Advanced Engineering Mathematics

Session 2

Linear Space

- A linear space over the field F, is a non-empty set S of vectors with a binary operation *addition* + : SxS->S and a *scalar multiplication* . : FxS->S such that:
  - (S,+) is an Abeliab group.
  - $(\alpha+\beta).x=\alpha.x+\beta.x$, $\alpha.(x+y)=\alpha.x+\alpha.y$, $\alpha.(\beta.x)=(\alpha\beta).x$, $1.x=x$

    (x and y are vectors whereas α,β are scalars)

Linear Space

- The set F^n of n-tuples of scalars form a linear space over F.
- The set F^x of functions f: X->F is a linear space, where x is a non-empty set and
  - $(f_1+f_2)(x)=f_1(x)+f_2(x)$,
  - $(\alpha f)(x)=\alpha f(x)$

$\Rightarrow$Let G be an open subset in R^n then the set C(G,R) of continuous functions f: X->R is a linear space.

Linear Independence

- Why?
- What?
  - The vectors are *linearly dependent* if there exists a *linear combination* of the vectors- with at least one nonzero coefficients α_k, that renders the zero vector.
  - Sinusoidal functions in C(0,1) are linearly independent. Why?

Dimension

- A measure of how many and what sort of vectors describe the space.
- A vector space S has dimension *n*, if:
  - It possesses a set of *n* independent vectors,
  - Every set of (*n*+1) vectors is dependent.

- If for every positive integer *k*, one can find *k* independent vectors in S, then S has infinite dimension.

Basis

- A certain set inside the vector space S, forms a basis provided that:
  - The vectors in the set be linearly independent.
  - Every arbitrary vector in S can be written as a linear combination of basis elements.

- Representation of each vector within S is unique in terms of the given basis.

Inner Product Space

- A linear space S is a *complex inner product space*, if for every ordered pair (x,y) of vectors; i.e. in SxS, there exists a unique scalar $<x,y>$ such that:
  - $<x,y>=<y,x>^*$
  - $<x+y,z>=<x,z>+<y,z>$
  - $<\alpha x,y>=\alpha<x,y>$
  - $<x,x> \;>=\; 0$, with equality iff $x=0$

Inner Product Space

- A *real inner product space* can be similarly defined.

- Cauchy-Schwarz-Bunjakowsky inequality reads as:

  $|\langle x,y\rangle| \le (\langle x,x\rangle)^{1/2} (\langle y,y\rangle)^{1/2}$

  Why?

Orthogonality

- Two vectors are orthogonal if their inner product is zero.
- A set of vectors is orthogonal if all members are mutually orthogonal.
- *Proper* orthogonal set is composed of linearly independent vectors.
- Orthonormal set.

Normed Linear Space

- A linear space S is a *normed linear space* if for every vector x in S, there exists a unique real number $||x||$ such that:
  - $||x|| >= 0$, with equality iff $x = 0$
  - $||\alpha x|| = |\alpha| \, ||x||$
  - $||x+y|| <= ||x|| + ||y||$ (triangle inequality)

Normed Linear Space

- A norm can be induced by the inner product.
- Why?

- Norm can be used to measure ‘closedness’, how? Why? What is it good for?

Normed Linear Space

- Convergence
- Cauchy sequence
- Completeness

- $C(\alpha,\beta)$ is incomplete, why?