Question 1

Consider the block diagram of Fig. 1, where

1. **$m(t)$ is a lowpass message with the bandwidth W_m.**
2. **The zero-order hold ADC has the sampling rate f_s and quantization level 2^ν.**
3. **In the PAM modulator, the bits are mapped to the bipolar NRZ signal**

$$u(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT)$$

, where the pulse $p(t) = \mathrm{sinc}(\frac{t}{T})$ and the binary polar symbols $a_k = 2(b_k - 0.5)A$. For simplicity, the symbols are assumed to be independent and identically distributed with $P\{a_k = A\} = P\{a_k = -A\} = 0.5$, the mean 0, and the autocorrelation function

$$R_a[n] = E\{a_{n+k}a_k\} = \begin{cases} A^2, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

4. **The FM modulator has the index β_f and its output is $v(t) = A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} u(\tau)d\tau)$.**
5. **The channel adds an AWGN noise $n_W(t)$ with the power spectral density $\frac{N_0}{2}$ to its input signal.**
6. **The FM demodulator is an ideal FM receiver.**
7. **The PAM demodulator works perfectly without any ISI or synchronization mismatch. The comparator threshold is set to its optimal value in the PAM demodulator.**
8. **The conditions of the perfect reconstruction are held in the DAC.**
9. **The required filters in the modulators and demodulators are ideal.**

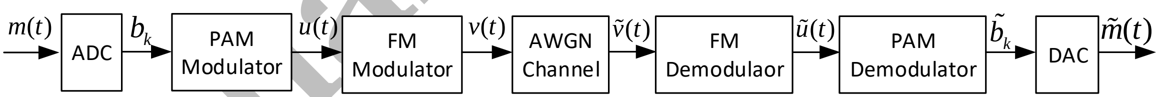

Figure 1: A mixed analog-digital communication system.

(a) Write an expression for the power spectral density $S_u(f)$ of $u(t)$ and determine the power content P_u of $u(t)$.

Since the symbol random process is zero-mean, independent, and consequently uncorrelated,

$$S_u(f) = \frac{R_a[0]}{T}|P(f)|^2 = \frac{R_a[0]}{T}|\mathcal{F}\{\mathrm{sinc}(\frac{t}{T})\}|^2 = \frac{A^2}{T}|T \sqcap (Tf)|^2 = TA^2 \sqcap (Tf)$$

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Therefore,

$$P_u = \int_{-\infty}^{\infty} S_u(f)df = TA^2\frac{1}{T} = A^2$$

(b) Determine 100% power bandwidth of $u(t)$, i.e., the bandwidth $[0, W_u]$ containing half of the power content of $u(t)$.

Clearly, the power spectral density is nonzero over the interval $[-\frac{1}{2T}, \frac{1}{2T}]$. So, $W_u = \frac{1}{2T}$.

(c) Determine the power P_v and bandwidth W_v of $v(t)$.

$v(t)$ is an FM signal, so $P_v = \frac{A_c^2}{2}$ and

$$W_v = 2(\beta_f + 1)W_u = \frac{\beta_f + 1}{T}$$

(d) Determine the output bit rate B_b of the ADC.

$$B_b = f_s \log_2(2^\nu) = \nu f_s$$

(e) Determine the bit rate B_u and symbol rate S_u of the PAM modulator.

Since the PAM modulator is binary,

$$B_u = S_u = \frac{1}{T}$$

(f) Specify the required conditions for perfect reconstruction of $\widetilde{m}(t)$ at the DAC.

The sampling rate should be atleast twice the bandwidth of sampled signal $m(t)$, i.e. $f_s \geq 2W_m$, and the frequency response corresponding to the zero-order hold pulse should be invertible over $[-W_m, W_m]$.

(g) Assuming high SNR conditions for the FM demodulator, find the SNR of $\tilde{u}(t)$.

$$(\frac{S}{N})_{\tilde{u}} = \frac{3k_f^2 A_c^2 P_u}{2N_0 W_u^3} = \frac{12T^3 A^2 k_f^2 A_c^2}{N_0}$$

(h) Assuming high SNR conditions for the FM demodulator, find the BER of $\tilde{b}_k$.

At output of the FM demodulator,

$$\widetilde{u}(t) = k_f u(t) + n(t) = k_f \sum_{k=0}^{\infty} a_k p(t - kT) + n(t)$$

, where $n(t)$ is the outpul noise of the FM demodulator with the power spectral density of $\frac{N_0}{A_c^2} f^2$. When $\widetilde{u}(t)$ passes the input filter of the PAM demodulator and is sampled at the regenerator to recover the Kth sample, the samples value equals

$$y(t_K) = k_f a_K + n(t_K)$$

since no ISI and synchronization mismatch occurs. Now,

$$\begin{aligned}
P_e &= P_0 P_{e|0} + P_1 P_{e|1} = \frac{1}{2}(P_{e|0} + P_{e|1}) \\
&= \frac{1}{2}(P[y(t_K) > V | a_K = -k_f A] + P[y(t_K) \le V | a_K = k_f A]) \\
&= \frac{1}{2}(P[n(t_K) - k_f A > V | a_K = 0] + P[n(t_K) + k_f A \le V | a_K = A]) \\
&= \frac{1}{2}(Q(\frac{V + k_f A}{\sigma}) + Q(\frac{k_f A - V}{\sigma}))
\end{aligned}$$

, where σ^2 is the variance of the noise $n(t_K)$. Optimizing the threshold,

$$\begin{aligned}
&\frac{dP_e}{dV} = 0 \\
\Rightarrow &-\frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{(V + k_f A)^2}{\sigma^2}} + \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{(k_f A - V)^2}{\sigma^2}} = 0 \\
\Rightarrow &(V + k_f A)^2 = (k_f A - V)^2 \\
\Rightarrow &4 k_f A V = 0 \\
\Rightarrow &V = 0 \\
\Rightarrow &P_{e_{\text{min}}} = Q(\frac{k_f A}{\sigma})
\end{aligned}$$

Finally,

$$\sigma^2 = E\{n(t_k)^2\} - 0^2 = R_n(0) = \int_{-\infty}^{\infty} S_n(f) df = \int_{-W_u}^{W_u} \frac{N_0}{A_c^2} f^2 df = \frac{2N_0}{3A_c^2} W_u^3 = \frac{N_0}{12 A_c^2 T^3}$$

So,

$$P_{e_{\text{min}}} = Q(\frac{k_f A}{\sqrt{\frac{N_0}{12 A_c^2 T^3}}}) = Q(\frac{k_f A_c A \sqrt{12 T^3}}{\sqrt{N_0}}) = Q(2 k_f A_c A T \sqrt{3 \frac{T}{N_0}})$$

(i) Assuming that the channel bandwidth is W_c, find the maximum bandwidth of $m(t)$ for which the system works properly.

The channel has to accommodate the FM modulated signal. So,

$$W_c \geq W_v = 2(\beta_f + 1)W_u = \frac{\beta_f + 1}{T} \Rightarrow T \geq (\beta_f + 1)\frac{1}{W_c}$$

On the other hand, the bit rate of the PAM should be equal or higher than its input rate. So,

$$B_u = \frac{1}{T} \geq B_b = \nu f_s \geq 2W_m \nu$$

So,

$$W_{m_{max}} = \frac{1}{2\nu T} = \frac{W_c}{2\nu(\beta_f + 1)}$$

(j) Find the maximum bandwidth of $m(t)$ for which the BER of $\tilde{b}_k$ is less than a given value of $P_{e_{th}}$.

Remember that $Q(x)$ is a decreasing function.

$$\begin{aligned}
&P_{e_{\text{min}}} \leq P_{e_{th}} \\
\Rightarrow &Q(2k_f A_c AT\sqrt{3\frac{T}{N_0}}) \leq P_{e_{th}} \\
\Rightarrow &2k_f A_c AT\sqrt{3\frac{T}{N_0}} \geq Q^{-1}(P_{e_{th}}) \\
\Rightarrow &T \geq \sqrt[3]{\frac{N_0}{12k_f^2 A_c^2 A^2}\left[Q^{-1}(P_{e_{th}})\right]^2}
\end{aligned}$$

On the other hand, the bit rate of the PAM should be equal or higher than its input rate. Further, perfect reconstruction conditions should be held. So,

$$B_u = \frac{1}{T} \geq B_b = \nu f_s \geq 2W\nu$$

Therefore,

$$W_{m_{max}} = \frac{1}{2\nu T} = \frac{1}{2\nu\sqrt[3]{\frac{N_0}{12k_f^2 A_c^2 A^2}\left[Q^{-1}(P_{e_{th}})\right]^2}}$$