
MATHEMATICAL QUESTIONS

Question 1

Show that if the periodic signal $x(t + T_0) = x(t)$ passes through an LTI system with the frequency response $H(f)$, then the power spectral density of the output signal equals

$$S_y(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 |H(n/T_0)|^2 \delta(f - n/T_0)$$

Note: Consult the slides on the calculation of the autocorrelation of the periodic signals.

Question 2

Although the solution of this question has been already posted to the course Telegram channel, it deserves to be repeated again due to its involved intuition. The real bandpass signal $x(t)$, whose frequency components are in the neighborhood of some f_0 (and $-f_0$), is passed through a filter with the transfer function $H(f)$, and the output is denoted by $y(t)$. The magnitude of the transfer function is denoted by $A(f)$, and its phase is denoted by $\theta(f)$. Assume that the transfer function of the filter is so smooth that in the bandwidth of the input signal, the magnitude of the transfer function is essentially constant and its phase can be approximated by its first-order Taylor-series expansion, i.e.,

$$A(f) \approx A(f_0)$$

and

$$\theta(f) \approx \theta(f_0) + (f - f_0)\theta'(f_0)$$

(a) Show that $Y_l(f)$, the Fourier transform of the lowpass equivalent of the output, can be written as

$$Y_l(f) \approx X_l(f)A(f_0)e^{j(\theta(f_0)+f\theta'(f_0))}$$

, where $X_l(f)$ is the Fourier transform of the lowpass equivalent of the input $x(t)$.

(b) Calculate that

$$y(t) \approx A(f_0)V_x(t - t_g) \cos(2\pi f_0(t - t_p) + \Theta_x(t - t_g))$$

, where $V_x(t)$ and $\Theta_x(t)$ are the envelop and phase of the input $x(t)$, $t_g = -\theta'(f_0)/(2\pi)$, and $t_p = -\theta(f_0)/(2\pi f_0)$.

(c) The quantities t_g and t_p are called envelope delay (or group delay) and phase delay, respectively. Can you interpret their role and justify this nomenclature?

Question 3

Intuitively, the bandwidth measures for the extent of significant spectral content of the real signal for positive frequencies. Different mathematical definitions might be used to calculate the bandwidth of the signal $|X(f)|$. Here are some of the common definitions:

1. **Absolute bandwidth** is the smallest positive frequency band, where, for frequencies outside it, $|X(f)|$ is zero.
2. **3-dB or half-power bandwidth** is the positive frequency band, where, for frequencies outside it, $|X(f)|$ is never greater than $1/\sqrt{2}$ times its maximum value.
3. **Null-to-null or zero-crossing bandwidth** is the frequency band, where the band edge frequencies create the first spectrum nulls. For the lowpass signals, the right side edge frequency only creates the null.
4. **Power bandwidth** is the positive frequency band in which 49.5% of the total power (or energy) resides.
5. **The Root Mean Square (RMS) bandwidth** is defined as $\sqrt{\int_0^{+\infty} f^2 |X(f)|^2 df / \int_0^{+\infty} |X(f)|^2 df}$ for a lowpass signal and $2\sqrt{\int_0^{+\infty} (f - f_0)^2 |X(f)|^2 df / \int_0^{+\infty} |X(f)|^2 df}$ for a bandpass signal centered around f_0 .

Fig. 1 shows a typical illustration of these definitions.

Note: If you need to numerically evaluate an integral or a function while answering the questions below, feel free to consult online calculators such as [Symbolab](#).

(a) Find the defined bandwidths for the signal $X(f) = A \Pi(\frac{f}{B})$.

(b) Find the defined bandwidths for the signal $X(f) = A \text{sinc}(\frac{f}{B})$.

(c) Which definition is better and more applicable? Why?

Question 4

Let Y be a positive-valued random variable, i.e., $f_Y(y) = 0$ for $y < 0$. You might consult Google to prove the following useful inequalities.

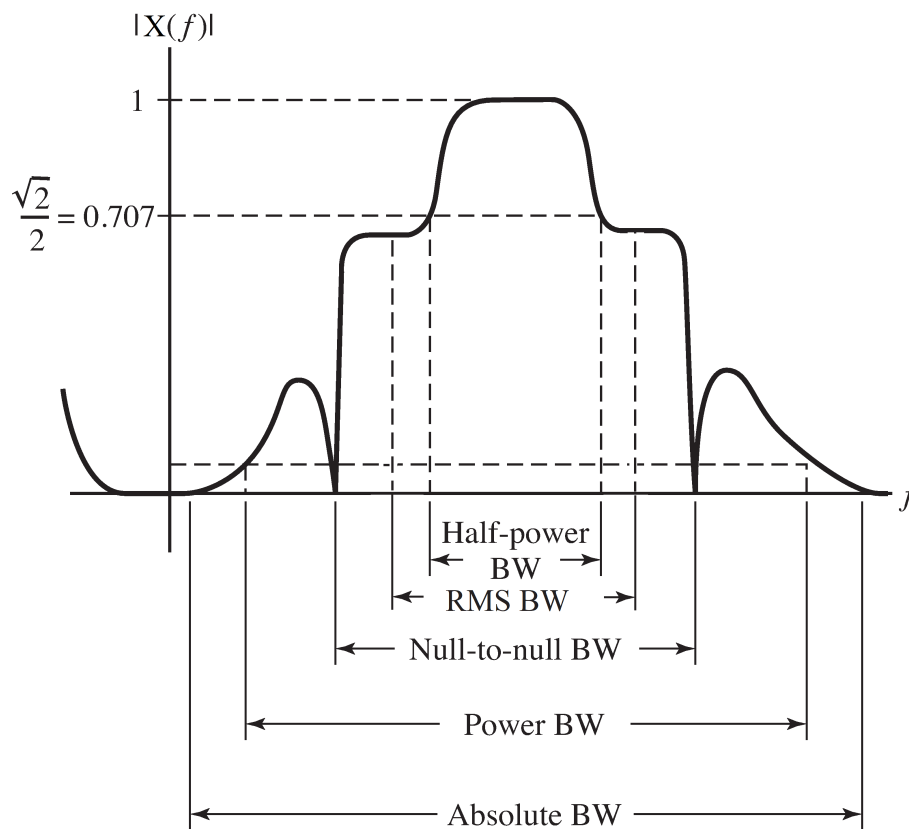


Figure 1: Various ways to define bandwidth for a real-valued signal.

(a) Let α be any positive constant, and prove the Markov inequality $P(Y > a) \leq \frac{E(Y)}{\alpha}$.

(b) Let X be any random variable with variance σ^2 , and define $Y = (X - E[X])^2$ and $\alpha = \epsilon^2$ for some positive ϵ . Derive the Chebychev inequality $P(|X - E(X)| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$.

Question 5

The characteristic function provides an alternative way for describing a random variable, similar to the CDF, which completely determines the behavior and properties of a random variable. The characteristic function of the random variable X is defined as

$$\phi_X(t) = E\{e^{jtX}\}$$

. Take a look at the free encyclopedia of wikipedia to answer the following questions.

(a) How does the characteristic function relate to the Fourier transform of the PDF $f_X(x)$?

(b) How can we find the mean and variance using the characteristic function?

(c) Calculate the characteristic function of the normal distribution $\mathcal{N}(\mu, \sigma^2)$ and use it to derive the mean and variance of the normal distribution.

SOFTWARE QUESTIONS

Question 6

MATLAB provides a useful function named *hilbert()*. Refer to the help page of the MATLAB software to learn more about *hilbert()* command and answer the questions below.

(a) How does the command of *hibert()* differ from the Hilbert transform introduced in the course lectures?

(b) What is Fast Fourier Transform and how does MATLAB use it to calculate the *hilbert()* command?

(c) Code an *mfile*, which takes an arbitrary signal as its input and plots the time- and frequency-domain graphs of the Hilbert transform of its input signal. Feel free to use MATLAB internal functions such as *hilbert()*.

BONUS QUESTIONS

Question 7

Return your answers by filling the \LaTeX template of the assignment.