# Analog Modulations

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# Overview

# Introduction to Modulation

# Message and Carrier

## Statement (Message)

*The message signal $m(t)$ is a real lowpass signal of bandwidth $W$ and power $P_m$, i.e.*

$$M(f) = 0, \quad |f| > W; \qquad P_m = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |m(t)|^2 dt$$

.

## Statement (Carrier)

*Carrier is a sinusoidal signal of the form $c(t) = A_c \cos(2\pi f_c t + \phi_c)$ with $f_c \gg W$.*

# Definition of Modulation

## Statement (Modulation)

*In modulation, the message signal $m(t)$ modulates the carrier signal $c(t)$ to generate the modulated signal $u(t)$ such that a feature of the carrier becomes a function of the message signal.*

## Statement (Amplitude Modulation)

*In amplitude modulation, the amplitude of the carrier is a function of the message as*

$$u(t) = f(m(t))\cos(2\pi f_c t + \phi_c)$$

## Statement (Angle Modulation)

*In angle modulation, the angle of the carrier is a function of the message as*

$$u(t) = A_c\cos(2\pi f_c t + f(m(t)))$$

# Advantages of Modulation

Modulation is performed to achieve,

1. To translate the frequency of the lowpass signal to the passband of the channel.
2. To simplify the structure of the transceiver by employing higher frequencies.
3. To accommodate for the simultaneous transmission of signals from several message sources, by means of multiplexing mechanisms.
4. To expand the bandwidth of the transmitted signal in order to increase its noise and interference immunity.

# Types of Modulation

Different analog modulation methods are,

1. Amplitude modulation
    1. Double-sideband (DSB)
    2. Conventional amplitude modulation (AM)
    3. Single-sideband (SSB)
    4. Vestigial-sideband (VSB)
2. Angle modulation
    1. Frequency modulation (FM)
    2. Phase modulation (PM)

# Performance of Modulation

The performance of the modulation is measured by,

1. Required bandwidth
2. Transmitted power
3. Transceiver complexity
4. Impairment immunity

✓ The immunity to AWGN noise, as a common impairment, is measured by Signal to Noise Ratio (SNR) at the output of the demodulator.

# Double Sideband Modulation

# DSB Modulation

## Statement (DSB)

*A DSB signal $u(t)$ is obtained by*

$$u(t) = m(t)c(t) = A_c m(t)\cos(2\pi f_c t)$$

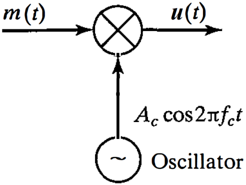


Figure: Block diagram of the DSB modulator.

# DSB Modulation

## Example (DSB signal)

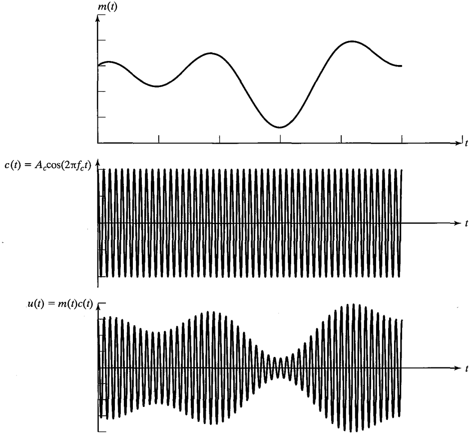


Figure: Examples of message, carrier, and DSB-modulated signals.

# Bandwidth of DSB Signal

## Statement (Spectrum of DSB Signal)

*The spectrum of the DSB modulated signal is*

$$U(f) = \frac{A_c}{2}\left[M(f - f_c) + M(f + f_c)\right]$$

## Statement (Bandwidth of DSB Signal)

*For a message signal having the bandwidth $W$, the corresponding DSB signal requires a bandwidth of $2W$.*

## Example (DSB spectrum)

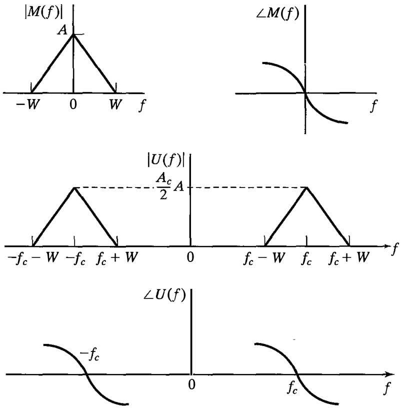


Figure: Spectrum of a message signal and its corresponding DSB-modulated signal.

# Bandwidth of DSB Signal

1. The frequency content of the DSB signal in the frequency band $|f| > f_c$ is called the upper sideband.
2. The frequency content of the DSB signal in the frequency band $|f| < f_c$ is called the lower sideband.
3. Either one of the DSB signal contains all the frequencies that are in the message.
4. Since the DSB signal contains both the upper and the lower sidebands, it is called a double-sideband signal.

# Bandwidth of DSB Signal

## Example (Sinusoidally-modulated DSB)

If $m(t) = a\cos(2\pi f_m t), \quad f_m \ll f_c$, the DSB signal is expressed in the time domain as

$$\begin{aligned} u(t) = m(t)c(t) &= aA_c \cos(2\pi f_m t)\cos(2\pi f_c t) \\ &= \frac{aA_c}{2}\cos(2\pi(f_c - f_m)t) + \frac{aA_c}{2}\cos(2\pi(f_c + f_m)t) \end{aligned}$$

## Example (Sinusoidally-modulated DSB)

If $m(t) = a\cos(2\pi f_m t), \quad f_m \ll f_c$, the DSB signal is expressed in the frequency domain as

$$\begin{aligned} U(f) = &\frac{aA_c}{4}\left[\delta(f - f_c + f_m) + \delta(f + f_c - f_m)\right] \\ &+ \frac{aA_c}{4}\left[\delta(f - f_c - f_m) + \delta(f + f_c + f_m)\right] \end{aligned}$$

# Bandwidth of DSB Signal

## Example (Sinusoidally-modulated DSB (cont.))

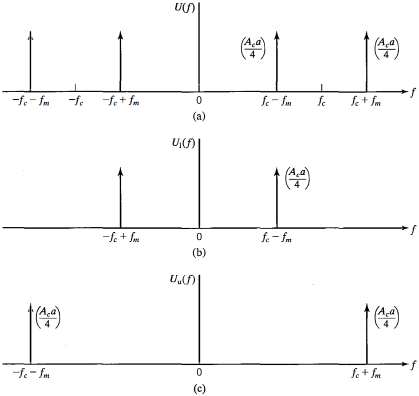


Figure: The (magnitude) spectrum of a DSB signal for (a) a sinusoidal message signal and (b) its lower and (c) upper sidebands.

**Example (Sinusoidally-modulated DSB (cont.))**

If $m(t) = a\cos(2\pi f_m t), \quad f_m \ll f_c$, $u(t) = u_l(t) + u_u(t)$, where the lower and upper sideband correspond to the signals

$$u_l(t) = \frac{aA_c}{2}\cos(2\pi(f_c - f_m)t)$$
$$u_u(t) = \frac{aA_c}{2}\cos(2\pi(f_c + f_m)t)$$

# Power of DSB Signal

## Statement (Power of DSB signal)

*The power content of the DSB signal equals* $P_u = \frac{A_c^2}{2} P_m$.

$$
\begin{aligned}
P_u &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} u^2(t) dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A_c^2 m^2(t) \cos^2(2\pi f_c t) dt \\
&= \frac{A_c^2}{2} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} m^2(t)(1 + \cos(4\pi f_c t)) dt \\
&= \frac{A_c^2}{2} P_m + \frac{A_c^2}{2} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} m^2(t) \cos(4\pi f_c t) dt \\
&= \frac{A_c^2}{2} P_m
\end{aligned}
$$

# Power of DSB Signal

## Example (Power of sinusoidally-modulated DSB)

If $m(t) = a\cos(2\pi f_m t), \quad f_m \ll f_c$, then

$$P_m = \frac{a^2}{2}$$

,

$$P_u = \frac{A_c^2}{2} P_m = \frac{a^2 A_c^2}{4}$$

, and

$$P_{u_l} = P_{u_u} = \frac{a^2 A_c^2}{8}$$

.

# DSB Demodulation

## Statement (DSB Demodulation)

*Suppose that the DSB signal $u(t)$ is transmitted through an ideal channel. Then, the received signal is $r(t) = u(t)$. The message can be demodulated by*

$$\widetilde{m}(t) = \frac{A_c}{2} m(t) \cos(\phi) = LPF\{r(t) \cos(2\pi f_c t + \phi)\}$$

*, where $\cos(2\pi f_c t + \phi)$ is a locally generated sinusoid and the ideal lowpass filter has the bandwidth $W$.*

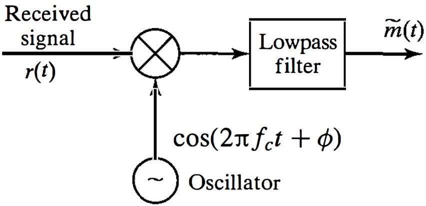


Figure: Block diagram of the basic DSB demodulator.

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$$\begin{aligned} r(t) \cos(2\pi f_c t + \phi) &= A_c m(t) \cos(2\pi f_c t) \cos(2\pi f_c t + \phi) \\ &= \frac{A_c}{2} m(t) \cos(\phi) + \frac{A_c}{2} m(t) \cos(4\pi f_c t + \phi) \end{aligned}$$

$$\widetilde{m}(t) = \text{LPF}\{r(t) \cos(2\pi f_c t + \phi)\} = \frac{A_c}{2} m(t) \cos(\phi)$$

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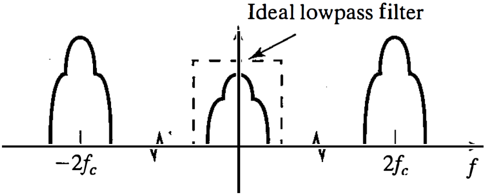


Figure: Frequency-domain representation of the DSB demodulation.

# DSB Demodulation

1. The power in the demodulated signal is decreased by a factor of $\cos^2(\phi)$.
2. If $\phi = 90^\circ$, the desired signal component vanishes.
3. A phase-coherent or synchronous demodulator is needed for recovering the message signal.
4. A synchronous demodulator uses a pilot tone or phase-locked loop (PLL) to lock to the phase of the carrier.
5. Since the process of modulation/demodulation involves the generation of new frequency components, modulators/demodulators are generally characterized as nonlinear and/or time-variant systems.

# Power-Law Amplitude Modulator

Power-law amplitude modulator exploits then voltage-current characteristic of a nonlinear device such as PN diode, which can be approximated as

$$v_o(t) = a_1 v_i(t) + a_2 v_i^2(t)$$

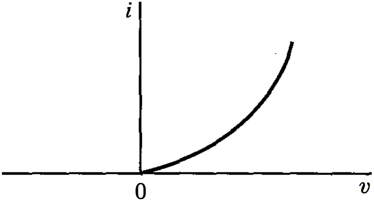


Figure: Voltage-current characteristic of aPN diode.

# Power-Law Amplitude Modulator

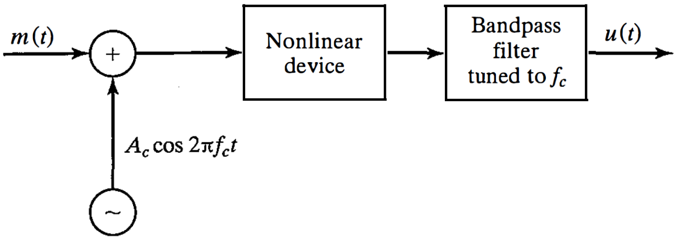


Figure: Block diagram of power-law amplitude modulator.

$$v_i(t) = m(t) + A_c \cos(2\pi f_c t)$$

$$v_o(t) = a_1[m(t) + A_c \cos(2\pi f_c t)] + a_2[m(t) + A_c \cos(2\pi f_c t)]^2$$

$$v_o(t) = a_1 m(t) + a_2 m^2(t) + a_2 A_c^2 \cos^2(2\pi f_c t) + A_c a_1[1 + \frac{2a_2}{a_1} m(t)] \cos(2\pi f_c t)$$

# Power-Law Amplitude Modulator

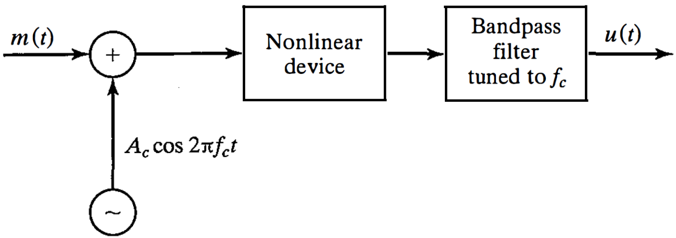


Figure: Block diagram of power-law amplitude modulator.

$$v_o(t) = a_1 m(t) + a_2 m^2(t) + a_2 A_c^2 \cos^2(2\pi f_c t) + A_c a_1 [1 + \frac{2a_2}{a_1} m(t)] \cos(2\pi f_c t)$$

Applying the bandpass filter with a bandwidth $2W$ centered at $f = f_c$

$$u(t) = A_c a_1 [1 + \frac{2a_2}{a_1} m(t)] \cos(2\pi f_c t)$$

, where $-1 < \frac{2a_2}{a_1} m(t) < 1$ by design.

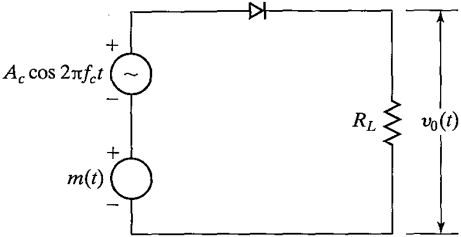


Figure: Schematic of switching amplitude modulator.

Assuming $A_c \gg |m(t)|$,

$$v_o(t) = \begin{cases} A_c \cos(2\pi f_c t) + m(t), & A_c \cos(2\pi f_c t) \geq 0 \\ 0, & A_c \cos(2\pi f_c t) < 0 \end{cases}$$

# Switching Amplitude Modulator

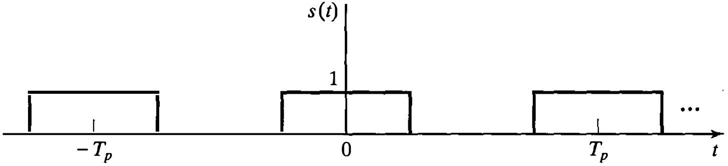


Figure: Equivalent periodic switching signal in switching amplitude modulator.

Assuming $A_c \gg |m(t)|$,

$$v_o(t) = \left[A_c \cos(2\pi f_c t) + m(t)\right] s(t)$$

$$v_o(t) = \left[A_c \cos(2\pi f_c t) + m(t)\right] \left[\frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos(2\pi f_c (2n-1) t)\right]$$

# Switching Amplitude Modulator

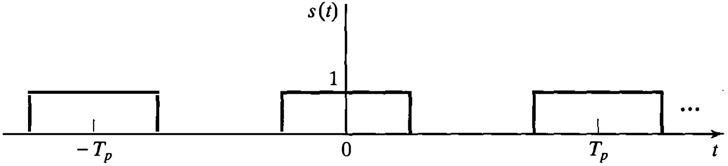


Figure: Equivalent periodic switching signal in switching amplitude modulator.

Assuming $A_c \gg |m(t)|$,

$$v_o(t) = \frac{A_c}{2}\left[1 + \frac{4}{\pi A_c} m(t)\right] \cos(2\pi f_c t) + \text{other terms}$$

Applying the bandpass filter with a bandwidth $2W$ centered at $f = f_c$

$$u(t) = \frac{A_c}{2}\left[1 + \frac{4}{\pi A_c} m(t)\right] \cos(2\pi f_c t)$$

# Balanced DSB Modulator (Balanced Mixer)

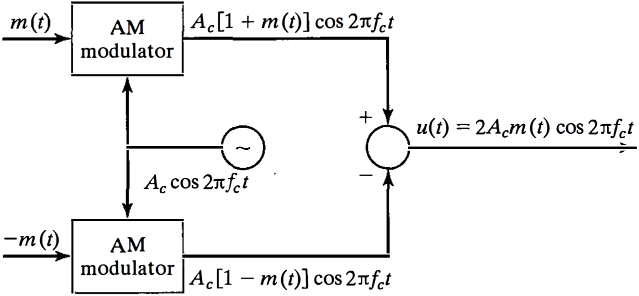


Figure: Block diagram of a balanced DSB modulator (balanced mixer).

# Ring DSB Modulator (Ring Mixer)

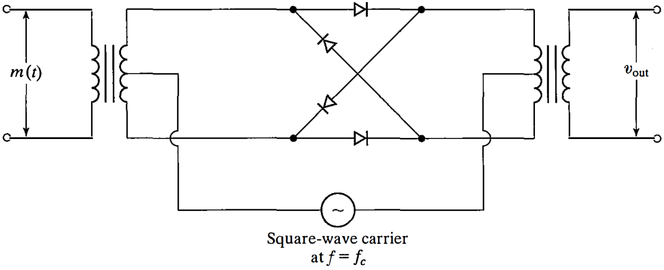


Figure: Block diagram of a ring DSB modulator (balanced mixer).

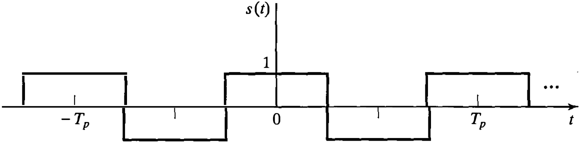


Figure: Periodic switching signal.

# Ring DSB Modulation (Ring Mixer)

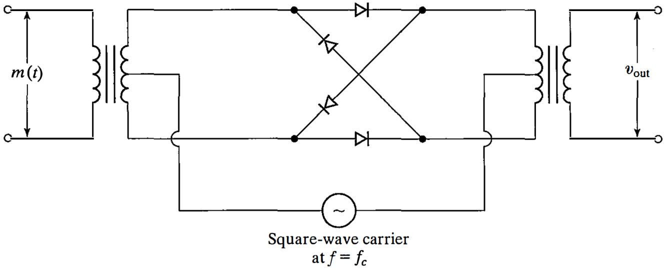


Figure: Block diagram of a ring DSB modulator (balanced mixer).

$$\text{if } |m(t)| \ll 1, \quad v_o(t) = \begin{cases} m(t), & s(t) \geq 0 \\ -m(t), & s(t) < 0 \end{cases} = m(t)s(t)$$

# Ring DSB Modulation (Ring Mixer)

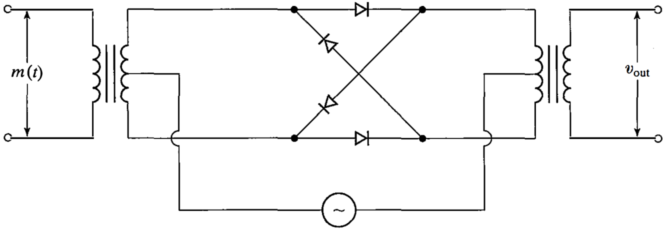


Figure: Block diagram of a ring DSB modulator (balanced mixer).

$$v_o(t) = m(t)s(t) = m(t)\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(2\pi f_c(2n-1)t)$$

Passing $v_o(t)$ through a bandpass filter with the center frequency $f = f_c$ and the bandwidth $2W$,

$$u(t) = \frac{4}{\pi}m(t)\cos(2\pi f_c t)$$

# Coherent Demodulator

✗ Pilot tone addition requires that certain portion of the transmitted signal power must be allocated to the transmission of the pilot.

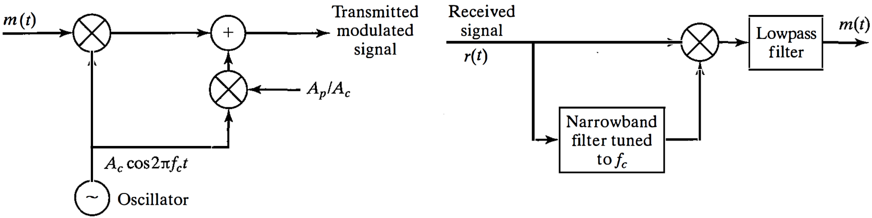


Figure: Pilot tone-based coherent demodulatoion of a DSB signal.

# Coherent Demodulator

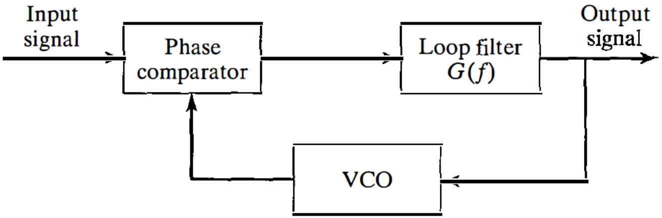


Figure: PLL block diagram.

# Coherent Demodulator

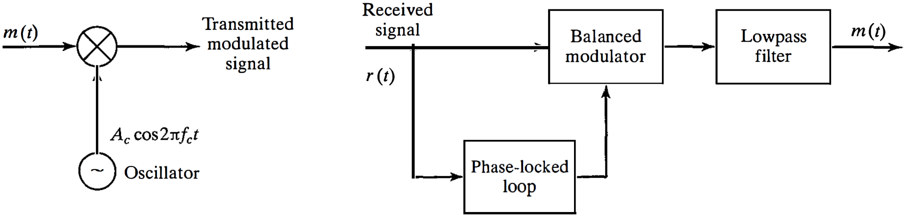


Figure: PLL-based demodulatoion of a DSB signal.

# DSB Modem

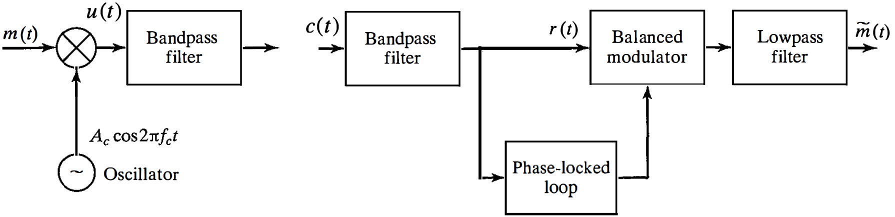


Figure: Block diagram of DSB modem.

# Effect of Noise on DSB Signal

## Statement (Effect of Noise on DSB signal)

*If a DSB signal passes an AWGN channel, the SNR at the output of the coherent DSB receiver is*

$$\left(\frac{S}{N}\right)_o = \frac{A_c^2 P_m}{2N_0 W} = \frac{P_R}{N_0 W}$$

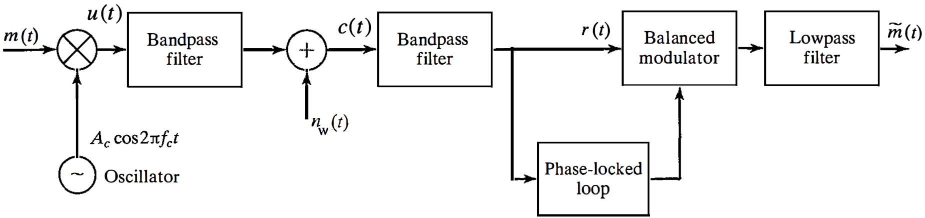


Figure: System model block diagram.

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The noisy DSB signal at the output of the channel is

$$c(t) = u(t) + n_w(t) = A_c m(t)\cos(2\pi f_c t) + n_w(t)$$

After the input BPF of the receiver,

$$r(t) = \text{BPF}\{c(t)\} = A_c m(t)\cos(2\pi f_c t) + n(t)$$

In terms of the in-phase and quadrature noise components,

$$r(t) = A_c m(t)\cos(2\pi f_c t) + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

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$$r(t) = A_c m(t)\cos(2\pi f_c t) + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

Now, we have

$$r(t)\cos(2\pi f_c t) = \frac{A_c}{2} m(t) + \frac{1}{2} n_c(t) + \text{double-frequency terms}$$

and consequently,

$$\text{LPF}\{r(t)\cos(2\pi f_c t)\} = \frac{A_c}{2} m(t) + \frac{1}{2} n_c(t)$$

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$$\text{LPF}\{r(t)\cos(2\pi f_c t)\} = \frac{A_c}{2} m(t) + \frac{1}{2} n_c(t)$$

$$P_o = \frac{A_c^2 P_m}{4}, \quad P_{n_o} = \frac{P_{n_c}}{4} = \frac{1}{4}\frac{N_0}{2} 2W \times 2 = \frac{N_0 W}{2}$$

$$\left(\frac{S}{N}\right)_o = \frac{P_o}{P_{n_o}} = \frac{A_c^2 P_m}{2N_0 W}$$

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$$\left(\frac{S}{N}\right)_o = \frac{A_c^2 P_m}{2N_0 W} = \frac{P_R}{N_0 W}$$

$$\left(\frac{S}{N}\right)_o = \frac{P_o}{P_{n_o}} = \frac{A_c^2 P_m}{2N_0 W}$$

$$P_R = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |A_c m(t) \cos(2\pi f_c t)|^2 dt = \frac{A_c^2}{2} P_m$$

$$\left(\frac{S}{N}\right)_o = \frac{P_o}{P_{n_o}} = \frac{A_c^2 P_m}{2N_0 W} = \frac{P_R}{N_0 W}$$

# Single Sideband Modulation

# SSB Modulation

✓ Only one sideband is enough to sent the message signal.

✓ The upper or lower sideband can be obtained by a bandpass filtering from the DSB signal.

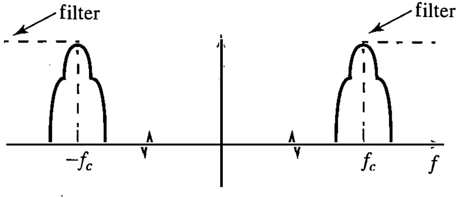


Figure: Upper sideband signal.

## Statement (Upper Sideband Signal)

*The upper sideband part of the DSB modulated signal* $u(t) = 2A_c m(t)\cos(2\pi f_c t)$ *is* $u_u(t) = A_c m(t)\cos(2\pi f_c t) - A_c \hat{m}(t)\sin(2\pi f_c t)$.

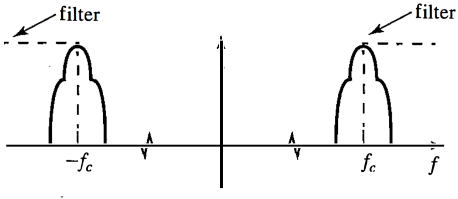


Figure: Upper sideband signal.

$$U_{DSB}(f) = A_c M(f - f_c) + A_c M(f + f_c)$$

$$H(f) = u(f - f_c) + u(-f - f_c)$$

$$U_u(f) = U_{DSB}(f)H(f)$$

## Statement (Upper Sideband Signal (cont.))

*The upper sideband part of the DSB modulated signal* $u(t) = 2A_c m(t)\cos(2\pi f_c t)$ *is* $u_u(t) = A_c m(t)\cos(2\pi f_c t) - A_c \hat{m}(t)\sin(2\pi f_c t)$.

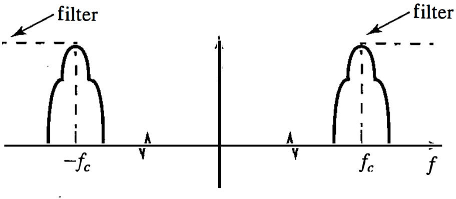


Figure: Upper sideband signal.

$$U_u(f) = U_{DSB}(f)H(f)$$

$$U_u(f) = (A_c M(f - f_c) + A_c M(f + f_c))(u(f - f_c) + u(-f - f_c))$$

$$U_u(f) = A_c M(f - f_c)u(f - f_c) + A_c M(f + f_c)u(-f - f_c)$$

## Statement (Upper Sideband Signal (cont.))

*The upper sideband part of the DSB modulated signal* $u(t) = 2A_c m(t)\cos(2\pi f_c t)$ *is* $u_u(t) = A_c m(t)\cos(2\pi f_c t) - A_c \hat{m}(t)\sin(2\pi f_c t)$.

$$U_u(f) = A_c M(f - f_c)u(f - f_c) + A_c M(f + f_c)u(-f - f_c)$$

$$U_u(f) = A_c M(f - f_c)\frac{1 - jj\mathrm{sgn}(f - f_c)}{2} + A_c M(f + f_c)\frac{1 - jj\mathrm{sgn}(-f - f_c)}{2}$$

$$U_u(f) = A_c M(f - f_c)\frac{1 - jj\mathrm{sgn}(f - f_c)}{2} + A_c M(f + f_c)\frac{1 + jj\mathrm{sgn}(f + f_c)}{2}$$

$$u_u(t) = \frac{A_c}{2}e^{j2\pi f_c t}(m(t) + j\hat{m}(t)) + \frac{A_c}{2}e^{-j2\pi f_c t}(m(t) - j\hat{m}(t))$$

# Upper Sideband Signal

**Statement (Upper Sideband Signal (cont.))**

*The upper sideband part of the DSB modulated signal* $u(t) = 2A_c m(t)\cos(2\pi f_c t)$ *is* $u_u(t) = A_c m(t)\cos(2\pi f_c t) - A_c \hat{m}(t)\sin(2\pi f_c t)$.

$$u_u(t) = \frac{A_c}{2} e^{j2\pi f_c t}(m(t) + j\hat{m}(t)) + \frac{A_c}{2} e^{-j2\pi f_c t}(m(t) - j\hat{m}(t))$$

$$u_u(t) = A_c m(t)\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2} - A_c \hat{m}(t)\frac{e^{j2\pi f_c t} - e^{-j2\pi f_c t}}{2j}$$

$$u_u(t) = A_c m(t)\cos(2\pi f_c t) - A_c \hat{m}(t)\sin(2\pi f_c t)$$

# Lower Sideband Signal

**Statement (Lower Sideband Signal)**

*The lower sideband part of the DSB modulated signal* $u(t) = 2A_c m(t)\cos(2\pi f_c t)$ *is* $u_l(t) = A_c m(t)\cos(2\pi f_c t) + A_c \hat{m}(t)\sin(2\pi f_c t)$.

$$u_{DSB}(t) = u_l(t) + u_u(t)$$

$$u_l(t) = 2A_c m(t)\cos(2\pi f_c t) - A_c m(t)\cos(2\pi f_c t) + A_c \hat{m}(t)\sin(2\pi f_c t)$$

$$u_l(t) = A_c m(t)\cos(2\pi f_c t) + A_c \hat{m}(t)\sin(2\pi f_c t)$$

# SSB Modulation

## Statement (SSB)

*An SSB signal $u(t)$ is obtained by*

$$u(t) = A_c m(t)\cos(2\pi f_c t) \mp A_c \hat{m}(t)\sin(2\pi f_c t)$$

*, where $\hat{m}(t)$ is the Hilbert transform of $m(t)$ and the plus and minus signs correspond to the lower and upper sideband, respectively. The spectrum of the modulated signal $U(f)$ equals $U_l(f)$ or $U_u(f)$ depending on the used sideband.*

The SSB signal can be generated

1. using Hilbert transform.
2. by filtering the DSB signal.

# SSB Modulation

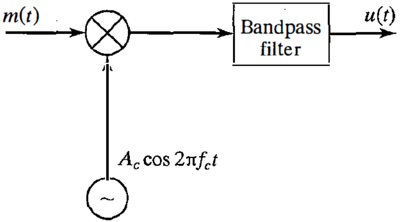


Figure: Generation of an SSB signal by filtering one of the sidebands of a DSB-SC signal.

✓ The sideband filter, which must have an extremely sharp cutoff in the vicinity of the carrier, is very hard to be implemented.

# SSB Modulation

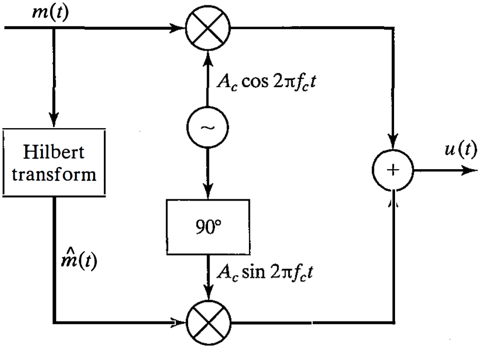


Figure: Generation of a lower SSB signal using Hilbert transform.

✓ The Hilbert filter may be hard to be implemented.

# SSB Modulation

**Example (Sinusoidally-modulated SSB)**

If $m(t) = \cos(2\pi f_m t), \quad f_m \ll f_c$, the SSB is expressed in the time domain as

$$\begin{aligned} u(t) &= A_c m(t)\cos(2\pi f_c t) \mp A_c \hat{m}(t)\sin(2\pi f_c t) \\ &= A_c \cos(2\pi f_m t)\cos(2\pi f_c t) \mp A_c \sin(2\pi f_m t)\sin(2\pi f_c t) \end{aligned}$$

, which equals to $u(t) = u_u(t) = A_c\cos(2\pi(f_c + f_m)t)$ or $u(t) = u_l(t) = A_c\cos(2\pi(f_c - f_m)t)$ when the upper or lower sideband is used, respectively.

# Bandwidth of SSB Signal

## Statement (Bandwidth of SSB Signal)

*For a message signal having the bandwidth $W$, the corresponding SSB signal requires a bandwidth of $W$.*

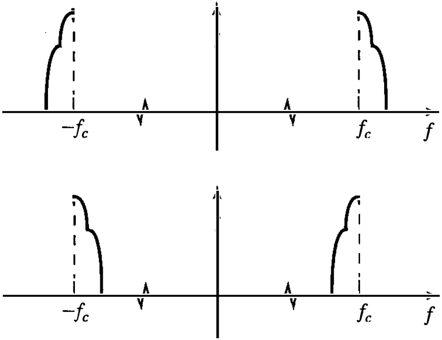


Figure: Bandwidth of USSB and LSSB signal.

# Power of SSB Signal

## Statement (Power of SSB signal)

*The power content of the SSB signal equals* $P_u = A_c^2 P_m$.

$$
\begin{aligned}
P_u &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} u^2(t)dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [A_c m(t)\cos(2\pi f_c t) \mp A_c \hat{m}(t)\sin(2\pi f_c t)]^2 dt \\
&= \frac{A_c^2}{2} P_m + \frac{A_c^2}{2} P_m = A_c^2 P_m
\end{aligned}
$$

# SSB Demodulation

## Statement (SSB demodulation)

*Suppose that the SSB signal $u(t)$ is transmitted through an ideal channel. Then, the received signal is $r(t) = u(t)$. The message can be demodulated by*

$$\widetilde{m}(t) = \frac{A_c}{2} m(t) = LPF\Big\{r(t)\cos(2\pi f_c t)\Big\}$$

*, where $\cos(2\pi f_c t)$ is a locally generated synchronous sinusoid and the ideal lowpass filter has the bandwidth $W$.*

$$r(t)\cos(2\pi f_c t + \phi) = \frac{A_c}{2} m(t)\cos(\phi) + \frac{A_c}{2}\hat{m}(t)\sin(\phi) + \text{double-frequency terms}$$

$$y_l(t) = \text{LPF}\Big\{r(t)\cos(2\pi f_c t + \phi)\Big\} = \frac{A_c}{2} m(t)\cos(\phi) + \frac{A_c}{2}\hat{m}(t)\sin(\phi)$$

$$\phi = 0 \Rightarrow y_l(t) = \frac{A_c}{2} m(t)$$

# SSB Demodulation

## Statement (SSB Demodulation)

*Suppose that the SSB signal $u(t)$ is transmitted through an ideal channel. Then, the received signal is $r(t) = u(t)$. The message can be demodulated by*

$$\widetilde{m}(t) = \frac{A_c}{2} m(t) = LPF\{r(t)\cos(2\pi f_c t)\}$$

*, where $\cos(2\pi f_c t)$ is a locally generated synchronous sinusoid and the ideal lowpass filter has the bandwidth $W$.*

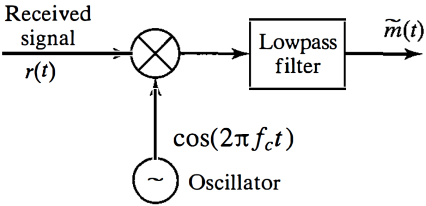


Figure: Block diagram of the basic SSB demodulator.

# SSB Demodulation

1. The phase offset not only attenuates the desired signal $m(t)$, but it also results in an undesirable distortion due to the presence of $\hat{m}(t)$.
2. A phase-coherent or synchronous demodulator is needed for recovering the message signal.
3. A synchronous demodulator uses a pilot tone or phase-locked loop (PLL) to lock to the phase of the carrier.

# Effect of Noise on SSB Signal

## Statement (Effect of Noise on SSB Signal)

*If a SSB signal passes an AWGN channel, the SNR at the output of the coherent SSB receiver is*

$$\left(\frac{S}{N}\right)_o = \frac{A_c^2 P_M}{N_0 W} = \frac{P_R}{N_0 W}$$

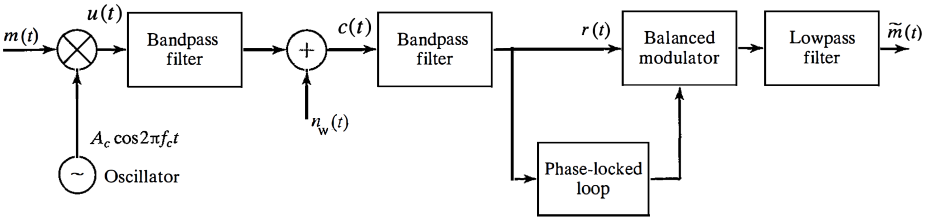


Figure: System model block diagram.

# Conventional Amplitude Modulation

# AM Modulation

## Statement (AM)

*A conventional AM signal* $u(t)$ *is obtained by*

$$u(t) = A_c[1 + am_n(t)]\cos(2\pi f_c t)$$

, *where* $0 < a < 1$ *is called the modulation index and* $m_n(t) = m(t)/\max|m(t)|$ *is the normalized message between* $[-1, 1]$.

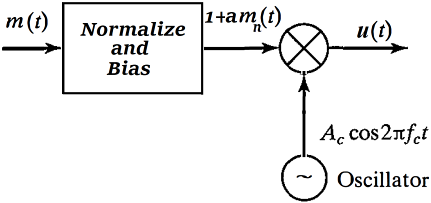


Figure: Block diagram of the conventional AM modulator.

# AM Modulation

## Example (AM signal)

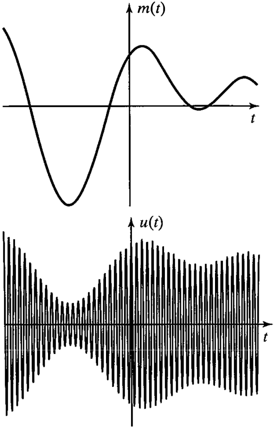


Figure: Examples of AM-modulated signals.

# Bandwidth of AM Signal

## Statement (Spectrum of AM Signal)

*The spectrum of the AM modulated signal is*

$$U(f) = \frac{A_c a}{2}\left[M_n(f - f_c) + M_n(f + f_c)\right] + \frac{A_c}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$

## Statement (Bandwidth of AM Signal)

*For a message signal having the bandwidth $W$, the corresponding AM signal requires a bandwidth of $2W$.*

## Example (AM spectrum)

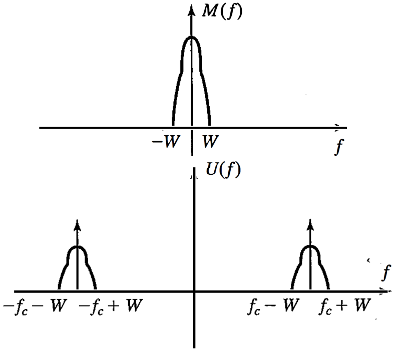


Figure: Spectrum of a message signal and its corresponding AM-modulated signal.

# Bandwidth of AM Signal

## Example (Sinusoidally-modulated AM)

If $m_n(t) = \cos(2\pi f_m t), \quad f_m \ll f_c$, the AM signal is

$$\begin{aligned} u(t) &= A_c[1 + a\cos(2\pi f_m t)]\cos(2\pi f_c t) \\ &= A_c\cos(2\pi f_c t) + \frac{aA_c}{2}\cos(2\pi(f_c - f_m)t) + \frac{aA_c}{2}\cos(2\pi(f_c + f_m)t) \end{aligned}$$

## Example (Sinusoidally-modulated AM (cont.))

If $m_n(t) = \cos(2\pi f_m t), \quad f_m \ll f_c$, the spectrum of the AM signal is

$$\begin{aligned} U(f) = \frac{A_c}{2}&\left[\delta(f - f_c) + \delta(f + f_c)\right] \\ &+ \frac{aA_c}{4}\left[\delta(f - f_c + f_m) + \delta(f + f_c - f_m)\right] \\ &+ \frac{aA_c}{4}\left[\delta(f - f_c - f_m) + \delta(f + f_c + f_m)\right] \end{aligned}$$

# Bandwidth of AM Signal

## Example (Sinusoidally-modulated AM (cont.))

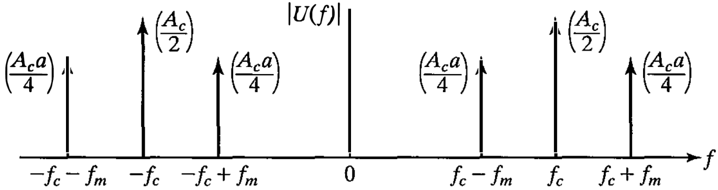


Figure: Spectrum of a sinusoidally-modulated AM signal.

# Bandwidth of AM Signal

## Example (Sinusoidally-modulated AM (cont.))

If $m_n(t) = \cos(2\pi f_m t), \quad f_m \ll f_c$, the lower and upper sideband correspond to the signals

$$u_l(t) = \frac{aA_c}{2}\cos(2\pi(f_c - f_m)t)$$
$$u_u(t) = \frac{aA_c}{2}\cos(2\pi(f_c + f_m)t)$$

# Power of AM Signal

## Statement (Power of AM signal)

*The power content of the AM signal equals*

$$P_u = \frac{A_c^2}{2}(1 + a^2 P_{m_n}) = \frac{A_c^2}{2}(1 + \frac{a^2 P_m}{\max^2 |m(t)|})$$

.

A conventional AM signal is similar to a DSB with the message $1 + am_n(t)$. So, $P_u = 0.5A_c^2 P_{1+am_n}$ . For a zero-DC message signal,

$$P_{1+am_n} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [1 + am_n(t)]^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [1 + a^2 m_n^2(t)] dt$$

$$= 1 + a^2 P_{m_n} = 1 + \frac{a^2 P_m}{\max^2 |m(t)|}$$

# Power of AM signal

**Example (Power of sinusoidally-modulated AM)**

If $m_n(t) = \cos(2\pi f_m t), \quad f_m \ll f_c$, then

$$P_{m_n} = \frac{1}{2}$$

,

$$P_u = \frac{A_c^2}{2}(1 + a^2 P_{m_n}) = \frac{A_c^2}{2} + \frac{a^2 A_c^2}{4}$$

, and

$$P_{u_l} = P_{u_u} = \frac{a^2 A_c^2}{8}$$

.

# AM Demodulation

## Statement (Coherent AM Demodulation)

*Suppose that the AM signal $u(t)$ is transmitted through an ideal channel. Then, the received signal is $r(t) = u(t)$. The message can be demodulated by*

$$\tilde{m}(t) = \frac{A_c}{2} a m_n(t) = LPF{+}DCR\{r(t)\cos(2\pi f_c t)\}$$

*, where $\cos(2\pi f_c t)$ is a locally generated synchronous sinusoid and the ideal lowpass filter has the bandwidth $W$.*

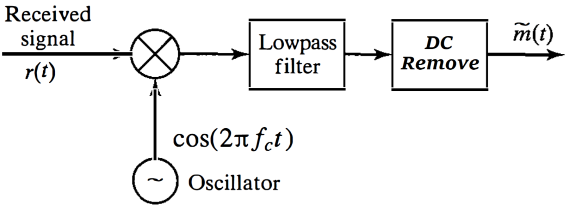


Figure: Block diagram of the basic coherent AM demodulator.

# AM Demodulation

## Statement (AM Envelope Demodulator)

*Suppose that the AM signal $u(t)$ is transmitted through an ideal channel. Then, the received signal is $r(t) = u(t)$. The received signal is demodulated by extracting the envelope $V_r(t)$ of the rectified version of $r(t)$ as $DCR\{V_r(t)\}$.*

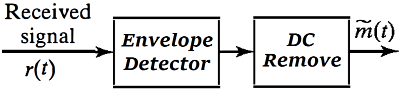


Figure: Block diagram of the AM envelope Demodulator.

$$u(t) = A_c[1 + am_n(t)]\cos(2\pi f_c t)$$

$$1 + am_n(t) \geq 0 \Rightarrow V_r(t) = |A_c[1 + am_n(t)]| = A_c[1 + am_n(t)]$$

$$\text{DCR}\{V_r(t)\} = A_c am_n(t)$$

# AM Demodulation

## Statement (Envelope AM demodulation)

*Suppose that the AM signal $u(t)$ is transmitted through an ideal channel. Then, the received signal is $r(t) = u(t)$. The received signal is demodulated by extracting the envelope of the rectified version of $r(t)$.*

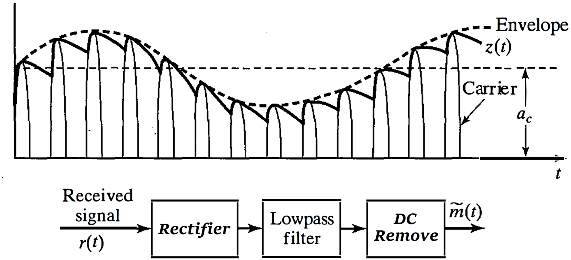


Figure: Envelope detection of an AM signal.

# Envelope Detector

✓ A simple envelope detector consists of a diode and an RC lowpass filter.

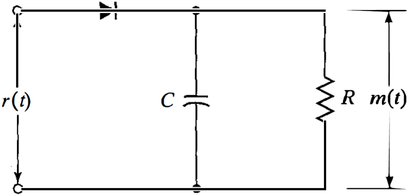


Figure: An envelope detector.

# Envelope Detector

✓ For good performance of the envelope detector, $\frac{1}{f_c} \ll RC \ll \frac{1}{W}$.

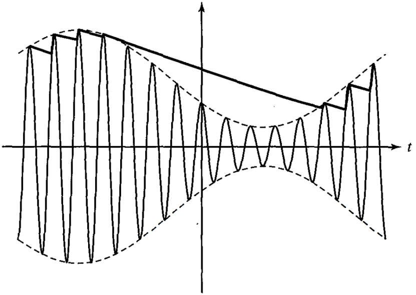


Figure: Effect of a large RC value on the performance of the envelope detector.

# Envelope Detector

✓ For good performance of the envelope detector, $\frac{1}{f_c} \ll RC \ll \frac{1}{W}$.

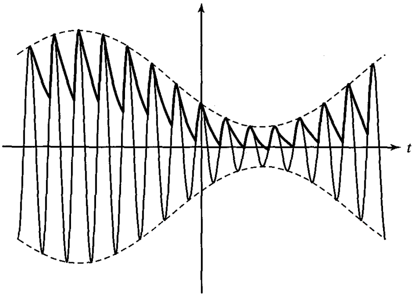


Figure: Effect of a small RC value on the performance of the envelope detector.

# Effect of Noise on AM signal

## Statement (Effect of Noise on Coherent AM)

*The SNR at the output of a coherent AM receiver is*

$$\left(\frac{S}{N}\right)_o = \frac{A_c^2 a^2 P_{m_n}}{2N_0 W} = \frac{a^2 P_{m_n}}{1 + a^2 P_{m_n}} \frac{P_R}{N_0 W}$$

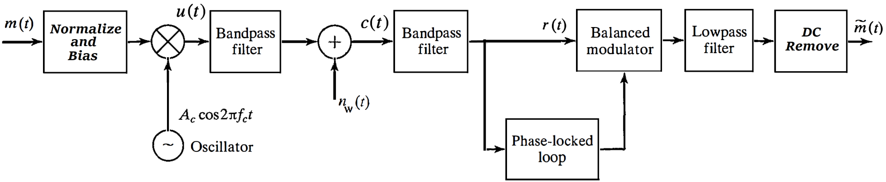


Figure: System model block diagram.

# Effect of Noise on AM signal

**Statement (Effect of Noise on Coherent AM)**

*The SNR at the output of a coherent AM receiver is*

$$\left(\frac{S}{N}\right)_o = \frac{A_c^2 a^2 P_{m_n}}{2N_0 W} = \frac{a^2 P_{m_n}}{1 + a^2 P_{m_n}} \frac{P_R}{N_0 W}$$

The noisy AM signal is expressed as

$$r(t) = A_c[1 + am_n(t)] \cos(2\pi f_c t) + n(t)$$

$$r(t) = \Big[A_c[1 + am_n(t)] + n_c(t)\Big] \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t)$$

$$r(t) \cos(2\pi f_c t) = \frac{1}{2} A_c[1 + am_n(t)] + \frac{1}{2} n_c(t) + \text{double-frequency terms}$$

$$\text{DCR\&LPF}\{r(t) \cos(2\pi f_c t)\} = \frac{1}{2} a A_c m_n(t) + \frac{1}{2} n_c(t)$$

# Effect of Noise on AM signal

## Statement (Effect of Noise on Coherent AM)

*The SNR at the output of a coherent AM receiver is*

$$\left(\frac{S}{N}\right)_o = \frac{A_c^2 a^2 P_{m_n}}{2N_0 W} = \frac{a^2 P_{m_n}}{1 + a^2 P_{m_n}} \frac{P_R}{N_0 W}$$

$$\text{DCR\&LPF}\{r(t)\cos(2\pi f_c t)\} = \frac{1}{2} a A_c m_n(t) + \frac{1}{2} n_c(t)$$

$$P_o = \frac{A_c^2 a^2 P_{m_n}}{4}, \quad P_{n_o} = \frac{P_{n_c}}{4} = \frac{1}{4} \frac{N_0}{2} 2W \times 2 = \frac{N_0 W}{2}$$

$$\left(\frac{S}{N}\right)_o = \frac{P_o}{P_{n_o}} = \frac{A_c^2 a^2 P_{m_n}}{2N_0 W}$$

# Effect of Noise on AM signal

## Statement (Effect of Noise on Coherent AM)

*The SNR at the output of a coherent AM receiver is*

$$\left(\frac{S}{N}\right)_o = \frac{A_c^2 a^2 P_{m_n}}{2N_0 W} = \frac{a^2 P_{m_n}}{1 + a^2 P_{m_n}} \frac{P_R}{N_0 W}$$

$$\left(\frac{S}{N}\right)_o = \frac{P_o}{P_{n_o}} = \frac{A_c^2 a^2 P_{m_n}}{2N_0 W}$$

$$P_R = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |r(t)|^2 dt = \frac{A_c^2}{2}[1 + a^2 P_{m_n}]$$

$$\left(\frac{S}{N}\right)_o = \frac{P_o}{P_{n_o}} = \frac{A_c^2 a^2 P_{m_n}}{2N_0 W} = \frac{a^2 P_{m_n}}{1 + a^2 P_{m_n}} \frac{\frac{A_c^2}{2}[1 + a^2 P_{m_n}]}{N_0 W} = \frac{a^2 P_{m_n}}{1 + a^2 P_{m_n}} \frac{P_R}{N_0 W}$$

**Statement (Effect of Noise on Envelope Detector)**

*At high SNR conditions, the SNR at the output of an envelope detector is approximately the same as that of the coherent AM receiver.*

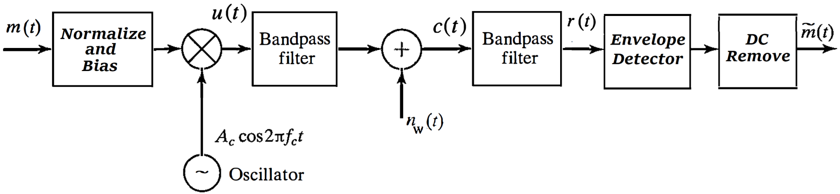


Figure: System model block diagram.

# Effect of Noise on AM signal

## Statement (Effect of Noise on Envelope Detector)

*At high SNR conditions, the SNR at the output of an envelope detector is approximately the same as that of the coherent AM receiver.*

$$r(t) = \big[A_c[1 + am_n(t)] + n_c(t)\big] \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t)$$

$$V_r(t) = \sqrt{\big[A_c[1 + am_n(t)] + n_c(t)\big]^2 + n_s^2(t)}$$

Assuming that the signal component is much stronger than the noise,

$$V_r(t) \approx A_c[1 + am_n(t)] + n_c(t)$$

# Effect of Noise on AM signal

## Statement (Effect of Noise on Envelope Detector)

*At low SNR conditions, no meaningful SNR can be defined at the output of an envelope detector.*

$$V_r(t) = \sqrt{\left[A_c[1 + am_n(t)] + n_c(t)\right]^2 + n_s^2(t)}$$

Assuming that the noise component is much stronger than the signal,

$$V_r(t) \approx \sqrt{[n_c^2(t) + n_s^2(t)]\left[1 + \frac{2A_c n_c(t)}{n_c^2(t) + n_s^2(t)}(1 + am_n(t))\right]}$$

$$V_r(t) \approx V_n(t)\left[1 + \frac{A_c n_c(t)}{V_n^2(t)}(1 + am_n(t))\right]$$

# Vestigial Sideband Modulation

# VSB Modulation

✓ The stringent filtering requirements in an SSB system can be relaxed by allowing vestige.

✓ The vestige simplifies the design of the sideband filter.

✓ VSB modulation is appropriate for signals that have a strong low-frequency component, such as video signals.

✗ The vestige needs a modest increase in the channel bandwidth.

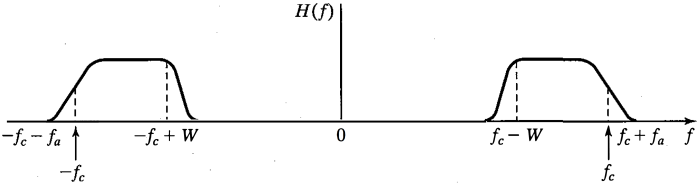


Figure: VSB filtering.

# VSB Modulation

## Statement (VSB)

*A VSB modulated signal is obtained by passing the DSB signal through a sideband filter* $H(f)$ *as* $u(t) = \left[A_c m(t)\cos(2\pi f_c t)\right] * h(t)$ *or equivalently,* $U(f) = \frac{A_c}{2}\left[M(f - f_c) + M(f + f_c)\right]H(f)$.

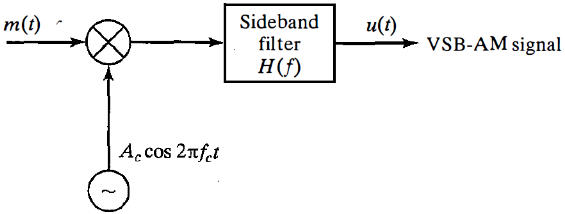


Figure: Generation of a VSB signal.

# Demodulation of VSB Signal

## Statement (Demodulation of VSB)

*To demodulate a VSB signal, we multiply it by the carrier component* $\cos(2\pi f_c t)$ *and pass the result through an ideal lowpass filter.*

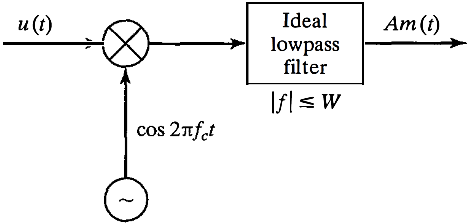


Figure: Block diagram of the basic VSB demodulator.

# Demodulation of VSB Signal

## Statement (Demodulation of VSB Signal)

*To demodulate a VSB signal, we multiply it by the carrier component* $\cos(2\pi f_c t)$ *and pass the result through an ideal lowpass filter.*

$$v(t) = u(t)\cos(2\pi f_c t)$$

$$V(f) = \frac{1}{2}\big[U(f - f_c) + U(f + f_c)\big]$$

$$V(f) = \frac{A_c}{4}\big[M(f-2f_c)+M(f)\big]H(f-f_c)+\frac{A_c}{4}\big[M(f+2f_c)+M(f)\big]H(f+f_c)$$

$$\widetilde{M}(f) = \frac{A_c}{4}M(f)\big[H(f - f_c) + H(f + f_c)\big]$$

$$H(f - f_c) + H(f + f_c) = C, \quad |f| \leq W$$

# Demodulation of VSB Signal

## Statement (VSB Filter Condition)

*The VSB sideband filter should satisfy*

$$H(f - f_c) + H(f + f_c) = C, \quad |f| \leq W$$

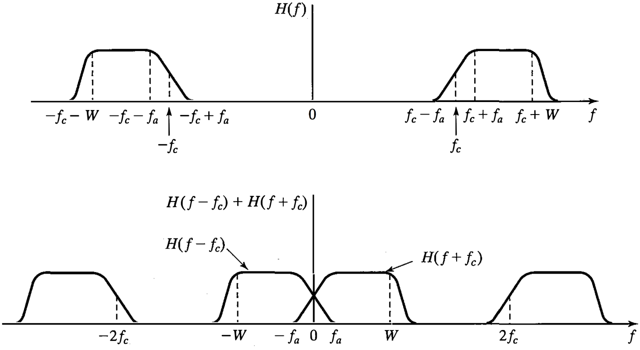


Figure: Frequency response of the VSB filter for selecting the upper sideband.

# VSB

1. The VSB sideband filter has odd symmetry about the carrier frequency $f_c$ in the frequency range $f_c - f_a < f < f_c + f_a$.
2. $f_a$ is a conveniently selected frequency that is some small fraction of $W$, i.e., $f_a \ll W$.
3. To avoid distortion of the message signal, the VSB filter should have a linear phase over its passband $f_c - f_a < |f| < f_c + W$.
4. Power, bandwidth, and SNR analysis of VSB is very similar to SSB provided that $f_a \ll W$.

# Frequency Modulation

# FM

## Statement (FM)

*An frequency-modulated signal is written as*

$$u(t) = A_c \cos(2\pi f_c t + \phi(t)) = A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau)$$

*, where $k_f$ is called frequency deviation constant. The instantaneous frequency of the modulated signal is defined as*

$$f_i(t) = \frac{1}{2\pi} \frac{d\left[2\pi f_c t + \phi(t)\right]}{dt} = f_c + k_f m(t)$$

## Example (FM signal)

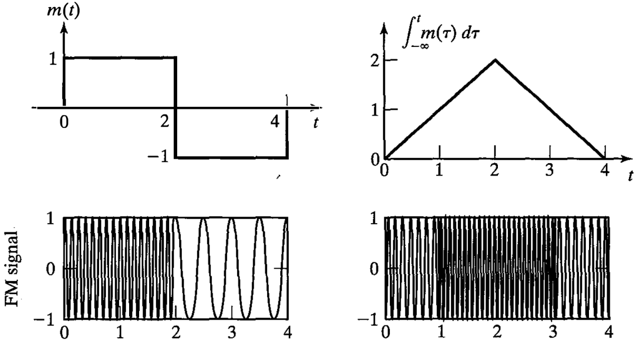


Figure: Frequency modulation of square and sawtooth waves.

## Statement (FM Modulation Index)

*The modulation index of the FM is defined as*

$$\beta_f = \frac{k_f \max\{|m(t)|\}}{W} = \frac{\Delta f_{max}}{W}$$

*, where $\Delta f_{max}$ is the maximum frequency deviation.*

**Example (Sinusoidally-modulated FM signal)**

For the message signal $m(t) = a\cos(2\pi f_m t)$, the FM signal is

$$u(t) = A_c \cos(2\pi f_c t + \frac{k_f a}{f_m} \sin(2\pi f_m t)) = A_c \cos(2\pi f_c t + \beta_f \sin(2\pi f_m t))$$

.

# FM

## Statement (Narrowband FM Modulation)

*Consider an FM system with $\phi(t) \ll 1$. Then,*

$$
\begin{aligned}
u(t) &= A_c \cos(2\pi f_c t + \phi(t)) \\
&= A_c \cos(2\pi f_c t) \cos(\phi(t)) - A_c \sin(2\pi f_c t) \sin(\phi(t)) \\
&\approx A_c \cos(2\pi f_c t) - A_c \phi(t) \sin(2\pi f_c t) \\
&= A_c \cos(2\pi f_c t) + A_c \phi(t) \cos(2\pi f_c t + \frac{\pi}{2}) \\
&= A_c \cos(2\pi f_c t) + A_c \left[2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau\right] \cos(2\pi f_c t + \frac{\pi}{2})
\end{aligned}
$$

## Statement (Narrowband FM Modulation)

*Although narrowband FM and conventional AM modulations share some similarities, they have some differences.*

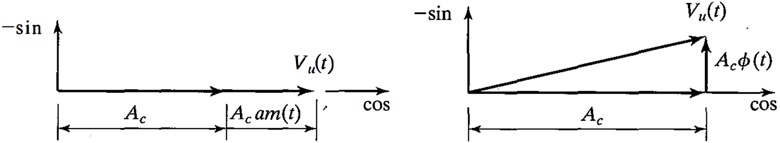


Figure: Phasor diagrams for the conventional AM and narrowband FM modulation.

# Bandwidth and Power of FM Signal

## Statement (FM by a Sinusoidal Signal)

*For the sinusoidal message $m(t) = a\cos(2\pi f_m t)$, the FM signal is*

$$u(t) = A_c \cos(2\pi f_c t + \beta_f \sin(2\pi f_m t)) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta_f) \cos(2\pi(f_c + nf_m)t)$$

*, where $J_n(\beta_f)$ is the Bessel function of the first kind of order $n$.*

$$u(t) = A_c \cos(2\pi f_c t + \beta_f \sin(2\pi f_m t)) = \Re\{A_c e^{j2\pi f_c t} e^{j\beta_f \sin(2\pi f_m t)}\}$$

$$= \Re\{A_c e^{j2\pi f_c t} \sum_{n=-\infty}^{\infty} J_n(\beta_f) e^{j2\pi n f_m t}\} = \sum_{n=-\infty}^{\infty} A_c J_n(\beta_f) \cos(2\pi(f_c + nf_m)t)$$

, where $J_n(\beta_f) = \frac{1}{2\pi} \int_0^{2\pi} e^{j(\beta_f \sin(u) - nu)} du$.

# Bandwidth and Power of FM Signal

1. $J_{-n}(\beta) = J_n(\beta)$ for an even $n$ and $J_{-n}(\beta) = -J_n(\beta)$ for an odd $n$.
2. $J_n(\beta) \approx \frac{\beta^n}{2^n n!}$ for a small $\beta$.
3. $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$.

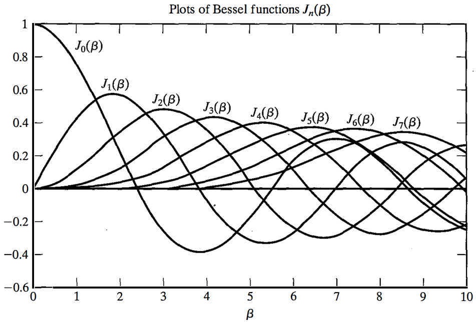


Figure: Bessel functions for various values of $n$.

# Bandwidth and Power of FM Signal

**Statement (Bandwidth of Sinusoidal FM)**

*For the sinusoidal message* $m(t) = a\cos(2\pi f_m t)$, *the actual bandwidth of the FM signal is infinite.*

**Statement (Power of Sinusoidal FM)**

*For the sinusoidal message* $m(t) = a\cos(2\pi f_m t)$, *the power of the FM signal is* $A_c^2/2$.

# Bandwidth and Power of FM Signal

For the sinusoidal message $m(t)$,

1. The modulated signal contains all the harmonics $f_c + nf_m$ for $n = 0, \pm 1, \pm 2, \cdots$.
2. The amplitude of the harmonic $f_c + nf_m$ for large $n$ is very small.
3. A finite effective bandwidth for the modulated signal can be defined.
4. For a small $\beta$, only the first harmonic is important.
5. For larger $\beta$, more harmonics should be considered to include 80%, 90%, and 98% of the total power.

| Power | $\beta = 0.5$ | $\beta = 1$ | $\beta = 2$ | $\beta = 5$ | $\beta = 8$ | $\beta = 10$ |
|---|---|---|---|---|---|---|
| 80% | – | 1 | 2 | 4 | 7 | 9 |
| 90% | 1 | 1 | 2 | 5 | 8 | 10 |
| 98% | 1 | 2 | 3 | 6 | 9 | 11 |

Table: Required number of harmonics.

# Bandwidth and Power of FM Signal

1. The 98%-power effective bandwidth of sinusoidal FM is approximately $B_c = 2(\beta_f + 1)f_m = 2(k_f a + f_m)$.
2. Increasing $a$, the amplitude of the modulating signal, increases the bandwidth $B_c$.
3. Increasing $f_m$, the frequency of the message signal, also increases the bandwidth $B_c$.
4. The number of harmonics, including the carrier, is $M_c = 2([\beta]+1)+1 = 2[\beta] + 3 = 2[\frac{k_f a}{f_m}] + 3$.
5. Increasing the amplitude $a$ increases the number of harmonics.
6. Increasing $f_m$ almost linearly decreases the number of harmonics.

# Bandwidth and Power of FM Signal

## Example (FM by a Sinusoidal Signal)

For the message $m(t) = \cos(20\pi t)$, the carrier $c(t) = 10\cos(2\pi f_c t)$, and the deviation constant $k_f = 50$, the bandwidth including 99% of the power is 120 Hz.

$$\beta = k_f \max\{|m(t)|\}/W = k_f \max\{|m(t)|\}/f_m = 5$$

$$u(t) = 10\cos(2\pi f_c t + 5\sin(20\pi t)) = \sum_{n=-\infty}^{\infty} 10 J_n(5)\cos(2\pi(f_c + 10n)t)$$

$$\sum_{n=-k}^{k} \frac{10^2}{2} J_n^2(5) \geq 0.99 \times \frac{10^2}{2} \Rightarrow k = 6$$

So, the the edge harmonics that should be considered are $f_c \pm 10k = f_c \pm 10 \times 6$ and the bandwidth is 120 Hz.

# Bandwidth and Power of FM Signal

## Example (FM by a Sinusoidal Signal (cont.))

For the message $m(t) = \cos(20\pi t)$, the carrier $c(t) = 10\cos(2\pi f_c t)$, and the deviation constant $k_f = 50$, the bandwidth including 99% of the power is 120 Hz.

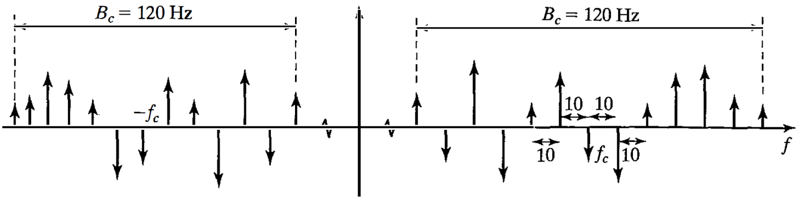


Figure: The harmonics present inside the effective bandwidth of the example.

# Bandwidth and Power of FM Signal

## Statement (Effective Bandwidth of FM (Carson's Rule))

*The effective bandwidth of an FM signal is approximately*

$$B_c = 2(\beta_f + 1)W$$

*, where $W$ is the frequency of the message signal $m(t)$.*

## Statement (Power of FM)

*The power content of an FM signal is $\frac{A_c^2}{2}$.*

# Bandwidth and Power of FM Signal

## Example (Carson's Rule)

Assuming that $m(t) = 10\text{sinc}(10^4 t)$, the transmission bandwidth of an FM-modulated signal with $k_f = 4000$ is $B_c = 90$ kHz.

$$M(f) = 10^{-3}\sqcap(10^{-4}f) \Rightarrow W = 5000 \text{ Hz}$$

$$\beta = \frac{k_f \max\{|m(t)|\}}{W} = \frac{4000 \times 10}{5000} = 8$$

$$B_c = 2(\beta + 1)W = 90000 \text{ Hz}$$

Two common approaches for FM modulation are

1. Modulation techniques
    1. VCO (Varactor-diode, Reactance tube)
    2. Indirect
2. Demodulation techniques
    1. FM to AM
    2. PLL (Feedback)

# VCO Modulator

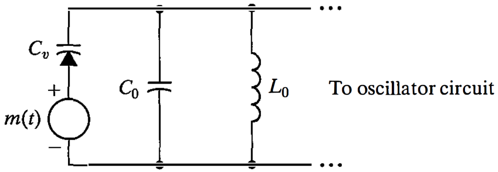


Figure: Varactor-diode implementation of an FM modulator.

$$C(t) = C_0 + K_0 m(t), \quad = \frac{1}{2\pi\sqrt{L_0 C_0}}$$

$$f_i(t) = \frac{1}{2\pi\sqrt{L_0(C_0 + K_0 m(t))}} = \frac{f_c}{\sqrt{1 + \frac{K_0}{C_0} m(t)}}$$

$$|\frac{K_0}{C_0} m(t)| \ll 1 \Rightarrow f_i(t) = f_c \frac{1}{\sqrt{1 + \frac{K_0}{C_0} m(t)}} \approx f_c(1 - \frac{K_0}{2C_0} m(t))$$

# Indirect Modulator

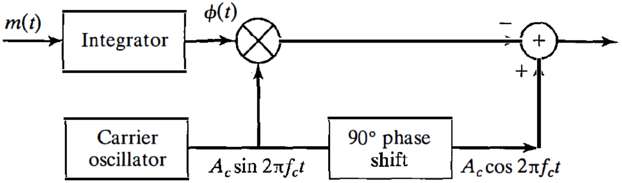


Figure: Generation of a narrowband FM signal.

$$u_n(t) = A_c \cos(2\pi f_c t + \phi(t)) \approx A_c \cos(2\pi f_c t) - A_c \phi(t) \sin(2\pi f_c t)$$

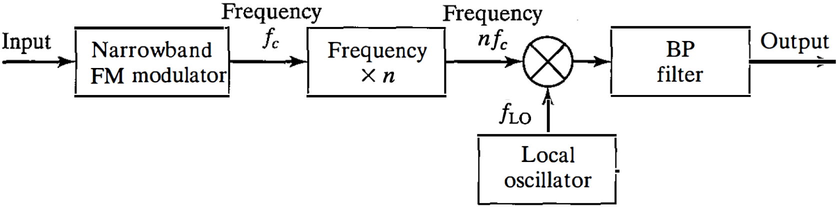


Figure: Indirect FM generation.

$$y(t) = A_c \cos(2\pi n f_c t + n\phi(t))$$

$$u(t) = A_c \cos(2\pi (n f_c - f_{LO}) t + n\phi(t))$$

# FM to AM Demodulator

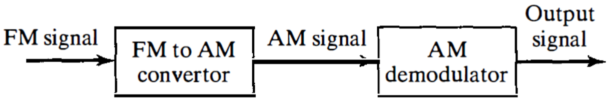


Figure: FM to AM demodulator with differentiator.

$$|H(f)| = V_0 + k(f - f_c), \quad |f - f_c| < \frac{B_c}{2}$$

$$u(t) = A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau)$$

$$v_o(t) = A_c(V_0 + k k_f m(t)) \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau)$$

# FM to AM Demodulator

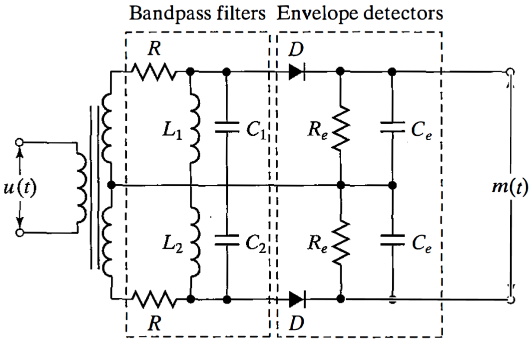


Figure: A balanced FM demodulator.

✗ The noise contained within $B_c$ is passed by the demodulator.

# FM to AM Demodulator

✓ In a balanced FM demodulator, two circuits tuned at two approperate frequencies $f_1$ and $f_2$ are used.

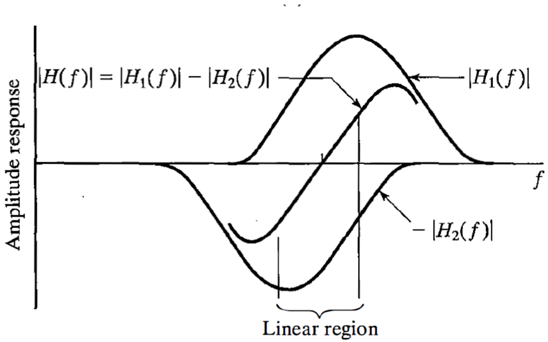


Figure: Differentiator in a balanced FM demodulator.

# PLL Demodulator

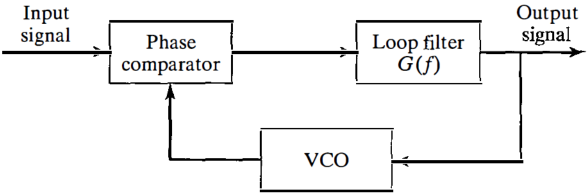


Figure: Block diagram of a PLL demodulator.

$$u(t) = A_c \cos(2\pi f_c t + \phi(t)) = A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau)$$

$$f_v(t) = f_c + k_v v(t)$$

$$y_v(t) = A_c \sin(2\pi f_c t + \phi_v(t)), \quad \phi_v(t) = 2\pi k_f \int_0^t v(\tau) d\tau$$

# PLL Demodulator

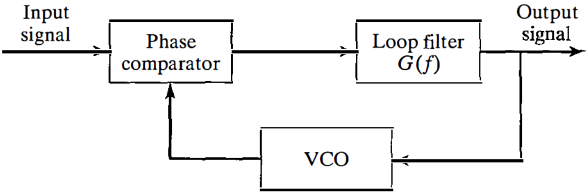


Figure: Block diagram of a PLL demodulator.

The phase comparator is basically a multiplier and a filter that rejects the signal component centered at $2f_c$. Hence, its output may be expressed as

$$e(t) = \frac{1}{2} A_v A_c \sin[\phi(t) - \phi_v(t)]$$

When the PLL is in lock position, the phase error is small. So,

$$\sin[\phi(t) - \phi_v(t)] \approx \phi(t) - \phi_v(t) = \phi_e(t)$$

# PLL Demodulator

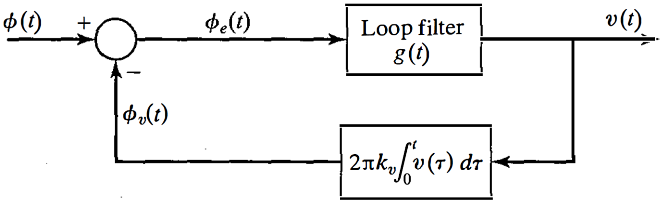


Figure: Linearized locked PLL demodulator.

$$\phi_e(t) = \phi(t) - 2\pi k_v \int_0^t v(\tau) d\tau$$

$$\frac{d\phi_e(t)}{dt} + 2\pi k_v v(t) = \frac{d\phi(t)}{dt}$$

$$\frac{d\phi_e(t)}{dt} + 2\pi k_v \int_0^\infty \phi_e(\tau) g(t-\tau) d\tau = \frac{d\phi(t)}{dt}$$

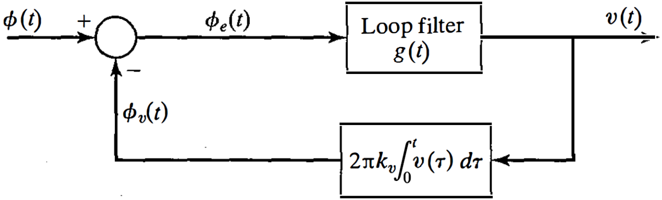


Figure: Linearized locked PLL demodulator.

$$j2\pi f \Phi_e(f) + 2\pi k_v \Phi_e(f) G(f) = j2\pi f \Phi(f) \Rightarrow \Phi_e(f) = \frac{1}{1 + \frac{k_v}{jf} G(f)} \Phi(f)$$

$$\left|\frac{k_v}{jf} G(f)\right| \gg 1 \Rightarrow V(f) = G(f)\Phi_e(f) = \frac{G(f)}{1 + \frac{k_v}{jf} G(f)} \Phi(f) \approx \frac{j2\pi f}{2\pi k_v} \Phi(f)$$

# PLL Demodulator

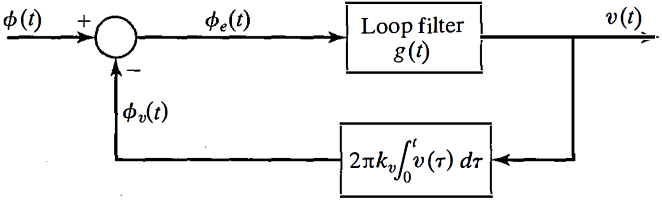


Figure: Linearized locked PLL demodulator.

$$v(t) = \frac{1}{2\pi k_v} \frac{d\phi(t)}{dt} = \frac{k_f}{k_v} m(t)$$

✓ The noise contained within $W$ is passed by the demodulator.

# Effect of Noise on FM signal

## Statement (Effect of Noise on FM Demodulator)

*At high SNR conditions, the SNR at the output of an FM demodulator is*

$$\left(\frac{S}{N}\right)_o = 3P_R\left(\frac{\beta_f}{\max|m(t)|}\right)^2\frac{P_m}{N_0 W}$$

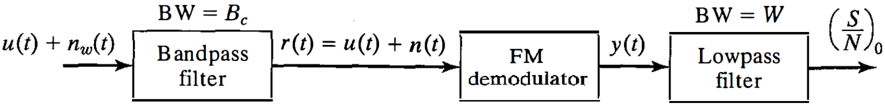


Figure: The block diagram of an FM demodulator.

# Effect of Noise on FM signal

**Statement (Effect of Noise on FM Demodulator)**

*At high SNR conditions, the SNR at the output of an FM demodulator is*

$$\left(\frac{S}{N}\right)_o = 3P_R\left(\frac{\beta_f}{\max|m(t)|}\right)^2 \frac{P_m}{N_0 W}$$

$$u(t) = A_c \cos(2\pi f_c t + \phi(t)) = A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau)$$

$$r(t) = u(t) + n(t) = u(t) + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

$$r(t) = u(t) + \sqrt{n_c^2(t) + n_s^2(t)} \cos(2\pi f_c t + \arctan(\frac{n_s(t)}{n_c(t)}))$$

$$r(t) = u(t) + V_n(t)\cos(2\pi f_c t + \Phi_n(t))$$

# Effect of Noise on FM signal

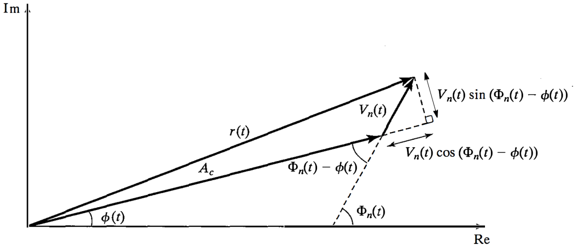


Figure: Phasor diagram of an FM signal when the signal is much stronger than the noise.

$$r(t) = u(t) + V_n(t)\cos(2\pi f_c t + \Theta_n(t))$$

If $V_n(t) \ll A_c$,

$$\begin{aligned} r(t) \approx & [A_c + V_n(t)\cos(\Phi_n(t) - \phi(t))] \\ & \times \cos\left(2\pi f_c t + \phi(t) + \arctan\left(\frac{V_n(t)\sin(\Phi_n(t) - \phi(t))}{A_c + V_n(t)\cos(\Phi_n(t) - \phi(t))}\right)\right) \end{aligned}$$

# Effect of Noise on FM signal

## Statement (Effect of Noise on FM Demodulator)

*At high SNR conditions, the SNR at the output of an FM demodulator is*

$$(\frac{S}{N})_o = 3P_R(\frac{\beta_f}{\max|m(t)|})^2\frac{P_m}{N_0 W}$$

$$\begin{aligned} r(t) =& [A_c + V_n(t)\cos(\Phi_n(t) - \phi(t))] \\ & \times \cos\left(2\pi f_c t + \phi(t) + \frac{V_n(t)}{A_c}\sin(\Phi_n(t) - \phi(t))\right) \end{aligned}$$

$$y(t) = k_f m(t) + \frac{1}{2\pi}\frac{d}{dt}Y_n(t)$$

$$Y_n(t) = \frac{V_n(t)}{A_c}\sin(\Phi_n(t) - \phi(t))$$

# Effect of Noise on FM signal

## Statement (Effect of Noise on FM Demodulator)

*At high SNR conditions, the SNR at the output of an FM demodulator is*

$$(\frac{S}{N})_o = 3P_R(\frac{\beta_f}{\max|m(t)|})^2 \frac{P_m}{N_0 W}$$

$$Y_n(t) = \frac{1}{A_c}\left[V_n(t)\sin(\Phi_n(t))\cos(\phi(t)) - V_n(t)\cos(\Phi_n(t))\sin(\phi(t))\right]$$

$$Y_n(t) = \frac{1}{A_c}\left[n_s(t)\cos(\phi(t)) - n_c(t)\sin(\phi(t))\right]$$

When we compare variations in $n_s(t)$ and $n_c(t)$, we can assume that $\phi(t)$ is almost constant. So,

$$Y_n(t) \approx \frac{1}{A_c}\left[n_s(t)\cos(\phi) - n_c(t)\sin(\phi)\right]$$

# Effect of Noise on FM signal

## Statement (Effect of Noise on FM Demodulator)

*At high SNR conditions, the SNR at the output of an FM demodulator is*

$$\left(\frac{S}{N}\right)_o = 3P_R\left(\frac{\beta_f}{\max|m(t)|}\right)^2 \frac{P_m}{N_0 W}$$

The power spectral density of $Y_n(t)$ is obtained as

$$Y_n(t) \approx \frac{1}{A_c}\left[n_s(t)\cos(\phi) - n_c(t)\sin(\phi)\right]$$

$$S_{Y_n}(f) \approx \left[\left(\frac{\cos(\phi)}{A_c}\right)^2 + \left(\frac{\sin(\phi)}{A_c}\right)^2\right] S_{n_c}(f) = \frac{S_{n_c}(f)}{A_c^2}$$

$$S_{Y_n}(f) \approx \frac{S_{n_c}(f)}{A_c^2} = \begin{cases} \frac{N_0}{A_c^2}, & |f| \le \frac{B_c}{2} \\ 0, & \text{otherwise} \end{cases}$$

# Effect of Noise on FM signal

## Statement (Effect of Noise on FM Demodulator)

*At high SNR conditions, the SNR at the output of an FM demodulator is*

$$\left(\frac{S}{N}\right)_o = 3P_R\left(\frac{\beta_f}{\max|m(t)|}\right)^2\frac{P_m}{N_0W}$$

The power spectral density of $\frac{1}{2\pi}\frac{d}{dt}Y_n(t)$ is obtained as

$$\frac{4\pi^2f^2}{4\pi^2}S_{Y_n}(f) \approx \begin{cases} \frac{N_0}{A_c^2}f^2, & |f| \leq \frac{B_c}{2} \\ 0, & \text{otherwise} \end{cases}$$

The power spectral density of the output noise is

$$S_{n_o}(f) \approx \frac{N_0}{A_c^2}f^2, \quad |f| \leq W$$

# Effect of Noise on FM signal

## Statement (Effect of Noise on FM Demodulator)

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$$\left(\frac{S}{N}\right)_o = 3P_R\left(\frac{\beta_f}{\max|m(t)|}\right)^2 \frac{P_m}{N_0 W}$$

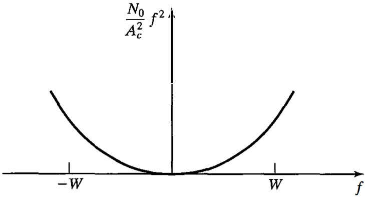


Figure: Noise power spectrum at demodulator output in FM.

# Effect of Noise on FM signal

## Statement (Effect of Noise on FM Demodulator)

*At high SNR conditions, the SNR at the output of an FM demodulator is*

$$\left(\frac{S}{N}\right)_o = 3P_R\left(\frac{\beta_f}{\max|m(t)|}\right)^2 \frac{P_m}{N_0 W}$$

$$P_{n_o} = \frac{2N_0 W^3}{3A_c^2}$$

$$P_{s_o} = k_f^2 P_m$$

$$\left(\frac{S}{N}\right)_o = \frac{3k_f^2 A_c^2}{2W^2}\frac{P_m}{N_0 W} = 3P_R\left(\frac{\beta_f}{\max|m(t)|}\right)^2 \frac{P_m}{N_0 W}$$

# Effect of Noise on FM signal

1. In FM, the output SNR is proportional to the square of the modulation index $\beta_f$.
2. The increase in the received SNR is obtained by increasing the bandwidth.
3. Increasing $\beta_f$ increases the noise power and therefore, the approximation $V_n(t) \ll A_c$ will no longer be valid. When this event, which is called threshold effect, occurs the signal will be lost in noise.
4. Increasing the transmitter power reduces the noise power and results in a better SNR.
5. In FM, the effect of noise is higher at higher frequencies.

# Effect of Noise on FM signal

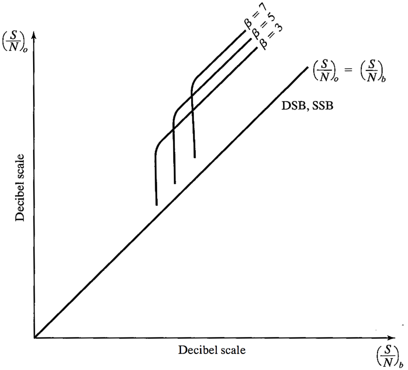


Figure: Output SNR of an FM system as a function of the baseband SNR $\left(\frac{S}{N}\right)_b = \frac{P_R}{N_0 W}$.

$$\left(\frac{S}{N}\right)_o = 3\frac{P_m \beta_f^2}{(\max|m(t)|)^2}\left(\frac{S}{N}\right)_b = \frac{3}{2}\beta_f^2\left(\frac{S}{N}\right)_b, \qquad \frac{P_m}{(\max|m(t)|)^2} = \frac{1}{2}$$

# Phase Modulation

# PM

## Statement (PM)

*A phase-modulated signal is written as*

$$u(t) = A_c \cos(2\pi f_c t + \phi(t)) = A_c \cos(2\pi f_c t + k_p m(t))$$

*, where $k_p$ is called phase deviation constant.*

## Example (PM signal)

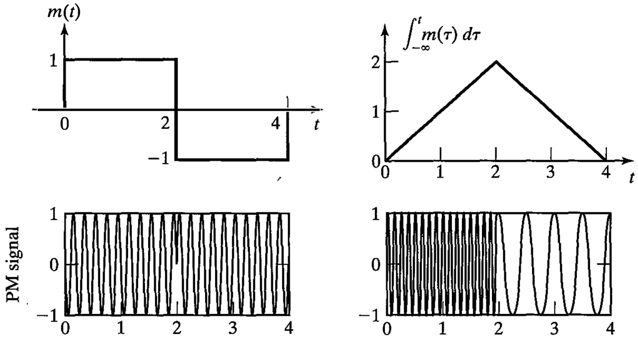


Figure: Phase modulation of square and sawtooth waves.

**Statement (PM Modulation Index)**

*The modulation index of the PM is defined as*

$$\beta_p = k_p \max\{|m(t)|\} = \Delta\phi_{max}$$

*, where $\Delta\phi_{max}$ is the maximum phase deviation.*

# PM Modulation

**Example (Sinusoidally-modulated PM signal)**

For the message signal $m(t) = a\sin(2\pi f_m t)$, the PM signal is

$$u(t) = A_c \cos(2\pi f_c t + k_p a \sin(2\pi f_m t)) = A_c \cos(2\pi f_c t + \beta_p \sin(2\pi f_m t))$$

.

# Relationship between PM and FM

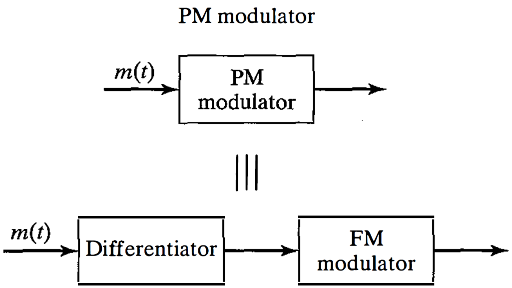


Figure: A comparison of frequency and phase modulators.

# Relationship between PM and FM

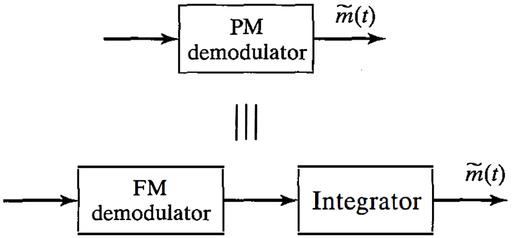


Figure: A comparison of frequency and phase demodulators.

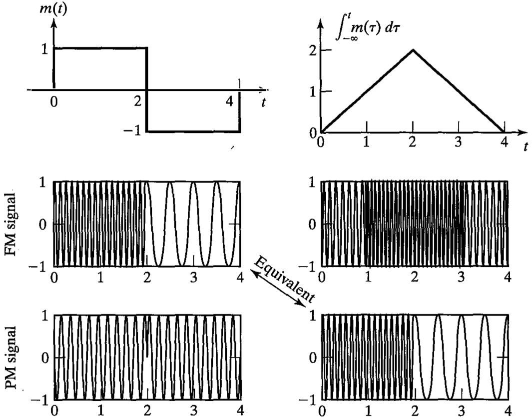


Figure: Frequency and phase modulations of square and sawtooth waves.

## Statement (PM by a Sinusoidal Signal)

*For the sinusoidal message* $m(t) = a\sin(2\pi f_m t)$, *the PM signal is*

$$u(t) = A_c \cos(2\pi f_c t + \beta_p \sin(2\pi f_m t)) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta_p)\cos(2\pi(f_c + nf_m)t)$$

, *where* $J_n(\beta_p)$ *is the Bessel function of the first kind of order* $n$.

# Bandwidth and Power of PM Signal

## Statement (Bandwidth of Sinusoidal PM)

*For the sinusoidal message* $m(t) = a\sin(2\pi f_m t)$, *the actual bandwidth of the PM signal is infinite.*

## Statement (Power of Sinusoidal PM)

*For the sinusoidal message* $m(t) = a\sin(2\pi f_m t)$, *the power of the PM signal is* $A_c^2/2$.

# Bandwidth and Power of PM Signal

1. The 98%-power effective bandwidth of sinusoidal PM is approximately $B_c = 2(\beta_p + 1)f_m = 2(k_p a + 1)f_m$.
2. Increasing $a$, the amplitude of the modulating signal, increases the bandwidth $B_c$.
3. Increasing $f_m$, the frequency of the message signal, also increases the bandwidth $B_c$.
4. The number of harmonics, including the carrier, is $M_c = 2([\beta]+1)+1 = 2[\beta] + 3 = 2[k_p a] + 3$.
5. Increasing the amplitude $a$ increases the number of harmonics.
6. Increasing $f_m$ does not change the number of harmonics.

# Bandwidth and Power of PM Signal

## Statement (Effective Bandwidth of PM (Carson's Rule))

*The effective bandwidth of a PM signal is approximately*

$$B_c = 2(\beta_p + 1)W$$

*, where $W$ is the frequency of the message signal $m(t)$.*

## Statement (Power of PM)

*The power content of a PM signal is $\frac{A_c^2}{2}$.*

# Effect of Noise on PM signal

## Statement (Effect of Noise on PM Demodulator)

*At high SNR conditions, the SNR at the output of a PM demodulator is*

$$\left(\frac{S}{N}\right)_o = P_R\left(\frac{\beta_p}{\max|m(t)|}\right)^2 \frac{P_m}{N_0 W}$$

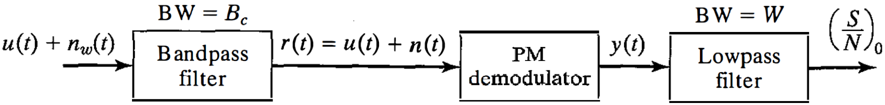


Figure: The block diagram of an PM demodulator.

# Effect of Noise on PM signal

1. In PM, the output SNR is proportional to the square of the modulation index $\beta_p$.
2. The increase in the received SNR is obtained by increasing the bandwidth.
3. Increasing $\beta_p$ increases the noise power and therefore, the approximation $V_n(t) \ll A_c$ will no longer be valid. When this event, which is called threshold effect, occurs the signal will be lost in noise.
4. Increasing the transmitter power reduces the noise power and results in a better SNR.

# Comparison of Analog Modulations

# Performance Comparison

1. Required bandwidth: $\text{SSB} \gtrsim \text{VSB} > \text{DSB} = \text{AM} \gg \text{FM} \approx \text{PM}$.
2. Transmitted power: $\text{FM} \approx \text{PM} > \text{DSB} \gtrsim \text{SSB} \approx \text{VSB} > \text{AM}$.
3. Transceiver complexity: $\text{AM} \gtrsim \text{FM} \approx \text{PM} > \text{DSB} > \text{VSB} > \text{SSB}$.
4. Noise immunity: $\text{FM} \gtrsim \text{PM} \gg \text{SSB} = \text{DSB} \approx \text{VSB} > \text{AM}$.

# The End