Analytical expression of giant Goos–Hänchen shift in terms of proper and improper modes in waveguide structures with arbitrary refractive index profile

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We analytically relate the giant Goos–Hänchen shift, observed at the interface of a high refractive index prism and a waveguide structure with an arbitrary refractive index profile, to the spatial resonance phenomenon. The proximity effect of the high refractive index prism on modal properties of the waveguide is discussed, and the observed shift is expressed in terms of proper and improper electromagnetic modes supported by the waveguide with no prism. The transversely increasing improper modes are shown playing an increasingly important role as the high refractive index prism comes closer to the waveguide.
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The lateral shift of totally reflected optical beams from a planar interface has been referred to as the Goos–Hänchen shift (GHS) and is observed in two different regimes. One corresponds to the nonresonant, usually small, GHS, which is due to the branch point singularity in the reflection coefficient of planar structures [1,2]. The other corresponds to the giant GHS, which is caused by the simple pole singularity in the reflection coefficient of planar structures [1,3]. In the latter regime, the structure basically comprises an optical prism with high refractive index n_p placed at the distance d_g above a waveguide with arbitrary refractive index profile. The waveguide has a pure guided mode, whose longitudinal propagation constant is β_g. The high refractive index prism in this structure transforms the waveguide into a leaky wave structure with a complex propagation constant β_p, whose real part is usually close to β_g. A number of works have focused on relating the giant GHS to the leaky mode propagation constant of the whole structure, β_p, or the unperturbed mode of the waveguide, β_g [1,4,5]. To the best of our knowledge, no formula is yet reported to express the GHS in terms of the electromagnetic modes in the general case. Here, the giant GHS is analytically linked up to the modes of the unperturbed waveguide with an arbitrary refractive index profile. It is found that the giant GHS depends not only on the transversely decreasing proper modes of the waveguide but also on its transversely increasing improper modes. The latter are shown to play an increasingly important role in the strong coupling regime as the optical prism comes nearer to the waveguide.

To obtain an analytic expression for the GHS, the transfer matrix of the whole structure, $\mathbf{Q}_T$, is first written as a multiplication of $\mathbf{Q}$, the transfer matrix of the waveguide with no prism, and $\mathbf{T}$, the transfer matrix of the gap between prism and waveguide having a thickness of d_g and refractive index of n_c:

$$\mathbf{Q}_T = \mathbf{Q} \times \mathbf{T}, \tag{1}$$

where

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \tag{2}$$

and q_{ij}s denote the element of the transfer matrix of the waveguide. The transfer matrix of the gap, $\mathbf{T}$, can be also written as

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} \left(1 + f\frac{\kappa_p}{\kappa_c}\right)\exp(-\kappa_c d_g) & \left(1 - f\frac{\kappa_p}{\kappa_c}\right)\exp(-\kappa_c d_g) \\ \left(1 - f\frac{\kappa_p}{\kappa_c}\right)\exp(\kappa_c d_g) & \left(1 + f\frac{\kappa_p}{\kappa_c}\right)\exp(\kappa_c d_g) \end{bmatrix}, \tag{3}$$

where κ_i with $i = p, c$ stands for the x component of the wave vector in either the prism ($i = p$) or cover ($i = c$) region, and the factor f is 1 for TE and ${n_c}^2/{n_p}^2$ for TM polarized waves, respectively.

The overall reflection coefficient of the structure, r, can then be straightforwardly written in terms of the elements of the $\mathbf{Q}_T$ matrix:

$$r = -\frac{Q_{21}}{Q_{22}}. \tag{4}$$

Here, the numerator and denominator of the overall reflection coefficient of the structure, i.e., Q_{21} and Q_{22}, are, respectively, the $(2,1)$th and $(2,2)$th element of the overall transfer matrix $\mathbf{Q}_T$. Now, insomuch as the incident spatial frequency, β, is close to the resonance spatial frequency, β_g, i.e., $|\beta - \beta_g| \ll 1$, Q_{21} and Q_{22} can be represented by their first-order Taylor series expansion around β_g:

$$Q_{21}(\beta) = Q_{21}(\beta_g) + \left.\frac{\partial Q_{21}}{\partial \beta}\right|_{\beta_g} (\beta - \beta_g), \tag{5a}$$

$$Q_{22}(\beta) = Q_{22}(\beta_g) + \left.\frac{\partial Q_{22}}{\partial \beta}\right|_{\beta_g} (\beta - \beta_g). \tag{5b}$$

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By writing $Q_{21}(\beta_g)$, $Q_{22}(\beta_g)$ in terms of q_{ij}s and t_{ij}s and using the fact that $q_{22}(\beta_g)=0$ (because β_g is an eigenmode of the structure with no prism), we have

$$Q_{21}(\beta)=\left.\frac{\partial Q_{21}}{\partial\beta}\right|_{\beta_g}(\beta-\beta_z), \tag{6a}$$

$$Q_{22}(\beta)=\left.\frac{\partial Q_{22}}{\partial\beta}\right|_{\beta_g}(\beta-\beta_p), \tag{6b}$$

where β_z and β_p denote the zero and the pole of the overall reflection of the structure and read as

$$\beta_z=\beta_g-\frac{Q_{21}(\beta_g)}{\left.\frac{\partial Q_{21}}{\partial\beta}\right|_{\beta_g}}=\beta_g-\frac{q_{21}(\beta_g)}{q_{22}{}'(\beta_g)}\times\left.\frac{t_{11}/t_{21}}{1+\left(\frac{t_{11}q_{21}{}'}{t_{21}q_{22}{}'}+\frac{t_{11}{}'q_{21}}{t_{21}q_{22}{}'}\right)}\right|_{\beta_g}, \tag{7a}$$

$$\beta_p=\beta_g-\frac{Q_{22}(\beta_g)}{\left.\frac{\partial Q_{22}}{\partial\beta}\right|_{\beta_g}}=\beta_g-\frac{q_{21}(\beta_g)}{q_{22}{}'(\beta_g)}\times\left.\frac{t_{12}/t_{22}}{1+\left(\frac{t_{12}q_{21}{}'}{t_{22}q_{22}{}'}+\frac{t_{12}{}'q_{21}}{t_{22}q_{22}{}'}\right)}\right|_{\beta_g}, \tag{7b}$$

and the prime denotes derivation with respect to β.

The overall reflection coefficient of the structure, r, is thus approximately represented by a zero and a simple pole in the complex plane of spatial frequency:

$$r=\frac{-\left.\frac{\partial Q_{21}}{\partial\beta}\right|_{\beta_g}}{\left.\frac{\partial Q_{22}}{\partial\beta}\right|_{\beta_g}}\times\frac{\beta-\beta_z}{\beta-\beta_p}. \tag{8}$$

The obtained pole-zero representation of the overall reflection coefficient in Eq. (8) can now be applied to the well-known Artmann formula [6], and the GHS can be written as

$$\mathrm{GHS}=\frac{\partial\phi}{\partial\beta}=\mathrm{Im}\left[\frac{d(\ln(r))}{d\beta}\right]=\mathrm{Im}\left[\frac{\beta_z-\beta_p}{(\beta-\beta_p)(\beta-\beta_z)}\right], \tag{9}$$

where Im represents the imaginary part of its argument and ϕ stands for the phase factor of the reflection coefficient.

It should be noticed that the modulus of the reflection coefficient, r, is 1 for lossless waveguides when the incident beam is totally reflected. The pole and zero of the overall reflection coefficient form a complex conjugate pair, and the maximum GHS is observed at $\beta=\mathrm{Re}[\beta_p]$ and amounts to $2/\mathrm{Im}[\beta_p]$]. For a lossy structure, on the other hand, $\mathrm{Re}[\beta_p]=\mathrm{Re}[\beta_z]$, and the maximum GHS is observed at $\beta=\mathrm{Re}[\beta_p]$ and amounts to $(\mathrm{Im}[\beta_z]-\mathrm{Im}[\beta_p])/(\mathrm{Im}[\beta_z]\mathrm{Im}[\beta_p])$.

The analytical expression given in Eq. (9) then relates the GHS to β_p and β_z, the propagation constants of the proper and improper modes in the whole structure, i.e., waveguide and prism. These propagation constants, β_p and β_z, are also related to the propagation constants of the proper and improper modes in the waveguide structure with no prism. The reflection coefficient at $x=d_g$, i.e., reflection from the cover-waveguide interface, denoted r_w, is written in terms of the elements of the **Q** matrix:

$$r_w=-\frac{q_{21}(\beta)}{q_{22}(\beta)}. \tag{10}$$

Given that β_g is the propagation constant of a proper waveguide mode, $q_{22}(\beta_g)=0$ and β_g is a simple pole whose residue is

$$\mathrm{Res}(\beta_g)=-\frac{q_{21}(\beta_g)}{q_{22}{}'(\beta_g)}. \tag{11}$$

Now by assuming that β_0 is the nearest zero of r_w to β_g, we have $q_{21}(\beta_0)=0$, and thus the first-order Taylor series approximation of $q_{21}(\beta)$ calculated around β_g and evaluated at β_o yields the following equation:

$$\beta_g-\beta_0=\frac{q_{21}(\beta_g)}{q_{21}{}'(\beta_g)}. \tag{12}$$

Interestingly, the right-hand side of Eq. (12) is the difference between propagation constants of proper and improper modes. Furthermore, the elements of the **T** matrix can be written as

$$\frac{t_{11}}{t_{21}}=-r_{\mathrm{pc}}^{-1}\exp(-2\kappa_c d_g), \tag{13a}$$

$$\frac{t_{12}}{t_{22}}=-r_{\mathrm{pc}}\exp(-2\kappa_c d_g), \tag{13b}$$

where r_{pc} denotes the Fresnel reflection coefficient at the interface of the prism and cover, when there is no waveguide in the structure.

It is now possible to apply Eqs. (11)–(13) in Eqs. (7) and approximate the zero and pole of the overall reflection coefficients in terms of d_g, of the x component of the wave vector in the cover region calculated at β_g, i.e., $\kappa_c(\beta_g)$, of proper and improper mode propagation constants in the waveguide with no prism, i.e., β_g and β_0, and of the residue of the reflection coefficient at the cover-waveguide interface:

$$\tilde{\beta}_z=\beta_g-\frac{\mathrm{Res}(\beta_g)\exp(-2\kappa_c(\beta_g)d_g)}{r_{\mathrm{pc}}(\beta_g)+\mathrm{Res}(\beta_g)/(\beta_g-\beta_0)\exp(-2\kappa_c(\beta_g)d_g)}, \tag{14a}$$

$$\tilde{\beta}_p=\beta_g-\frac{\mathrm{Res}(\beta_g)r_{\mathrm{pc}}(\beta_g)\exp(-2\kappa_c(\beta_g)d_g)}{1+r_{\mathrm{pc}}(\beta_g)\mathrm{Res}(\beta_g)/(\beta_g-\beta_0)\exp(-2\kappa_c(\beta_g)d_g)}. \tag{14b}$$

In obtaining these expressions, $r_{\mathrm{pc}}(\beta)$ and $\kappa_c(\beta)$ have been, respectively, replaced by $r_{\mathrm{pc}}(\beta_g)$ and $\kappa_c(\beta_g)$, and

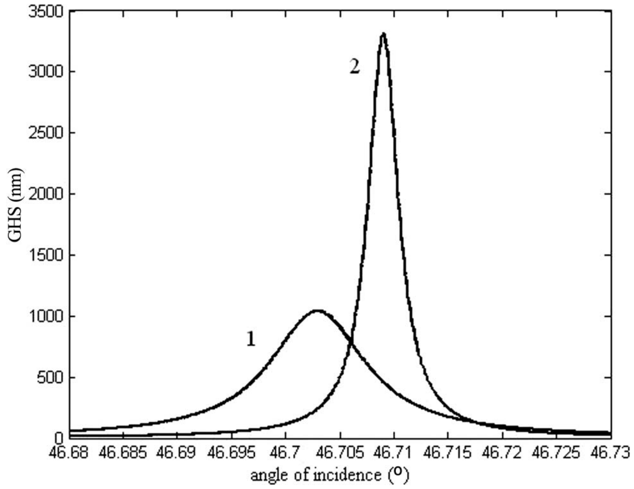

Fig. 1. GHS in nanometers versus angle of incidence for the lossless structure with (curve 1) $d_g = 0.4\lambda_0$ and (2) $d_g = 0.55\lambda_0$. Artmann's formula (solid curve) and the proposed approximation using β_p and β_z given in Eqs. (14) (dashed curve).

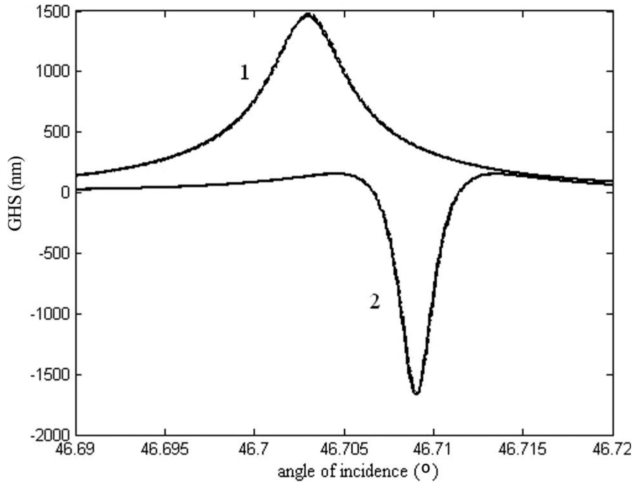

Fig. 2. GHS in nanometers versus angle of incidence for the lossy structure with (curve 1) $d_g = 0.4\lambda_0$ and (2) $d_g = 0.55\lambda_0$. Artmann's formula (solid curve) and the proposed approximation using β_p and β_z given in Eq. (14) (dashed curve).

also the derivatives of $t_{11}(\beta)$ and $t_{12}(\beta)$ with respect to spatial frequency β have been neglected. These simplifications, valid at the vicinity of singularity point β_g, do not cause a significant error because $r_{\mathrm{pc}}(\beta)$, $\kappa_c(\beta)$, $t_{11}(\beta)$, and $t_{12}(\beta)$ are all slowly varying functions of the spatial frequency β. It is possible to derive these expressions by using the zero-pole approximation of the waveguide reflection coefficient, r_w. The approximate expressions in Eqs. (14), and consequently the GHS obtained by applying them to Eq. (9), are therefore as valid as the zero-pole approximation of the reflection coefficient at the cover-waveguide region.

It is also easy to show that the approximate expressions for β_z and β_p in Eqs. (14) can be further simplified to

$$\tilde{\beta}_z = \beta_g - \mathrm{Res}(\beta_g) \times r_{\mathrm{pc}}^{-1}(\beta_g) \exp(-2\kappa_c(\beta_g) d_g), \tag{15a}$$

$$\tilde{\beta}_p = \beta_g - \mathrm{Res}(\beta_g) \times r_{\mathrm{pc}}(\beta_g) \exp(-2\kappa_c(\beta_g) d_g), \tag{15b}$$

for the weak coupling regime, when $|\exp(-2\kappa_c(\beta_g)d_g) /(\beta_g - \beta_o)| = \delta \ll 1$. The approximate expressions in Eqs. (15) are equivalent to the single pole approximation of the waveguide reflection coefficient. The presence of the improper mode, coinciding with the zero of the reflection coefficient, is therefore neglected in the weak coupling regime. These expressions show that the proximity of pole, β_g, and zero, β_0, in determining the coupling strength of the waveguide structure and prism is as essential as the gap width d_g.

As an example, a monochromatic S polarized wave is incident upon a graded refractive index profile:

$$n(x) = \left[n_f^2 - (n_f^2 - n_s^2) \frac{x}{d} \right]^{1/2}, \tag{16}$$

where $n_f = 2.210$, and $n_s = 2.177$, the normalized frequency is $V = 3.78$, and the asymmetry parameter is $a = 2.27$. A high refractive index prism with $n_p = 3$ is placed over the waveguide structure, leaving a gap of d_g between prism–cover and cover–waveguide interfaces. Two different gap widths $d_g = 0.4\lambda_0$ and $d_g = 0.55\lambda_0$ are considered, and the GHS for wide enough incident beams is plotted versus angle of incidence in Fig. 1. The proposed approximation coincides with Artmann's formula. A slight amount of loss is then added to the waveguide, and $n_f = 2.210$ is changed to $n_f = 2.210 - 0.0002j$. The GHS is plotted versus angle of incidence in Fig. 2. Once again, the applicability of the proposed expressions is shown. Figure 2 shows that the presented approximations are still valid when the waveguide is lossy and β_g is not a real number.

These examples show that the proposed approximations are as accurate as Artmann's formula for the near resonant GHS. They, however, provide a good physical insight by relating the GHS to the leaky mode characteristics and to the coupling parameters. They also enable us to perceive two different coupling regimes: weak and strong.

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