Image Restoration

- Image Degradation Model (Linear/Additive)

\[
g(x, y) = h(x, y) \ast f(x, y) + \eta(x, y)
\]

\[
G(u, v) = H(u, v) \cdot F(u, v) + N(u, v)
\]
Medical Image Analysis and Processing

Image Restoration

- **Source of noise**
  - Image acquisition (digitization)
  - Image transmission

- **Spatial properties of noise**
  - Statistical behavior of the gray-level values of pixels
  - Noise parameters, correlation with the image

- **Frequency properties of noise**
  - Fourier spectrum
  - Ex. white noise (a constant Fourier spectrum)
Image Restoration

- Noise Model

\[ p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(z-\mu)^2}{2\sigma^2} \right) \]

\[ p(z) = \frac{a^b z^{b-1}}{(b-1)!} e^{-az} u(z) \]

\[ p(z) = \frac{1}{b-a} (u(z-a) - u(z-b)) \]

\[ p(z) = 2 \frac{(z-a)}{b} \exp \left( -\frac{(z-a)^2}{b} \right) u(z-a) \]

\[ p(z) = 2 \frac{(z-a)}{b} \exp \left( -\frac{(z-a)^2}{b} \right) u(z-a) \]

\[ p(z) = 1 \frac{(z-a)}{b-a} \]

\[ p(z) = P_a \delta(z-a) + P_b \delta(z-b) \]

E. Fatemizadeh, Sharif University of Technology, 2011
Image Restoration

• Test Pattern
  – Histogram has three Spikes!

FIGURE 5.3 Test pattern used to illustrate the characteristics of the noise PDFs shown in Fig. 5.2.
Image Restoration

- Noisy Images
  - Gaussian
  - Rayleigh
  - Gamma

**FIGURE 5.4** Images and histograms resulting from adding Gaussian, Rayleigh, and gamma noise to the image in Fig. 5.3.
Image Restoration

- Noisy Images
  - Exponential
  - Uniform
  - Salt & Pepper

FIGURE 5.4 (Continued) Images and histograms resulting from adding exponential, uniform, and impulse noise to the image in Fig. 5.3.
Medical Image Analysis and Processing

Image Restoration

- Periodic Noise:
  - Electronic Devices

FIGURE 5.5
(a) Image corrupted by sinusoidal noise.
(b) Spectrum (each pair of conjugate impulses corresponds to one sine wave).
(Original image courtesy of NASA.)
Image Restoration

• Periodic noise
  – Observe the frequency spectrum

• Random noise with PDFs
  – Case 1: imaging system is available
    • Capture images of “flat” environment
  – Case 2: noisy images available
    • Take a strip from constant area
    • Draw the histogram and observe it
    • Measure the mean and variance
Image Restoration

- Medical Example:
  - MRI Artifact:
    - Phantom:

Phantom  Gibbs  Noise
Medical Image Analysis and Processing

Image Restoration

• Medical Example:
  – CT Metal Artifact:
Image Restoration

- Noise Estimation:
  - Shape: Histogram of a subimage (Background)

**FIGURE 5.6** Histograms computed using small strips (shown as inserts) from (a) the Gaussian, (b) the Rayleigh, and (c) the uniform noisy images in Fig. 5.4.
Image Restoration

- **Noise-only spatial filter:**
  \[ g(x,y) = f(x,y) + \eta(x,y) \]

- **Adaptive, local noise reduction:**
  - If \( \sigma^2_\eta \) is small, return \( g(x,y) \)
  - If \( \sigma^2_L \gg \sigma^2_\eta \), return value close to \( g(x,y) \)
  - If \( \sigma^2_L \approx \sigma^2_\eta \), return the arithmetic mean \( m_L \)

\[
\hat{f}(x, y) = g(x, y) - \frac{\sigma^2_\eta}{\sigma^2_L}(g(x, y) - m_L)
\]
Medical Image Analysis and Processing

Image Restoration

**Example:**

**FIGURE 5.13**
(a) Image corrupted by additive Gaussian noise of zero mean and variance 1000.
(b) Result of arithmetic mean filtering.
(c) Result of geometric mean filtering.
(d) Result of adaptive noise reduction filtering. All filters were of size $7 \times 7$. 

Original Noisy  A- Mean  G- Mean  Local
Image Restoration

- **Linear Degradation:**

  \[
  g(x, y) = H[f(x, y)] + \eta(x, y)
  \]

  \[
  L\text{-System:} \quad \begin{cases}
  H[f(x, y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha, \beta)H[\delta(x-\alpha, y-\beta)] \, d\alpha \, d\beta \\
  h(x, y, \alpha, \beta) = H[\delta(x-\alpha, y-\beta)]
  \end{cases}
  \]

  \[
  LSI\text{-System:} \quad \begin{cases}
  H[f(x, y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha, \beta)h(x-\alpha, y-\beta) \, d\alpha \, d\beta \\
  g(x, y) = f(x, y) * h(x, y) + \eta(x, y) \\
  G(u, v) = F(u, v)H(u, v) + N(u, v)
  \end{cases}
  \]
Image Restoration

• Degradation Estimation:
  – Image Observation:
    • Look at the image and
  – Experiments:
    • Acquire image using well defined object (Flat, pinhole, and etc.)
  – Modeling:
    • Introduce certain model for certain degradation using physical knowledge.
Image Restoration

• Degradation (Using Observation/PSF)

\[ H_s(u,v) = \frac{H_s(u,v)}{\hat{F_s}(u,v)} \]

\[ H_s(u,v) = \frac{G(u,v)}{A_{PSF}} \]

FIGURE 5.24
Degradation estimation by impulse characterization.
(a) An impulse of light (shown magnified).
(b) Imaged (degraded) impulse.
Image Restoration

• Atmospheric Turbulence:
Image Restoration

- Modeling of turbulence in atmospheric images:

\[ H(u, v) = \exp \left( -k \left( u^2 + v^2 \right)^{-5/6} \right) \]
Image Restoration

- Motion Blurring Modeling:

\[
g(x, y) = \int_{0}^{T} f(x - x_0(t), y - y_0(t))dt
\]

\[
G(u, v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( \int_{0}^{T} f(x - x_0(t), y - y_0(t))dt \right) e^{-j2\pi(ux + vy)} dxdy
\]

\[
G(u, v) = F(u, v) \int_{0}^{T} e^{-j2\pi(u\dot{x_0}(t) + v\dot{y_0}(t))} dt = F(u, v) H(u, v)
\]
Medical Image Analysis and Processing

Image Restoration

• Linear one/Two dimensional motion blurring:

\[ x_0(t) = at/T , \quad t_{\text{Max}} = T \implies H(u,v) = \frac{T}{\pi u a} \sin(\pi u a) e^{-j \pi u a} \]

\[ x_0(t) = at/T , \quad y_0(t) = bt/T \implies \]

\[ H(u,v) = \frac{T}{\pi(u a + v b) \sin(\pi(u a + v b))} e^{-j \pi(u a + v b)} \]
Image Restoration

- Motion Blurring Example:
Image Restoration

- MR Motion Artifact:
Image Restoration

• Motion Blurring Discrete Modeling:

\[ \frac{1}{9} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \frac{1}{25} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \frac{1}{49} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \]
Image Restoration

• Inverse Filtering:
  – Without Noise:

\[ \hat{F}(u,v) = \frac{G(u,v)}{\hat{H}(u,v)} = \frac{F(u,v)H(u,v)}{\hat{H}(u,v)} \approx F(u,v) \]

⇒ Problem of division by zero!
Image Restoration

• Inverse Filtering:
  – With Noise:

\[
\hat{F}(u,v) = \frac{G(u,v)}{\hat{H}(u,v)} = \frac{F(u,v)H(u,v) + N(u,v)}{\hat{H}(u,v)} \approx F(u,v) + \frac{N(u,v)}{\hat{H}(u,v)}
\]

⇒ Problem of division by zero!
⇒ Impossible to recover even if \(H(.,.)\) is known!!
Image Restoration

- **Pseudo Inverse (Constrained) Filtering:**
  - Set infinite (large) value to zero;
  - Multiply $H(u,v)$ by a I/G/B LPF

$$
\hat{F}(u,v) = \begin{cases} 
\frac{G(u,v)}{\hat{H}(u,v)} & \hat{H}(u,v) \geq H_{THR} \\
\frac{G(u,v)}{H_{THR}} & \hat{H}(u,v) < H_{THR}
\end{cases} \quad H_{THR} \geq 0
$$

$$
\hat{F}(u,v) = \begin{cases} 
\frac{G(u,v)}{\hat{H}(u,v)} & \hat{H}(u,v) \geq H_{THR} \\
T & \hat{H}(u,v) < H_{THR}
\end{cases}
$$
Image Restoration

**FIGURE 5.27**
Restoring Fig. 5.25(b) with Eq. (5.7-1).
(a) Result of using the full filter. (b) Result with $H$ cut off outside a radius of 40; (c) outside a radius of 70; and (d) outside a radius of 85.
Image Restoration

• Phase Problem:
  – Look at this formulation:

\[
\hat{F}(u, v) = G(u, v) \begin{cases} 
\frac{1}{\hat{H}(u, v)} & \left| \hat{H}(u, v) \right| \geq H_{THR} \\
\frac{1}{H_{THR}} & \left| \hat{H}(u, v) \right| < H_{THR}
\end{cases}
\]

  – *We preserve the Correct Phase!*
Image Restoration

• Phase Problem:
Image Restoration

• Wiener Filtering:

\[ g(x, y) = s(x, y) + \eta(x, y) \iff G(u, v) = S(u, v) + N(u, v) \]

\[ \hat{F}(u, v) = W(u, v) \cdot G(u, v) \]

\[ E(u, v) = F(u, v) - \hat{F}(u, v) = F(u, v) - W(u, v) \cdot G(u, v) \]

\[ E\left\{ |E(u, v)|^2 \right\} = E \left\{ (F - WG)(F - WG)^* \right\} \]
Image Restoration

- Wiener Filtering in 2D case:

\[
E \left\{ \left| E(u, v) \right|^2 \right\} = P_{FF} + WP_{GG} W^* - W^* P_{FG} - WP_{GF}
\]

\[
\frac{\partial E \left\{ \left| E(u, v) \right|^2 \right\}}{\partial W} = 0 \Rightarrow W(u, v) = \frac{P_{FG}(u, v)}{P_{GG}(u, v)}
\]

\[
P_{XX}(u, v) = E \left\{ \left| X \right|^2 \right\} : \text{Spectral Estimation}
\]

\[
P_{XY}(u, v) = E \left\{ XY^* \right\} : \text{Cross Spectral Estimation}
\]

\[
P_{XY}(u, v) = P_{YX}^*(u, v)
\]
Image Restoration

- Wiener Filtering in 2D case:
  - Special Cases:
    - Noise Only:

\[
g(x, y) = f(x, y) + \eta(x, y) \Leftrightarrow G(u, v) = F(u, v) + N(u, v)
\]

Uncorrelated Noise and Image:

\[
W(u, v) = \frac{P_{FF}(u, v)}{P_{FF}(u, v) + P_{NN}(u, v)}
\]
Image Restoration

- Degradation plus Noise:
  \[ g(x, y) = f(x, y) * h(x, y) + \eta(x, y) \]
  \[ G(u, \nu) = F(u, \nu) H(u, \nu) + N(u, \nu) \]

Uncorrelated Noise and Image:

\[
W(u, \nu) = \frac{P_{FF} H^*}{P_{FF} |H|^2 + P_{NN}} = \frac{H^*}{|H|^2 + \frac{P_{NN}}{P_{FF}}}
\]

\[
= \frac{1}{H} \frac{|H|^2}{|H|^2 + \frac{P_{NN}}{P_{FF}}} = \frac{1}{H} \frac{|H|^2}{|H|^2 + \left(\frac{P_{FF}}{P_{NN}}\right)^{-1}}
\]
Image Restoration

- Degradation plus Noise:
  - White Noise

\[
\frac{1}{H} \left| H \right|^2 + \frac{\left| H \right|^2}{K} \quad \text{Select Interactively}
\]
Image Restoration

• Wiener Filter is known as:
  – Wiener-Hopf
  – Minimum Mean Square Error
  – Least Square Error

• Problems with Wiener:
  – $P_{FF}$
  – $P_{NN}$
Image Restoration

- Phase in Wiener Filter:

\[ W = \frac{1}{H} \frac{|H|^2}{|H|^2 + \frac{P_{NN}}{P_{FF}}} \Rightarrow \left\{ \begin{array}{l}
|W| = \frac{|H|}{|H|^2 + \frac{P_{NN}}{P_{FF}}} \\
\angle W = -\angle H = \angle \left( \frac{1}{H} \right) \end{array} \right. \]

- No Phase compensation!
Image Restoration

- Wiener Filter vs. Inverse Filter:

\[ W = \frac{H^*}{|H|^2 + \frac{P_{NN}}{P_{FF}}} \Rightarrow \lim_{P_{NN} \to 0} W = \frac{H^*}{|H|^2} = \begin{cases} 
\frac{1}{H} & H \neq 0 \\
0 & H = 0 
\end{cases} \]
Medical Image Analysis and Processing

Image Restoration

**Full Inverse**  **Pseudo Inverse**  **Wiener**

![Image Restoration Results](image.png)

**FIGURE 5.28** Comparison of inverse- and Wiener filtering. (a) Result of full inverse filtering of Fig. 5.25(b). (b) Radially limited inverse filter result. (c) Wiener filter result.
Image Restoration

Inverse

Motion Blurring + Noise

Wiener

Noise Decrease

FIGURE 5.29 (a) Image corrupted by motion blur and additive noise. (b) Result of inverse filtering. (c) Result of Wiener filtering. (d)-(f) Same sequence, but with noise variance one order of magnitude less. (g)-(i) Same sequence, but noise variance reduced by five orders of magnitude from (a). Note in (b) how the deblurred image is quite visible through a “curtain” of noise.
Iterative Wiener Filter:

- We formulate for noise-only case:

0. \( i = 0 \)

1. \( P_{FF}^{(i)} = P_{GG} \)

2. \( W^{(i+1)} = \frac{P_{FF}^{(i)}}{P_{FF}^{(i)} + P_{NN}} \)

3. \( F^{(i+1)} = W^{(i+1)} G^{(i+1)} \)

4. \( P_{FF}^{(i+1)} = E \left\{ \left| F^{(i+1)} \right|^2 \right\} \)

5. Repeat 2, 3, 4 until convergence.
Image Restoration

• Adaptive Wiener Filter:
  – Image are Non Stationary!
  – Need Adaptive WF which is locally optimal.
  – Assume small region which image are stationary

• Image Model in each region:
  \[ f(x, y) = \mu_f(x, y) + \sigma_f \eta(x, y) \]
  \( \eta \) : zero-mean white noise with unit variance!
  \( \mu_f, \sigma_f \) : Constant over each region.

• Noise Image:
  \[ g(x, y) = f(x, y) + v(x, y), \quad \sigma_v : \text{Constant over each region} \]
Image Restoration

- Local Wiener Filter in each region:

\[
W_a(u, v) = \frac{P_{ff}}{P_{ff} + P_{vv}} = \frac{\sigma_f^2}{\sigma_f^2 + \sigma_v^2}
\]

\[
w_a(x, y) = \frac{\sigma_f^2}{\sigma_f^2 + \sigma_v^2} \delta(x, y)
\]

\[
\hat{f}(x, y) = (g(x, y) - \mu_f) \ast w_a(x, y) + \mu_f
\]

\[
\hat{f}(x, y) = (g(x, y) - \mu_f) \frac{\sigma_f^2}{\sigma_f^2 + \sigma_v^2} + \mu_f
\]

\[\mu_f(x, y): \text{Low-pass filtered on noisy image.}\]

\[\mu_f(x, y) = \mu_g(x, y), \quad \text{Zero mean assumption}\]

\[g(x, y) - \mu_f(x, y): \text{Hi-pass filtered on noisy image.}\]

\[
\therefore \hat{f}(x, y) = HP(x, y) \frac{\sigma_f^2(x, y)}{\sigma_f^2(x, y) + \sigma_v^2} + LP(x, y)
\]
Image Restoration

- **Parameter Estimation:**

  \[
  \sigma^2_g = \text{Local Noisy Image Variance}
  \]

  \[
  \sigma^2_v = \text{Variance in a smooth image region or background}
  \]

  \[
  \sigma^2_f(x, y) = \sigma^2_g(x, y) - \sigma^2_v
  \]
Image Restoration

• Results:

Example simulations

noiseless image \( f(m,n) \)

noise (variance = 100) \( v(m,n) \)

noisy signal \( g(m,n) \)
Image Restoration

Results:

generation of the adaptive Wiener Filter

\[
\frac{\sigma_f^2(m, n)}{\sigma_f^2(m, n) + \sigma_v^2}
\]
Image Restoration

• Results:

\[
LP(m, n) + \frac{\sigma_f^2(m, n)}{\sigma_f^2(m, n) + \sigma_v^2} HP(m, n)
\]

\[
HP(m, n)
\]

\[
LP(m, n)
\]
Image Restoration

Results:

Another example with higher noise (variance = 225)

\[
g(m, n) = LP(m, n) + \frac{\sigma_f^2(m, n)}{\sigma_f^2(m, n) + \sigma_v^2} HP(m, n)
\]
Image Restoration

- Matlab Image Restoration Command:
  - `deconvblind`: Restore image using blind deconvolution
  - `deconvlucy`: Restore image using accelerated Richardson-Lucy algorithm
  - `deconvreg`: Restore image using Regularized filter
  - `deconvwnr`: Restore image using Wiener filter
  - `wiener2`: Perform 2-D adaptive noise-removal filtering
  - `edgetaper`: Taper the discontinuities along the image edges