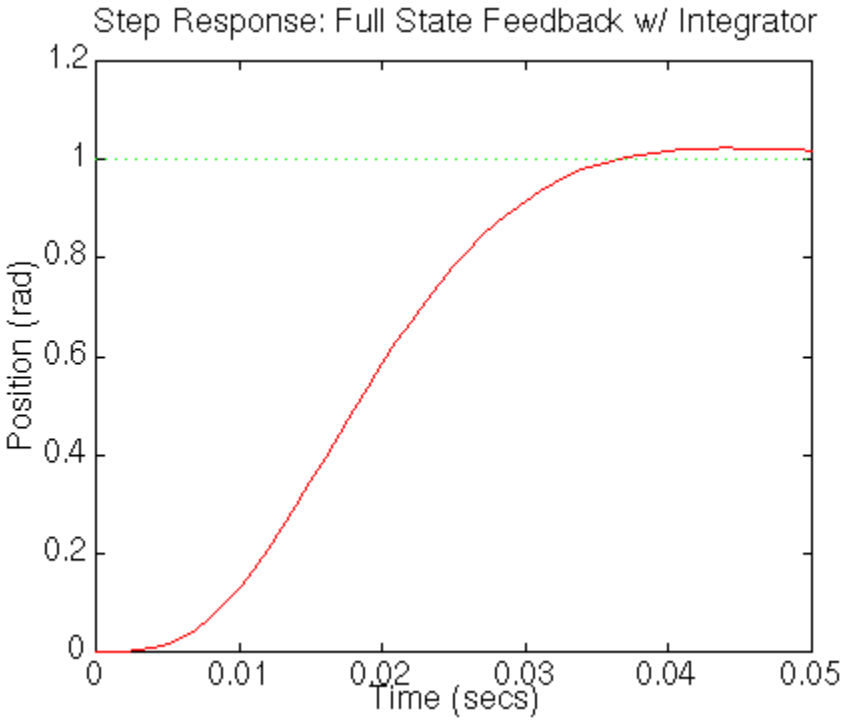
$$\dot{x}_a = (A_a - B_{au}K)\,x_a + B_a r$$

$$\theta = C_a x_a$$

Now, the integral of the output will be fed back, and will be used by the controller to remove steady state error to a disturbance. We can now redesign our controller. Since we need to place one closed-loop pole for each pole in the plant, we will place another pole at -300, which will be faster than the rest of the poles. Since the closed-loop system matrix depends on B_{au}, we will use B_{au} in the place command rather that B_a. Add the following to your m-file:

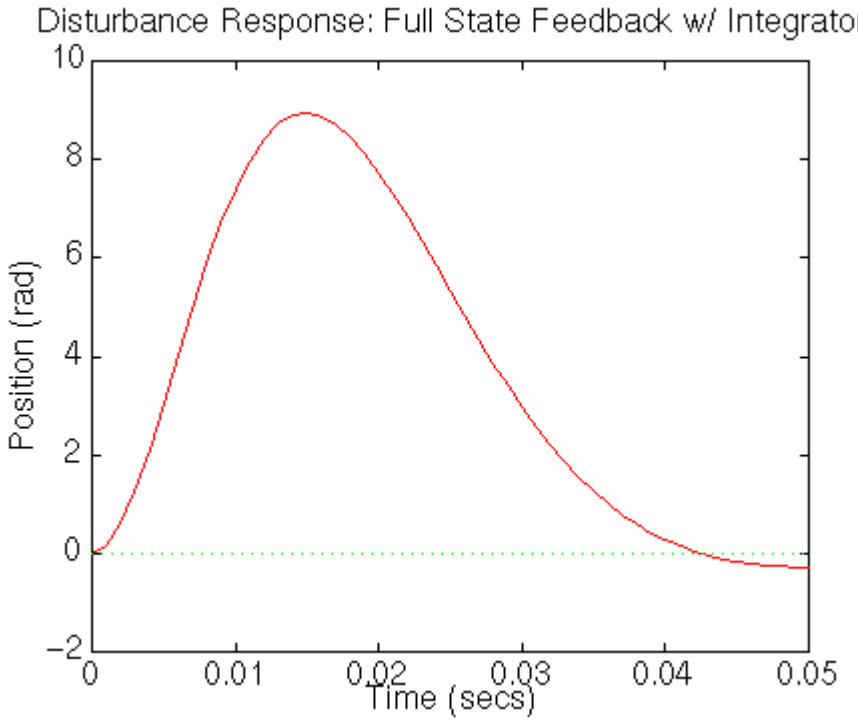
```
Aa=[0 1 0 0
    0 0 1 0
    0 0 -b/J K/J
    0 0 -K/L -R/L];
Ba=[ -1 ; 0 ; 0 ; 0];
Bau=[0 ; 0 ; 0 ; 1/L ];
Ca=[0 1 0 0];
Da=[0];

p4=-300;
Kc=place(Aa,Bau,[p1,p2,p3,p4]);

t=0:0.001:.05;
step(Aa-Bau*Kc,Ba,Ca,Da,1,t)
```

Run your m-file (or just these new lines) and you will get the following output.



To look at the disturbance response, we apply a similar B matrix as we did previously when simulating the disturbance response.

```
step(Aa-Bau*Kc,[0 ; 0 ; 1/J ; 0] ,Ca,Da,1,t)
```



We can see that all of the design specifications have been met by this controller.