Probability, Random Variables, and Stochastic Processes

Mohammad Hadi

mohammad.hadi@sharif.edu

@MohammadHadiDastgerdi

Fall 2020

Overview

Probability

Sample Space, Events, and Probability

- A random experiment is any experiment whose outcome cannot be predicted with certainty.
- A random experiment has certain outcomes $\omega \in \Omega$.
- The set of all possible outcomes is called the sample space Ω.
- A sample space is discrete if the number of its elements are finite or countably infinite, otherwise it is a nondiscrete sample space.
- Events are subsets of the sample space, i.e., $E \subset \Omega$.
- Events are disjoint if their intersection is empty. i.e. $E_i \cap E_j = \emptyset$.

Sample Space, Events, and Probability

Definition (Probability Axioms)

A probability P is defined as a set function assigning nonnegative values to all events E such that

1. $0 \leq P(E) \leq 1$ for all events.
2. $P(\Omega) = 1$.
3. For disjoint events $E_1, E_2, \cdots,$ $P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$.

1. $P(E^c) = 1 - P(E), \quad E^c = \Omega \setminus E$.
2. $P(\emptyset) = 0$.
3. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$.
4. $E_1 \subseteq E_2 \Rightarrow P(E_1) \leq P(E_2)$.

Conditional Probability

Definition (Conditional Probability)

The conditional probability of the event E_1 given the event E_2 is defined by

$$P(E_1|E_2) = \begin{cases} \frac{P(E_1 \cap E_2)}{P(E_2)} & , \quad P(E_2) \neq 0 \\ 0 & , \quad P(E_2) = 0 \end{cases}$$

Conditional Probability

1. The events E_1 and E_2 are said to be independent if $P(E_1|E_2) = P(E_1)$.
2. For independent events, $P(E_1 \cap E_2) = P(E_1)P(E_2)$.
3. If the events $\{E_i\}_{i=1}^n$ are disjoint and their union is the entire sample space, then they make a partition of the sample space Ω.
4. The total probability theorem states that for an event A, $P(A) = \sum_{i=1}^n P(E_i)P(A|E_i)$.
5. Bayes's rule gives the conditional probabilities $P(E_i|A)$ by

$$P(E_i|A) = \frac{P(E_i)P(A|E_i)}{P(A)} = \frac{P(E_i)P(A|E_i)}{\sum_{i=1}^n P(E_i)P(A|E_i)}$$

Random Variables

Random Variables

Definition (Random Variable)

A random variable is a mapping from the sample space Ω to the set of real numbers.

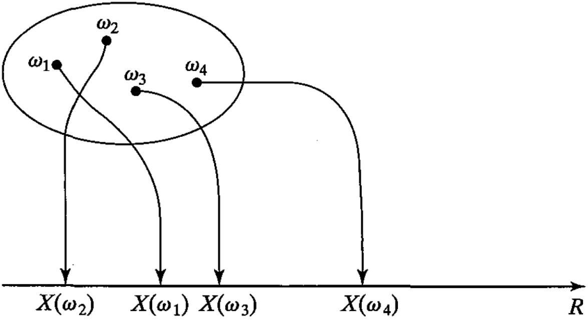

Figure: A random variable as a mapping from Ω to $\mathbb{R}$.

Random Variables

Definition (Cumulative Distribution Function (CDF))

The cumulative distribution function or CDF of a random variable X is defined as

$$F_X(x) = P\{\omega \in \Omega : X(\omega) \leq x\} = p\{X \leq x\}$$

1. $0 \leq F_X(x) \leq 1$.
2. $F_X(-\infty) = 0, \quad F_X(\infty) = 1$.
3. $P(a < X \leq b) = F_X(b) - F_X(a)$.

Random Variables

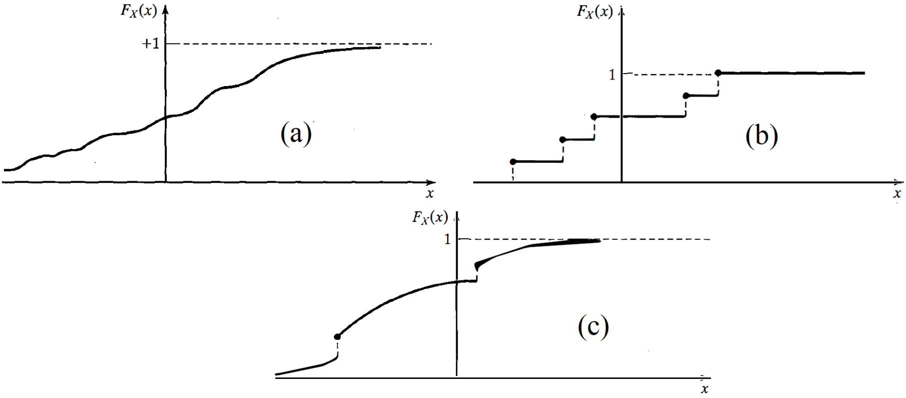

Figure: CDF for a (a) continuous (b) discrete (c) mixed random variable.

Random Variables

Definition (Probability Density Function (PDF))

The probability density function or PDF of a random variable X is defined as

$$f_X(x) = \frac{dF_X(x)}{dx}$$

1. $f_X(x) \geq 0$.
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$.
3. $P(a < X \leq b) = \int_a^b f_X(x)dx$.
4. $F_X(x) = \int_{-\infty}^{x^+} f_X(u)du$.

Random Variables

Definition (Probability Mass Function (PMF))

The probability mass function or PMF of a discrete random variable X is defined as

$$p_i = P\{X = x_i\}$$

1. $p_i \geq 0$.
2. $\sum_i p_i = 1$.

Important Random Variables

Statement (Bernoulli Random Variable)

The Bernoulli random variable is a discrete random variable taking two values 1 *and* 0*, with probabilities* p *and* $1 - p$*.*

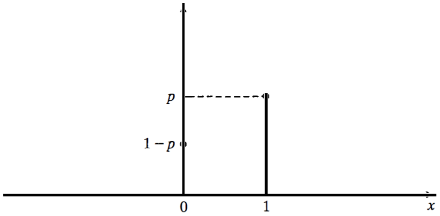

Figure: The PMF for the Bernoulli random variable.

Important Random Variables

Statement (Binomial Random Variable)

The binomial random variable is a discrete random variable giving the number of 1 *'s in n independent Bernoulli trials. The PMF is given by*

$$P\{X = k\} = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & 0 \le k \le n \\ 0, & \textit{otherwise} \end{cases}$$

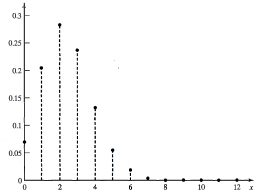

Figure: The PMF for the binomial random variable.

Statement (Uniform Random Variable)

The Uniform random variable is a continuous random variable taking values between a and b with equal probabilities for intervals of equal length. The density function is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & otherwise \end{cases}$$

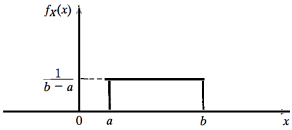

Figure: The PDF for the uniform random variable.

Important Random Variables

Statement (Gaussian Random Variable)

The Gaussian, or normal, random variable $\mathcal{N}(m, \sigma^2)$ *is a continuous random variable described by the density function*

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

, where m*,* σ*, and* σ^2 *are named mean, standard deviation, and variance.*

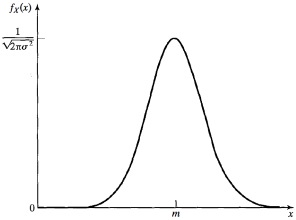

Figure: The PDF for the Gaussian random variable.

Statement (Q Function)

Assuming that X is a standard normal random variable $\mathcal{N}(0,1)$, the function $Q(x)$ is defined as

$$Q(x) = P\{X > x\} = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

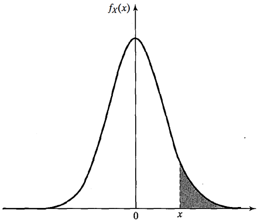

Figure: The Q-function as the area under the tail of a standard normal random variable.

The Q function has the following properties,

1. $Q(-\infty) = 1, \quad Q(0) = 0.5, \quad Q(+\infty) = 0.$
2. $Q(-x) = 1 - Q(x).$

The important bounds on the Q function are

1. $Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}, \quad x \geq 0.$
2. $Q(x) < \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}, \quad x \geq 0.$
3. $Q(x) > \frac{1}{\sqrt{2\pi}x}(1 - \frac{1}{x^2})e^{-\frac{x^2}{2}}, \quad x > 1.$

For an $\mathcal{N}(m, \sigma^2)$ random variable,

1. $F_X(x) = P\{X \leq x\} = 1 - Q(\frac{x-m}{\sigma}).$

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0.0	5.000000×10^{-01}	2.4	8.197534×10^{-03}	4.8	7.933274×10^{-07}
0.1	4.601722×10^{-01}	2.5	6.209665×10^{-03}	4.9	4.791830×10^{-07}
0.2	4.207403×10^{-01}	2.6	4.661189×10^{-03}	5.0	2.866516×10^{-07}
0.3	3.820886×10^{-01}	2.7	3.466973×10^{-03}	5.1	1.698268×10^{-07}
0.4	3.445783×10^{-01}	2.8	2.555131×10^{-03}	5.2	9.964437×10^{-06}
0.5	3.085375×10^{-01}	2.9	1.865812×10^{-03}	5.3	5.790128×10^{-08}
0.6	2.742531×10^{-01}	3.0	1.349898×10^{-03}	5.4	3.332043×10^{-08}
0.7	2.419637×10^{-01}	3.1	9.676035×10^{-04}	5.5	1.898956×10^{-08}
0.8	2.118554×10^{-01}	3.2	6.871378×10^{-04}	5.6	1.071760×10^{-08}
0.9	1.840601×10^{-01}	3.3	4.834242×10^{-04}	5.7	5.990378×10^{-09}
1.0	1.586553×10^{-01}	3.4	3.369291×10^{-04}	5.8	3.315742×10^{-09}
1.1	1.356661×10^{-01}	3.5	2.326291×10^{-04}	5.9	1.817507×10^{-09}
1.2	1.150697×10^{-01}	3.6	1.591086×10^{-04}	6.0	9.865876×10^{-10}
1.3	9.680049×10^{-02}	3.7	1.077997×10^{-04}	6.1	5.303426×10^{-10}
1.4	8.075666×10^{-02}	3.8	7.234806×10^{-05}	6.2	2.823161×10^{-10}
1.5	6.680720×10^{-02}	3.9	4.809633×10^{-05}	6.3	1.488226×10^{-10}
1.6	5.479929×10^{-02}	4.0	3.167124×10^{-05}	6.4	7.768843×10^{-11}
1.7	4.456546×10^{-02}	4.1	2.065752×10^{-05}	6.5	4.016001×10^{-11}
1.8	3.593032×10^{-02}	4.2	1.334576×10^{-05}	6.6	2.055790×10^{-11}
1.9	2.871656×10^{-02}	4.3	8.539898×10^{-06}	6.7	1.042099×10^{-11}
2.0	2.275013×10^{-02}	4.4	5.412542×10^{-06}	6.8	5.230951×10^{-12}
2.1	1.786442×10^{-02}	4.5	3.397673×10^{-06}	6.9	2.600125×10^{-12}
2.2	1.390345×10^{-02}	4.6	2.112456×10^{-06}	7.0	1.279813×10^{-12}
2.3	1.072411×10^{-02}	4.7	1.300809×10^{-06}		

Table: Table of the Q Function.

Example (Q Function)

X is a Gaussian random variable with mean 1 and variance 4. Therefore,

$$\begin{aligned} P(5 < X < 7) &= F_X(7) - F_X(5) \\ &= 1 - Q(\frac{7-1}{2}) - [1 - Q(\frac{5-1}{2})] \\ &= Q(2) - Q(3) \approx 0.0214 \end{aligned}$$

Functions of a Random Variable

Statement (Functions of a Random Variable)

The CDF of the random variable $Y = g(X)$ is

$$F_Y(y) = P\{\omega \in \Omega : g(X(\omega)) \leq y\}$$

. In the special case that, for all y, the equation $g(x) = y$ has a countable number of solutions $\{x_i\}$, and for all these solutions, $g'(x_i)$ exists and is nonzero,

$$f_Y(y) = \sum_i \frac{f_X(x_i)}{|g'(x_i)|}$$

.

Functions of a Random Variable

Example (Linear function of a normal variable)

if X is $\mathcal{N}(m, \sigma^2)$, then $Y = aX + b$ is also a Gaussian random variable of the form $\mathcal{N}(am + b, a^2\sigma^2)$.

If $y = ax + b = g(x)$, then $x = (y - b)/a$ and $g'(x) = a$. So,

$$
\begin{aligned}
f_Y(y) &= \frac{f_X(x)}{|g'(x)|}\Big|_{x=(y-b)/a} \\
&= \frac{1}{a}\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}\Big|_{x=(y-b)/a} \\
&= \frac{1}{\sqrt{2\pi}a\sigma} e^{-\frac{(\frac{y-b}{a}-m)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}a\sigma} e^{-\frac{(y-b-am)^2}{2a^2\sigma^2}}
\end{aligned}
$$

Statistical Averages

Definition (Mean of Function)

The mean, expected value, or expectation of the random variable $Y = g(X)$ is defined as

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Definition (Mean of Function)

The mean, expected value, or expectation of the discrete random variable $Y = g(X)$ is defined as

$$E\{g(X)\} = \sum_i g(x_i) P\{X = x_i\}$$

Statistical Averages

Definition (Mean)

The mean, expected value, or expectation of the random variable X is defined as

$$E\{X\} = m_X = \int_{-\infty}^{\infty} x f_X(x) dx$$

Definition (Mean)

The mean, expected value, or expectation of the discrete random variable X is defined as

$$E\{X\} = m_X = \sum_i x_i P\{X = x_i\}$$

1. $E(cX) = cE(X)$.
2. $E(X + c) = c + E(X)$.
3. $E(c) = c$.

Statistical Averages

Definition (Variance)

The variance of the random variable X is defined as

$$\sigma_X^2 = V(X) = E\{(X - E\{X\})^2\} = E\{X^2\} - (E\{X\})^2$$

1. $V(cX) = c^2 V(X)$.
2. $V(X + c) = V(X)$.
3. $V(c) = 0$.

Important Random Variables

Example (Bernoulli random variable)

If X is a Bernoulli random variable, $E(X) = p$ and $V(X) = p(1-p)$.

Example (Binomial random variable)

If X is a Binomial random variable, $E(X) = np$ and $V(X) = np(1-p)$.

Example (Uniform random variable)

If X is a Uniform random variable, $E(X) = \frac{a+b}{2}$ and $V(X) = \frac{(b-a)^2}{12}$.

Example (Gaussian random variable)

If X is a Gaussian random variable, $E(X) = m$ and $V(X) = \sigma^2$.

Bi-variate Random Variables

Definition (Joint CDF)

Let X and Y represent two random variables. For these two random variables, the joint CDF is defined as

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$$

1. $F_X(x) = F_{X,Y}(x, \infty)$.
2. $F_Y(x) = F_{X,Y}(\infty, y)$.
3. If X and Y are statistically independent, $F_{X,Y}(x, y) = F_X(x)F_Y(y)$.

Bi-variate Random Variables

Definition (Joint PDF)

Let X and Y represent two random variables. For these two random variables, the joint PDF is defined as

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

1. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$.
2. $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$.
3. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$.
4. $P\{(x,y) \in A\} = \int \int_{(x,y)\in A} f_{X,Y}(x,y)dxdy$.
5. $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)dudv$.
6. If X and Y are statistically independent, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

Definition (Conditional PDF)

The conditional PDF of the random variable Y, given that the value of the random variable X is equal to x, is defined as

$$f_{Y|X}(y|x) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_X(x)}, & f_X(x) \neq 0 \\ 0, & f_X(x) = 0 \end{cases}$$

Bi-variate Random Variables

Definition (Mean)

The expected value of $g(X,Y)$ is defined as $E\{g(X,Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)dxdy$

Definition (Correlation)

$R(X,Y) = E(XY)$ is called the correlation f X and Y.

Definition (Covariance)

The covariance of X and Y is defined as $C(X,Y) = E(XY) - E(X)E(Y)$.

Definition (Correlation Coefficient)

The correlation coefficient of X and Y is defined as $\rho_{X,Y} = C(X,Y)/(\sigma_X \sigma_Y)$.

Bi-variate Random Variables

1. If $\rho_{X,Y} = C(X,Y) = 0$. i.e., $E(XY) = E(X)E(Y)$, then X and Y are called uncorrelated.
2. If X and Y are independent, $E(XY) = E(X)E(Y)$, i.e., X and Y are uncorrelated.
3. $|\rho_{X,Y}| \leq 1$.
4. If $\rho_{X,Y} = 1$, then $Y = aX + b$, where a is a positive.
5. If $\rho_{X,Y} = -1$, then $Y = aX + b$, where a is a negative.

Example (Moment calculation)

Assume that $X \sim \mathcal{N}(3,4)$ and $Y \sim \mathcal{N}(-1,2)$ are independent. If $Z = X - Y$ and $W = 2X + 3Y$, then

$$
\begin{aligned}
E(Z) &= E(X) - E(Y) = 3 + 1 = 4 \\
E(W) &= 2E(X) + 3E(Y) = 6 - 3 = 3 \\
E(X^2) &= V(X) + (E(X))^2 = 4 + 9 = 13 \\
E(Y^2) &= V(Y) + (E(Y))^2 = 2 + 1 = 3 \\
E(XY) &= E(X)E(Y) = -3 \\
C(W,Z) &= E(WZ) - E(W)E(Z) = E(2X^2 - 3Y^2 + XY) - 12 = 2
\end{aligned}
$$

Bi-variate Random Variables

Statement (Multiple Functions of Multiple Random Variables)

If $Z = g(X, Y)$ *and* $W = h(X, Y)$ *and the set of equations*

$$\begin{cases} g(x, y) = z \\ h(x, y) = w \end{cases}$$

has a countable number of solutions $\{(x_i, y_i)\}$*, and if at these points the determinant of the Jacobian matrix*

$$J(x, y) = \begin{bmatrix} \partial z/\partial x & \partial z/\partial y \\ \partial w/\partial x & \partial w/\partial y \end{bmatrix}$$

is nonzero, then

$$f_{Z,W}(z, w) = \sum_i \frac{f_{X,Y}(x_i, y_i)}{|det J(x_i, y_i)|}$$

.

Bi-variate Random Variables

Example (Magnitude and phase of two i.i.d Gaussian variables)

If X and Y are independent and identically distributed zero-mean Gaussian random variables with the variance σ^2, i.e., $X \sim \mathcal{N}(0, \sigma^2) \perp\!\!\!\perp Y \sim \mathcal{N}(0, \sigma^2)$, then the random variables $V = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan \frac{Y}{X}$ are independent and have Rayleigh and uniform distribution, respectively, i.e., $V = \sqrt{X^2 + Y^2} \sim \mathcal{R}(\sigma) \perp\!\!\!\perp \Theta = \arctan \frac{Y}{X} \sim \mathcal{U}[0, 2\pi]$.

$V = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan \frac{Y}{X}$ and

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Example (Magnitude and phase of two i.i.d Gaussian variables)

If $X \sim \mathcal{N}(0, \sigma^2) \perp\!\!\!\perp Y \sim \mathcal{N}(0, \sigma^2)$, then $V = \sqrt{X^2 + Y^2} \sim \mathcal{R}(\sigma) \perp\!\!\!\perp \Theta = \arctan \frac{Y}{X} \sim \mathcal{U}[0, 2\pi]$.

$$J(x, y) = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} \Rightarrow |\det J(x, y)| = \frac{1}{\sqrt{x^2 + y^2}} = \frac{1}{v}$$

$$\begin{cases} \sqrt{x^2 + y^2} = v \\ \arctan \frac{y}{x} = \theta \end{cases} \Rightarrow \begin{cases} x = v \cos\theta \\ y = v \sin\theta \end{cases}$$

$$f_{V,\Theta}(v, \theta) = v f_{X,Y}(v\cos\theta, v\sin\theta) = \frac{v}{2\pi\sigma^2} e^{-\frac{v^2}{2\sigma^2}}$$

Example (Magnitude and phase of two i.i.d Gaussian variables)

If $X \sim \mathcal{N}(0, \sigma^2) \perp\!\!\!\perp Y \sim \mathcal{N}(0, \sigma^2)$, then $V = \sqrt{X^2 + Y^2} \sim \mathcal{R}(\sigma) \perp\!\!\!\perp \Theta = \arctan \frac{Y}{X} \sim \mathcal{U}[0, 2\pi]$.

$$f_\Theta(\theta) = \int_{-\infty}^{\infty} f_{V,\Theta}(v, \theta) dv = \frac{1}{2\pi}, 0 \leq \theta \leq 2\pi$$

$$f_V(v) = \int_{-\infty}^{\infty} f_{V,\Theta}(v, \theta) d\theta = \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}}, v \geq 0$$

The magnitude and the phase are independent random variables since

$$f_{V,\Theta}(v, \theta) = f_\Theta(\theta) f_V(v)$$

Bi-variate Random Variables

Statement (Jointly Gaussian Random Variables)

Jointly Gaussian random variables X and Y are distributed according to a joint PDF of the form

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$
$$\times \exp\left\{ -\frac{1}{2(1-\rho^2)}\left[\frac{(x-m_1)^2}{\sigma_1^2} + \frac{(y-m_2)^2}{\sigma_2^2} - \frac{2\rho(x-m_1)(y-m_2)}{\sigma_1\sigma_2}\right]\right\}$$

✓ Two uncorrelated jointly Gaussian random variables are independent. Therefore, for jointly Gaussian random variables, independence and uncorrelatedness are equivalent.

Multi-variate Random Variables

Definition (Multi-variate CDF)

Let $\boldsymbol{X} = (X_1, \cdots, X_n)^T$ represent n random variables. For these random vector , the CDF is defined as

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = F_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = P(X_1 \leq x_1, \cdots, X_n \leq x_n)$$

Definition (Multi-variate PDF)

Let $\boldsymbol{X} = (X_1, \cdots, X_n)^T$ represent n random variables. For these random vector , the PDF is defined as

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = \frac{\partial^n F_{X_1,\cdots,X_n}(x_1, \cdots, x_n)}{\partial x_1 \cdots \partial x_n}$$

Multi-variate Random Variables

Definition (Joint Multi-variate CDF)

Let $\boldsymbol{X} = (X_1, \cdots, X_n)^T$ and $\boldsymbol{Y} = (Y_1, \cdots, Y_m)^T$ represent two random vectors. For these random vector , the joint CDF is defined as

$$F_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{y}) = P(X_1 \leq x_1, \cdots, X_n \leq x_n, Y_1 \leq y_1, \cdots, Y_m \leq y_m)$$

Definition (Joint Multi-variate PDF)

Let $\boldsymbol{X} = (X_1, \cdots, X_n)^T$ and $\boldsymbol{Y} = (Y_1, \cdots, Y_m)^T$ represent two random vectors. For these random vector , the joint PDF is defined as

$$f_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\partial^{n+m} F_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{y})}{\partial x_1 \cdots \partial x_n \partial y_1 \cdots \partial y_m}$$

Multi-variate Random Variables

Definition (Mean)

The expected value of $\boldsymbol{X}$ is defined as $E(\boldsymbol{X}) = (E\{X_1\}, \cdots, E\{X_n\})$

Definition (Correlation)

$R(\boldsymbol{X}, \boldsymbol{Y}) = E(\boldsymbol{X}\boldsymbol{Y}^T)$ is called the correlation matrix of X and Y.

Definition (Covariance)

The covariance of X and Y is defined as $C(\boldsymbol{X}, \boldsymbol{Y}) = E\big((\boldsymbol{X} - E(\boldsymbol{X}))(\boldsymbol{Y} - E(\boldsymbol{Y}))^T\big) = E(\boldsymbol{X}\boldsymbol{Y}^T) - E(\boldsymbol{X})E(\boldsymbol{Y})^T$.

Multi-variate Random Variables

1. If $f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$, then $\boldsymbol{X}$ is called mutually independent.
2. If $C(\boldsymbol{X}, \boldsymbol{X})$ is a diagonal matrix, then $\boldsymbol{X}$ is called mutually uncorrelated.
3. If $\boldsymbol{X}$ is independent, then, $\boldsymbol{X}$ is uncorrelated.
4. If $f_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{y}) = f_{\boldsymbol{X}}(\boldsymbol{x}) f_{\boldsymbol{Y}}(\boldsymbol{y})$, then $\boldsymbol{X}$ and $\boldsymbol{Y}$ are called independent.
5. If $C(\boldsymbol{X}, \boldsymbol{Y}) = \mathbf{0}$, then $\boldsymbol{X}$ and $\boldsymbol{Y}$ are called uncorrelated.
6. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are uncorrelated.

Statement (Jointly Gaussian Random Variables)

Jointly Gaussian random variables $\boldsymbol{X} = (X_1, \cdots, X_n)^T$ *are distributed according to a joint PDF of the form*

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = (2\pi|\boldsymbol{\Sigma}|)^{-\frac{n}{2}} \exp\left[\frac{-1}{2}(\boldsymbol{x}-\boldsymbol{m})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{m})\right]$$

, where $\boldsymbol{m} = E(\boldsymbol{X})$ *and* $\boldsymbol{\Sigma} = C(\boldsymbol{X},\boldsymbol{X})$ *are the mean vector and covariance matrix and* $|\boldsymbol{\Sigma}|$ *is the determinant of* $\boldsymbol{\Sigma}$.

✓ Uncorrelated jointly Gaussian random variables are independent. Therefore, for jointly Gaussian random variables, independence and uncorrelatedness are equivalent.

Theorem (Central Limit Theorem)

If $\{X_i\}_{i=1}^{n}$ are n i.i.d. (independent and identically distributed) random variables, which each have the mean m and variance σ^2, then $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to $\mathcal{N}(m, \frac{\sigma^2}{n})$.

✓ The central limit theorem states that the sum of many i.i.d. random variables converges to a Gaussian random variable.

Random Processes

Random Processes

✓ A random process is a set of possible realizations of signal waveforms.

Example (Sample random process)

$$X(t) = A\cos(2\pi f_0 t + \Theta), \quad \Theta \sim U[0, 2\pi].$$

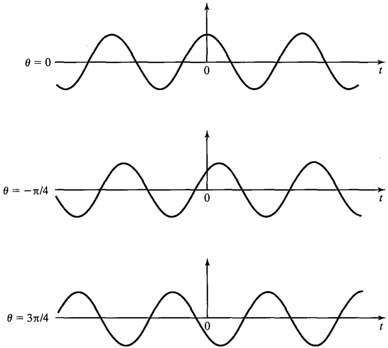

Figure: Sample functions of the example random process.

Random Processes

Example (Sample random process)

$X(t) = X, \quad X \sim U[-1, 1].$

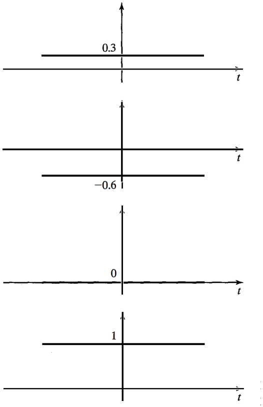

Figure: Sample functions of the example random process.

✓ A random process is denoted by $x(t;\omega)$, where $\omega \in \Omega$ is a random variable.

✓ For each ω_i, there exists a deterministic time function $x(t;\omega_i)$, which is called a sample function or a realization.

✓ For the different outcomes at a fixed time t_0, the numbers $x(t_0;\omega)$ constitute a random variable denoted by $X(t_0)$.

✓ At each time instant t_0 and for each $\omega_i \in \Omega$, we have the number $x(t_0;\omega_i)$.

Random Processes

Example (Sample random process)

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ denote the sample space corresponding to the random experiment of throwing a die. For all $\omega \in \Omega$, let $x(t; \omega) = \omega e^{-t} u(t)$ denote a random process. Then $X(1)$ is a random variable taking values $\{e^{-1}, 2e^{-1}, 3e^{-1}, 4e^{-1}, 5e^{-1}, 6e^{-1}\}$ and each has probability $1/6$.

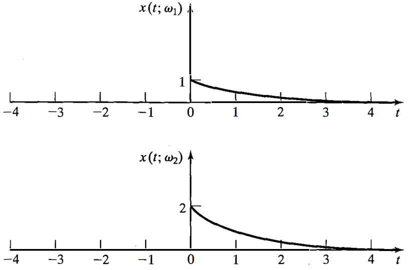

Figure: Sample functions of a random process.

Definition (Mean Function)

The mean, or expectation, of the random process $X(t)$ is a deterministic function of time denoted by $m_X(t)$ that at each time instant to equals the mean of the random variable $X(t_0)$. That is, $m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx, \forall t$.

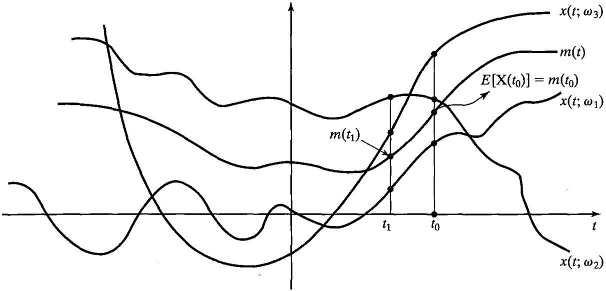

Figure: The mean of a random process.

Statistical Averages

Definition (Autocorrelation Function)

The autocorrelation function of the random process $X(t)$ is defined as

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2) dx_1 dx_2$$

.

Example (Statistical averages)

If $X(t) = A\cos(2\pi f_0 t + \Theta)$, $\Theta \sim U[0, 2\pi]$, then $m_X(t) = 0$ and $R_X(t_1, t_2) = \frac{A^2}{2}\cos(2\pi f_0(t_1 - t_2))$.

$$m_x(t) = E[X(t)] = E[A\cos(2\pi f_0 t + \Theta)] = \int_0^{2\pi} A\cos(2\pi f_0 t + \theta)\frac{1}{2\pi}d\theta = 0$$

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= E[A\cos(2\pi f_0 t_1 + \Theta)A\cos(2\pi f_0 t_2 + \Theta)] \\ &= E[\frac{A^2}{2}\cos(2\pi f_0(t_1 - t_2)) + \frac{A^2}{2}\cos(2\pi f_0(t_1 + t_2) + 2\Theta)] \\ &= \frac{A^2}{2}\cos(2\pi f_0(t_1 - t_2)) \end{aligned}$$

Example (Statistical averages)

If $X(t) = X, \quad X \sim U[-1, 1]$, then $m_X(t) = 0$ and $R_X(t_1, t_2) = \frac{1}{3}$.

$$m_x(t) = E[X(t)] = E[X] = \frac{-1+1}{2} = 0$$

$$R_X(t_1, t_2) = E[X^2] = \frac{(1-(-1))^2}{12} = \frac{1}{3}$$

Wide-Sense Stationary Processes

Definition (Wide-Sense Stationary (WSS))

A process $X(t)$ is WSS if the following conditions are satisfied

1. $m_x(t) = E[X(t)]$ is independent of t.
2. $R_X(t_1, t_2)$ depends only on the time difference $\tau = t_1 - t_2$ and not on t_1 and t_2 individually.

1. $R_X(t_1, t_2) = R_X(t_2, t_1)$.
2. If $X(t)$ is WSS, $R_X(\tau) = R_X(-\tau)$.

Wide-Sense Stationary Processes

Example (WSS)

If $X(t) = A\cos(2\pi f_0 t + \Theta)$, $\Theta \sim U[0, 2\pi]$, then $m_X(t) = 0$ and $R_X(t_1, t_2) = \frac{A^2}{2}\cos(2\pi f_0(t_1 - t_2))$ and therefore, $X(t)$ is WSS.

$$m_x(t) = E[A\cos(2\pi f_0 t + \Theta)] = \int_0^{2\pi} A\cos(2\pi f_0 t + \theta)\frac{1}{2\pi}d\theta = 0$$

$$\begin{aligned} R_X(t_1, t_2) &= E[A\cos(2\pi f_0 t_1 + \Theta)A\cos(2\pi f_0 t_2 + \Theta)] \\ &= E[\frac{A^2}{2}\cos(2\pi f_0(t_1 - t_2)) + \frac{A^2}{2}\cos(2\pi f_0(t_1 + t_2) + 2\Theta)] \\ &= \frac{A^2}{2}\cos(2\pi f_0(t_1 - t_2)) \end{aligned}$$

Wide-Sense Stationary Processes

Example (WSS)

If $X(t) = A\cos(2\pi f_0 t + \Theta)$, $\quad \Theta \sim U[0, \pi]$, then $m_X(t) = -2\frac{A}{\pi}\sin(2\pi f_0 t)$ and $R_X(t_1, t_2) = \frac{A^2}{2}\cos(2\pi f_0(t_1 - t_2))$ and therefore, $X(t)$ is not WSS.

$$m_x(t) = E[A\cos(2\pi f_0 t + \Theta)] = \int_0^{\pi} A\cos(2\pi f_0 t + \theta)\frac{1}{\pi}d\theta = -2\frac{A}{\pi}\sin(2\pi f_0 t)$$

$$\begin{aligned} R_X(t_1, t_2) &= E[A\cos(2\pi f_0 t_1 + \Theta)A\cos(2\pi f_0 t_2 + \Theta)] \\ &= E[\frac{A^2}{2}\cos(2\pi f_0(t_1 - t_2)) + \frac{A^2}{2}\cos(2\pi f_0(t_1 + t_2) + 2\Theta)] \\ &= \frac{A^2}{2}\cos(2\pi f_0(t_1 - t_2)) \end{aligned}$$

Multiple Random Processes

Definition (Independent Processes)

Two random processes $X(t)$ and $Y(t)$ are independent if for all positive integers m, n, and for all $t_1, t_2, \cdots, t_n$ and $\tau_1, \tau_2, \cdots, \tau_m$ the random vectors $\left(X(t_1), X(t_2), \cdots, X(t_n)\right)$ and $\left(Y(\tau_1), Y(\tau_2), \cdots, Y(\tau_m)\right)$ are independent.

Definition (Uncorrelated Processes)

Two random processes $X(t)$ and $Y(t)$ are uncorrelated if for all positive integers m, n, and for all $t_1, t_2, \cdots, t_n$ and $\tau_1, \tau_2, \cdots, \tau_m$ the random vectors $\left(X(t_1), X(t_2), \cdots, X(t_n)\right)$ and $\left(Y(\tau_1), Y(\tau_2), \cdots, Y(\tau_m)\right)$ are uncorrelated.

Multiple Random Processes

1. The independence of random processes implies that they are uncorrelated.
2. The uncorrelatedness generally does not imply independence.
3. For the important class of Gaussian processes, the independence and uncorrelatedness are equivalent.

Multiple Random Processes

Definition (Cross Correlation)

The cross correlation between two random processes $X(t)$ and $Y(t)$ is defined as $R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$.

Definition (Jointly WSS)

Two random processes $X(t)$ and $Y(t)$ are jointly wide-sense stationary, or simply jointly stationary, if both $X(t)$ and $Y(t)$ are individually stationary and the cross-correlation $R_{XY}(t_1, t_2)$ depends only on $\tau = t_1 - t_2$.

1. $R_{XY}(t_1, t_2) = R_{YX}(t_2, t_1)$.
2. For jointly WSS random processes $X(t)$ and $Y(t)$, $R_{XY}(\tau) = R_{YX}(-\tau)$.

Example (Jointly WSS)

Assuming that the two random processes $X(t)$ and $Y(t)$ are jointly stationary, determine the autocorrelation of the process $Z(t) = X(t) + Y(t)$.

$$
\begin{aligned}
R_Z(t+\tau, t) &= E[Z(t+\tau)Z(t)] \\
&= E[(X(t+\tau) + Y(t+\tau))(X(t) + Y(t))] \\
&= R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{XY}(-\tau)
\end{aligned}
$$

Random Processes and Linear Systems

Statement (LTI System with Random Input)

If a stationary process $X(t)$ with mean m_x and autocorrelation function $R_X(\tau)$ is passed through an LTI system with impulse response $h(t)$, the input and output processes $X(t)$ and $Y(t)$ will be jointly stationary with

$$m_Y = m_x \int_{-\infty}^{\infty} h(t)dt$$

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau)$$

$$R_Y(\tau) = R_{XY}(\tau) * h(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

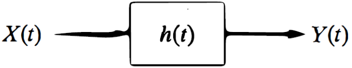

Figure: A random process passing through an LTI system.

Random Processes and Linear Systems

Statement (LTI System with Random Input)

If a stationary process $X(t)$ with mean m_x and autocorrelation function $R_X(\tau)$ is passed through an LTI system with impulse response $h(t)$, the input and output processes $X(t)$ and $Y(t)$ will be jointly stationary.

$$
\begin{aligned}
E[Y(t)] &= E[\int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau] \\
&= \int_{-\infty}^{\infty} E[X(\tau)]h(t-\tau)d\tau] \\
&= \int_{-\infty}^{\infty} m_X h(t-\tau)d\tau \\
&= m_X \int_{-\infty}^{\infty} h(u)du = m_Y
\end{aligned}
$$

Random Processes and Linear Systems

Statement (LTI System with Random Input)

If a stationary process $X(t)$ with mean m_x and autocorrelation function $R_X(\tau)$ is passed through an LTI system with impulse response $h(t)$, the input and output processes $X(t)$ and $Y(t)$ will be jointly stationary.

$$
\begin{aligned}
E[X(t_1)Y(t_2)] &= E[X(t_1)\int_{-\infty}^{\infty} X(s)h(t_2 - s)ds] \\
&= \int_{-\infty}^{\infty} E[X(t_1)X(s)]h(t_2 - s)ds \\
&= \int_{-\infty}^{\infty} R_X(t_1 - s)h(t_2 - s)ds \\
&= \int_{-\infty}^{\infty} R_X(t_1 - t_2 - u)h(-u)du = R_X(\tau) * h(-\tau) = R_{XY}(\tau)
\end{aligned}
$$

Statement (LTI System with Random Input)

If a stationary process $X(t)$ with mean m_x and autocorrelation function $R_X(\tau)$ is passed through an LTI system with impulse response $h(t)$, the input and output processes $X(t)$ and $Y(t)$ will be jointly stationary.

$$
\begin{aligned}
E[Y(t_1)Y(t_2)] &= E[Y(t_2)\int_{-\infty}^{\infty} X(s)h(t_1 - s)ds] \\
&= \int_{-\infty}^{\infty} E[X(s)Y(t_2)]h(t_1 - s)ds \\
&= \int_{-\infty}^{\infty} R_{XY}(s - t_2)h(t_1 - s)ds \\
&= \int_{-\infty}^{\infty} R_{XY}(u)h(t_1 - t_2 - u)du = R_{XY}(\tau) * h(\tau) = R_Y(\tau)
\end{aligned}
$$

Example (Differentiateor)

Assume a stationary process passes through a differentiator. What are the mean and autocorrelation functions of the output? What is the cross correlation between the input and output?

Since $h(t) = \delta'(t)$,

$$m_Y = m_x \int_{-\infty}^{\infty} h(t)dt = m_x \int_{-\infty}^{\infty} \delta'(t)dt = 0$$

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau) = R_X(\tau) * \delta'(-\tau) = -R_X(\tau) * \delta'(\tau) = -\frac{dR_X(\tau)}{d\tau}$$

$$R_Y(\tau) = R_{XY}(\tau) * h(\tau) = -\frac{dR_X(\tau)}{d\tau} * \delta'(\tau) = -\frac{d^2 R_X(\tau)}{d\tau^2}$$

Example (Hilbert Transform)

Assume a stationary process passes through a Hilbert filter. What are the mean and autocorrelation functions of the output? What is the cross correlation between the input and output?

Assume that $R_X(\tau)$ has no DC component. Since $h(t) = 1/(\pi t)$,

$$m_Y = m_x \int_{-\infty}^{\infty} h(t)dt = m_x \int_{-\infty}^{\infty} \frac{1}{\pi t} dt = 0$$

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau) = R_X(\tau) * \frac{-1}{\pi\tau} = -\widehat{R}_X(\tau)$$

$$R_Y(\tau) = R_{XY}(\tau) * h(\tau) = -\widehat{R}_X(\tau) * \frac{1}{\pi\tau} = -\widehat{\widehat{R}}_X(\tau) = R_X(\tau)$$

Power Spectral Density of Stationary Processes

Definition (Truncated Fourier Transform)

The truncated Fourier transform of a realization of the random process $X(t;\omega_i)$ over an interval $[-T/2, T/2]$ is defined by

$$X_T(f;\omega_i) = \int_{-T/2}^{T/2} x(t;\omega_i) e^{-j2\pi f t} dt$$

Definition (Power Spectral Density)

The power spectral density of the random process $X(t)$ is defined by

$$S_X(f) = \lim_{T\to\infty} \frac{1}{T} E[|X_T(f;\omega)|^2]$$

Power Spectral Density of Stationary Processes

Theorem (Wiener-Khinchin)

For a stationary random process $X(t)$, the power spectral density is the Fourier transform of the autocorrelation function, i.e.,

$$S_X(f) = \mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau$$

.

Power Spectral Density of Stationary Processes

Definition (Power)

The power in the random process $X(t)$ is obtained by

$$P_X = \int_{-\infty}^{\infty} S_X(f)df = \mathcal{F}^{-1}[S_X(f)]|_{\tau=0} = R_X(0)$$

.

Definition (Cross Power Spectral Density)

For the jointly stationary random processes $X(t)$ and $Y(t)$, the cross power spectral density is the Fourier transform of the cross correlation function, i.e.,

$$S_{XY}(f) = \mathcal{F}[R_{XY}(\tau)] = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j2\pi f\tau}d\tau$$

.

Power Spectral Density of Stationary Processes

Example (Wiener-Khinchin)

If $X(t) = A\cos(2\pi f_0 t + \Theta), \quad \Theta \sim U[0, 2\pi]$, then $R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$ and therefore, $S_X(f) = \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$ and $P_X = \frac{A^2}{2}$.

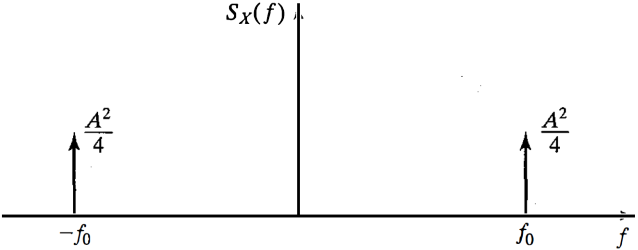

Figure: Power spectral density of the example random process.

Example (Wiener-Khinchin)

If $X(t) = X, \quad X \sim U[-1,1]$, then $R_X(\tau) = \frac{1}{3}$ and therefore, $S_X(f) = \frac{1}{3}\delta(f)$ and $P_X = \frac{1}{3}$.

Power Spectral Density of Stationary Processes

Statement (LTI System with Random Input)

If a stationary process $X(t)$ with mean m_x and autocorrelation function $R_X(\tau)$ is passed through an LTI system with impulse response $h(t)$ and frequency response $H(f)$, the input and output processes $X(t)$ and $Y(t)$ will be jointly stationary with

$$m_Y = m_x \int_{-\infty}^{\infty} h(t)dt \leftrightarrow m_y = m_x H(0)$$

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau) \leftrightarrow S_{XY}(f) = H^*(f) S_X(f)$$

$$R_{YX}(\tau) = R_{XY}(-\tau) \leftrightarrow S_{YX}(f) = S_{XY}^*(f) = H(f) S_X(f)$$

$$R_Y(\tau) = R_{XY}(\tau) * h(\tau) = R_X(\tau) * h(\tau) * h(-\tau) \leftrightarrow S_Y(f) = |H(f)|^2 S_X(f)$$

Power Spectral Density of Stationary Processes

Statement (LTI System with Random Input)

If a stationary process $X(t)$ with mean m_X and autocorrelation function $R_X(\tau)$ is passed through an LTI system with impulse response $h(t)$ and frequency response $H(f)$, the input and output processes $X(t)$ and $Y(t)$ will be jointly stationary.

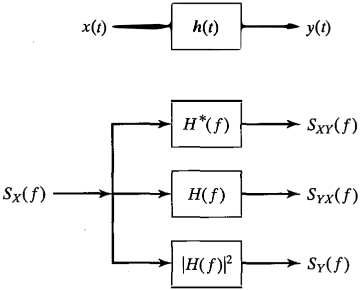

Figure: Input-output relations for the power spectral density and the cross-spectral density.

Power Spectral Density of Stationary Processes

Example (Power spectral densities for a differentiator)

If $X(t) = A\cos(2\pi f_0 t + \Theta), \quad \Theta \sim U[0, 2\pi]$ passes through a differentiator, we have $S_Y(f) = \pi^2 f_0^2 A^2[\delta(f - f_0) + \delta(f + f_0)]$ and $S_{XY}(f) = \frac{j\pi A^2 f_0}{2}[\delta(f + f_0) - \delta(f - f_0)]$.

$$S_Y(f) = 4\pi^2 f^2 \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)] = \pi^2 f_0^2 A^2[\delta(f - f_0) + \delta(f + f_0)]$$

$$S_{XY}(f) = -j2\pi f \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)] = \frac{j\pi A^2 f_0}{2}[\delta(f + f_0) - \delta(f - f_0)]$$

Example (Power spectral densities for a differentiator)

If $X(t) = X, \quad X \sim U[-1, 1]$ passes through a differentiator, we have $S_Y(f) = S_{XY}(f) = 0$.

$$S_Y(f) = 4\pi^2 f^2 \frac{1}{3}\delta(f) = 0$$

$$S_{XY}(f) = -j2\pi f \frac{1}{3}\delta(f) = 0$$

Example (Power Spectral Density of a Sum Process)

Let $Z(t) = X(t) + Y(t)$, where $X(t)$ and $Y(t)$ are jointly stationary random processes. Also assume that $X(t)$ and $Y(t)$ are uncorrelated and at least one of them has zero mean. Then, $S_Z(f) = S_X(f) + S_Y(f)$.

Since $R_{XY}(\tau) = m_X m_Y = 0$,
$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + R_{XY}(\tau) + R_{XY}(-\tau) = R_X(\tau) + R_Y(\tau)$. So,

$$S_Z(f) = \mathcal{F}\{R_Z(\tau)\} = S_X(f) + S_Y(f)$$

Gaussian, White, and Bandpass Processes

Definition (Gaussian Random Process)

A random process $X(t)$ is a Gaussian process if for all n and all $(t_1, t_2, \cdots, t_n)$, the random variables $\{X(t_i)\}_{i=1}^{n}$ have a jointly Gaussian density function.

For a Gassian random process,

1. At any time instant t_0, the random variable $X(t_0)$ is Gaussian.
2. At any two points t_1, t_2, random variables $(X(t_1), X(t_2))$ are distributed according to a two-dimensional jointly Gaussian distribution.

Gaussian Processes

Example (Gaussian Random Process)

Let $X(t)$ be a zero-mean stationary Gaussian random process with the power spectral density $S_X(f) = 5 \sqcap (f/1000)$. Then, $X(3) \sim \mathcal{N}(0, 5000)$.

$$m = m_{X(3)} = m_X = 0$$

$$\sigma^2 = V[X(3)] = E[X^2(3)] - (E[X(3)])^2 = E[X(3)X(3)] = R_X(0) = P_X$$

$$\sigma^2 = P_X = \int_{-\infty}^{\infty} S_X(f)df = 5000$$

Gaussian Processes

Definition (Jointly Gaussian Random Processes)

The random processes $X(t)$ and $Y(t)$ are jointly Gaussian if for all n, m and all $(t_1, t_2, \cdots, t_n)$ and $(\tau_1, \tau_2, \cdots, \tau_m)$, the random vector $(X(t_1), X(t_2), \cdots, X(t_n), Y(\tau_1), Y(\tau_2), \cdots, Y(\tau_m))$ is distributed according to an $n + m$ dimensional jointly Gaussian distribution.

For jointly Gassian random processes,

1. If the Gaussian process $X(t)$ is passed through an LTI system, then the output process $Y(t)$ will also be a Gaussian process. Moreover, $X(t)$ and $Y(t)$ will be jointly Gaussian processes.
2. For jointly Gaussian processes, uncorrelatednesss and independence are equivalent.

Example (Jointly Gaussian Random Processes)

Let $X(t)$ be a zero-mean stationary Gaussian random process with the power spectral density $S_X(f) = 5\sqcap(f/1000)$. If $X(t)$ passes a differentiator, the output random process $Y(3) \sim \mathcal{N}(0, 1.6 \times 10^{10})$.

Since $H(f) = 2\pi f$,

$$m = m_{Y(3)} = m_X H(0) = 0$$

$$\sigma^2 = V[Y(3)] = E[Y^2(3)] - (E[Y(3)])^2 = E[Y(3)Y(3)] = R_Y(0) = P_Y$$

$$\sigma^2 = P_Y = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df = 1.6 \times 10^{10}$$

Definition (White Random Process)

A random process $X(t)$ is called a white process if it has a flat power spectral density, i.e., if $S_X(f) = \frac{N_0}{2}$ equals the constant $\frac{N_0}{2}$ for all f.

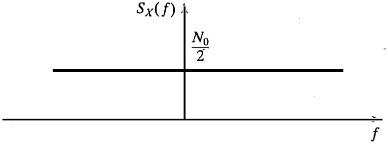

Figure: Power spectrum of a white process.

White Processes

1. The power content of a white process

$$P_X = \int_{-\infty}^{\infty} S_X(f) df = \int_{-\infty}^{\infty} \frac{N_0}{2} df = \infty$$

.

2. A white process is not a meaningful physical process.
3. The autocorrelation function of a white process is

$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(f)\} = \frac{N_0}{2}\delta(\tau)$$

.

White Processes

1. If we sample a zero-mean white process at two points t_1 and t_2 ($t_1 \neq t_2$), the resulting random variables will be uncorrelated.
2. If the zero-mean random process is white and also Gaussian, any pair of random variables $X(t_1)$, $X(t_2)$, where $t_1 \neq t_2$, will also be independent.

Bandpass Processes

Definition (Lowpass Random Process)

A WSS random process $X(t)$ is called lowpass if its autocorrelation $R_X(\tau)$ is a lowpass signal.

Definition (Bandpass Random Process)

A zero-mean real WSS random process $X(t)$ is called bandpass if its autocorrelation $R_X(\tau)$ is a bandpass signal.

✓ For a bandpass process, the power spectral density is located around frequencies $\pm f_c$, and for lowpass processes, the power spectral density is located around zero frequency.

Bandpass Processes

Definition (In-phase/Quadrature Random Process)

The in-phase and quadrature components of a bandpass random process $X(t)$ are defined as

$$X_c(t) = X(t)\cos(2\pi f_c t) + \hat{X}(t)\sin(2\pi f_c t)$$
$$X_s(t) = \hat{X}(t)\cos(2\pi f_c t) - X(t)\sin(2\pi f_c t)$$

Definition (Lowpass Equivalent Random Process)

The lowpass equivalent random process of a bandpass random process $X(t)$ is defined as

$$X_l(t) = X_c(t) + jX_s(t)$$

Bandpass Processes

Theorem (In-phase/Quadrature Random Process)

For the in-phase and quadrature components of a bandpass random process $X(t)$,

1. $X_c(t)$ *and* $X_s(t)$ *are jointly WSS zero-mean random processes.*
2. $X_c(t)$ *and* $X_s(t)$ *are both lowpass processes.*
3. $X_c(t)$ *and* $X_s(t)$ *have the same power spectral density as*

$$S_{X_c}(f) = S_{X_s}(f) = [S_X(f+f_c) + S_X(f-f_c)] \sqcap (\frac{f}{2f_c})$$

4. *The cross-spectral density of the components are*

$$S_{X_cX_s}(f) = -S_{X_sX_c}(f) = j[S_X(f+f_c) - S_X(f-f_c)] \sqcap (\frac{f}{2f_c})$$

Theorem (Lowpass Equivalent Random Process)

For the lowpass equivalent of a bandpass random process $X(t)$,

1. $$S_{X_l}(f) = 4S_X(f+f_c)u(f+f_c)$$

2. $$S_X(f) = \frac{1}{4}\left[S_{X_l}(f-f_c) + S_{X_l}(-f-f_c)\right]$$

3. $$R_{X_l}(\tau) = 2(R_X(\tau) + j\widehat{R_X}(\tau))e^{-j2\pi f_c \tau}$$

Bandpass Processes

Example (In-phase autocorrelation)

The autocorrelation of the in-phase component of a bandpass random process $X(t)$ is $R_{X_c}(\tau) = R_X(\tau)\cos(2\pi f_c\tau) + \widehat{R_X}(\tau)\sin(2\pi f_c\tau)$.

$$
\begin{aligned}
R_{X_c}(t+\tau, t) &= E\{X_c(t+\tau)X_c(t)\} \\
&= E\{[X(t+\tau)\cos(2\pi f_c(t+\tau)) + \hat{X}(t+\tau)\sin(2\pi f_c(t+\tau))] \\
&\times [X(t)\cos(2\pi f_c t) + \hat{X}(t)\sin(2\pi f_c t)]\} \\
&= R_X(\tau)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\
&+ R_{X\hat{X}}(t+\tau, t)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\
&+ R_{\hat{X}X}(t+\tau, t)\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\
&+ R_{\hat{X}\hat{X}}(t+\tau, t)\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\
&= R_X(\tau)\cos(2\pi f_c\tau) + \widehat{R_X}(\tau)\sin(2\pi f_c\tau)
\end{aligned}
$$

Thermal and Filtered Noise

Thermal Noise

✓ The thermal noise, which is produced by the random movement of electrons due to thermal agitation, is usually modeled by a white Gaussian process.

Thermal Noise

Statement (Thermal Noise)

Quantum mechanical analysis of the thermal noise shows that it has a power spectral density given by $S_n(f) = 0.5hf/(e^{\frac{hf}{KT}} - 1)$, *which can be approximated by* $KT/2 = N_0/2$ *for* $f < 2$ *THz, where* $h = 6.6 \times 10^{-34}$ $J \times sec$ *denotes Planck's constan,* $K = 1.38 \times 10^{-23}$ J/K *is Boltzmann's constant, and* T *denotes the temperature in degrees Kelvin. Further, the noise originates from many independent random particle movements.*

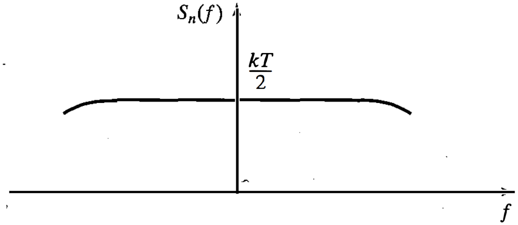

Figure: Power spectrum of thermal noise.

Thermal and Filtered Noise Model

Statement (Thermal Noise Model)

The thermal noise is assumed to have the following properties,

1. *Thermal noise is a stationary process.*
2. *Thermal noise is a zero-mean process.*
3. *Thermal noise is a Gaussian process.*
4. *Thermal noise is a white process with a PSD* $S_n(f) = \frac{KT}{2} = \frac{N_0}{2}$.

Statement (Filtered Noise Process)

The PSD of an ideally bandpass filtered noise is

$$S_X(f) = \frac{N_0}{2}|H(f)|^2$$

Filtered Noise Model

Example (Filtered Noise Process)

If the Gaussian white noise passes through the shown filter, the PSD of the filtered noise is

$$S_X(f) = \frac{N_0}{2}|H(f)|^2 = \begin{cases} \frac{N_0}{2}, & |f - f_c| \leq W \\ 0, & \text{otherwise} \end{cases}$$

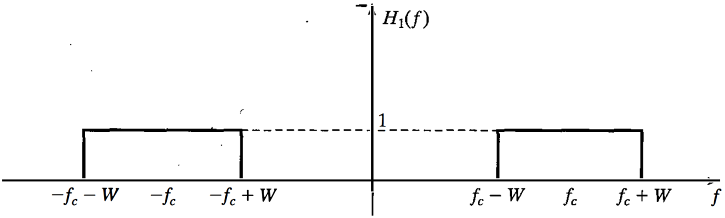

Figure: Filter transfer function $H(f)$.

Filtered Noise Model

For a filtered white Gaussian noise, the following properties for $X_c(t)$ and $X_s(t)$ can be proved.

1. $X_c(t)$ and $X_s(t)$ are zero-mean, lowpass, jointly WSS, and jointly Gaussian random processes.
2. If the power in process $X(t)$ is P_X, then the power in each of the processes $X_c(t)$ and $X_s(t)$ is also P_x.
3. Processes $X_c(t)$ and $X_s(t)$ have a common power spectral density, i.e., $S_{X_c}(f) = S_{X_s}(f) = [S_X(f + f_c) + S_X(f - f_c)] \sqcap (\frac{f}{2f_c})$.
4. If f_c and $-f_c$ are the axis of symmetry of the positive and negative frequencies, respectively, then $X_c(t)$ and $X_s(t)$ will be independent processes.

Filtered Noise Model

Example (Filtered Noise Process)

For the bandpass white noise at the output of filter given below, power spectral density of the process $Z(t) = aX_c(t) + bX_s(t)$ is $S_Z(f) = N_0(a^2 + b^2) \sqcap (\frac{f}{2W}))$.

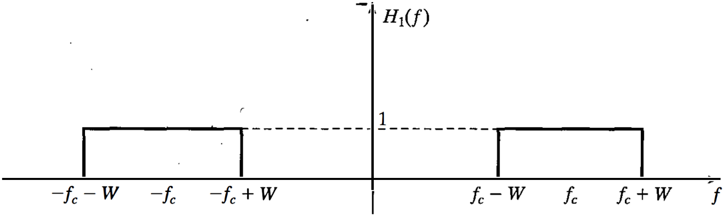

Figure: Filter transfer function $H(f)$.

Filtered Noise Model

Example (Filtered Noise Process (cont.))

For the bandpass white noise at the output of filter given below, power spectral density of the process $Z(t) = aX_c(t) + bX_s(t)$ is $S_Z(f) = N_0(a^2 + b^2) \sqcap \left(\frac{f}{2W}\right)$.

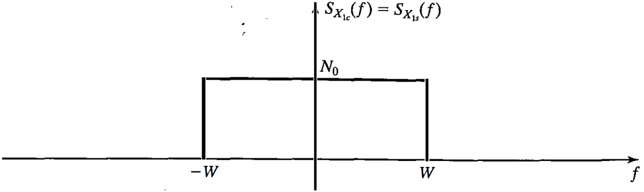

Figure: Power spectral densities of the in-phase and quadrature components of the example filtered noise.

Filtered Noise Model

Example (Filtered Noise Process (cont.))

For the bandpass white noise at the output of filter given below, power spectral density of the process $Z(t) = aX_c(t) + bX_s(t)$ is $S_Z(f) = N_0(a^2 + b^2) \sqcap (\frac{f}{2W})$.

Since f_c is the axis of symmetry of the noise power spectral density, the in-phase and quadrature components of the noise will be independent with zero mean. So,

$$R_Z(\tau) = E\{[aX_c(t+\tau)+bX_s(t+\tau)][aX_c(t)+bX_s(t)]\} = a^2 R_{X_c}(\tau)+b^2 R_{X_s}(\tau)$$

Since $S_{X_c}(f) = S_{X_s}(f) = N_0 \sqcap (\frac{f}{2W})$,

$$S_Z(f) = a^2 S_{X_c}(f) + b^2 S_{X_s}(f) = N_0(a^2 + b^2) \sqcap (\frac{f}{2W})$$

Noise Equivalent Bandwidth

Definition (Noise Equivalent Bandwidth)

The noise equivalent bandwidth of a filter with the frequency response $H(f)$ is defined as $B_{neq} = \frac{\int_{-\infty}^{\infty} |H(f)|^2 df}{2H_{max}^2}$, where H_{max} denotes the maximum of $|H(f)|$ in the passband of the filter.

✓ The power content of the filtered noise is $P_X = \int_{-\infty}^{\infty} |H(f)|^2 S_n(f) df = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df = N_0 B_{neq} H_{max}^2$

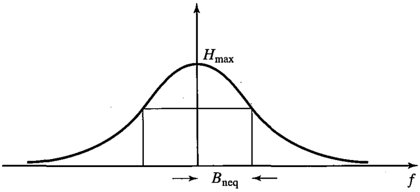

Figure: Noise equivalent bandwidth of a typical filter.

Noise Equivalent Bandwidth

Example (Noise Equivalent Bandwidth)

The noise equivalent bandwidth of a lowpass RC filter is $\frac{1}{4RC}$.

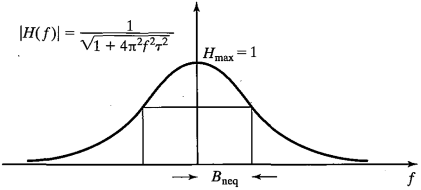

Figure: Frequency response of a lowpass f RC filter.

$$H(f) = \frac{1}{1 + j2\pi fRC} \Rightarrow |H(f)| = \frac{1}{\sqrt{1 + 4\pi^2 f^2 R^2 C^2}} \Rightarrow H_{max} = 1$$

$$B_{neq} = \frac{\int_{-\infty}^{\infty} |H(f)|^2 df}{2H_{max}^2} = \frac{\frac{1}{2RC}}{2} = \frac{1}{4RC}$$

The End