Signals and Linear Systems

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Overview

Signals

Basic Operations on Signals

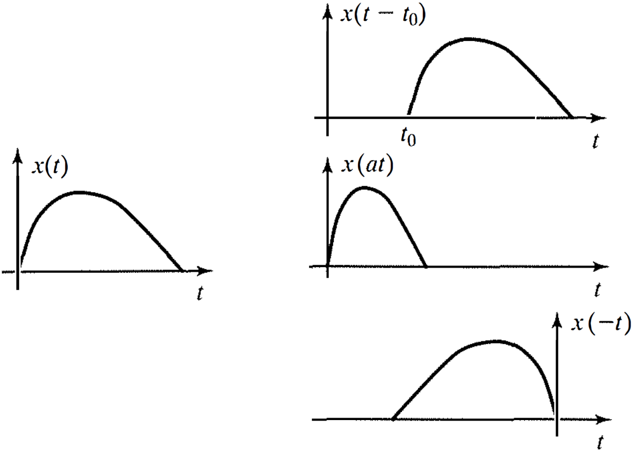


Figure: Time shifting, time scaling, time reversal.

$$x(t) \to x(t - t_0); \quad x(t) \to x(at); \quad x(t) \to x(-t)$$

Classification of Signals

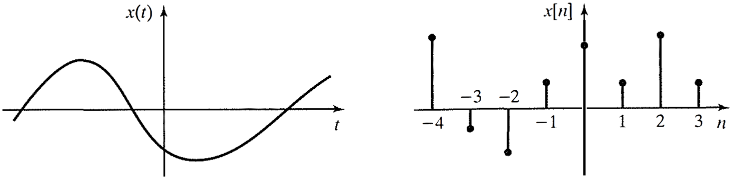


Figure: Continuous-time and discrete-time signals.

$$x(t), t \in \mathbb{R}; \quad x[n], n \in \mathbb{Z}$$

Classification of Signals

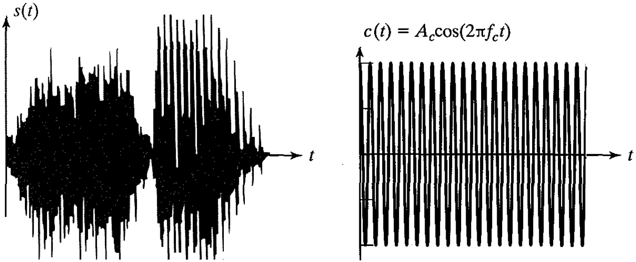


Figure: Random and deterministic signals.

$$x(t,\omega) \in \mathbb{R}, t \in \mathbb{R}, \omega \sim P[\Omega = \omega]; \quad x(t) \in \mathbb{R}, t \in \mathbb{R}$$

$$s(t) = \text{Audio Signal}; \quad c(t) = A_c \cos(2\pi f_c t)$$

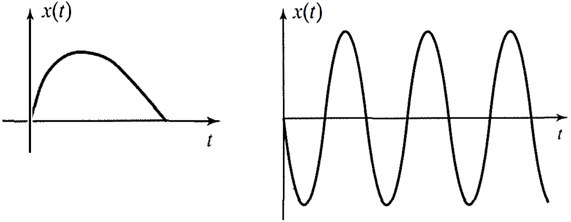


Figure: Nonperiodic and periodic signals.

$$\nexists T_0 : x(t + T_0) = x(t); \quad \exists T_0 : x(t + T_0) = x(t)$$

Classification of Signals

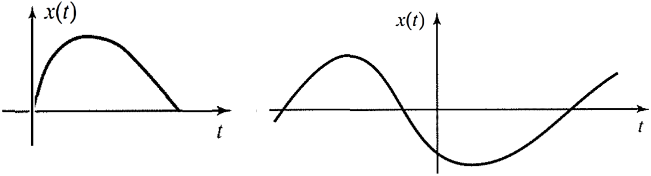


Figure: Causal and noncausal signals.

$$\forall t < 0 : x(t) = 0; \quad \exists t < 0 : x(t) \neq 0$$

Classification of Signals

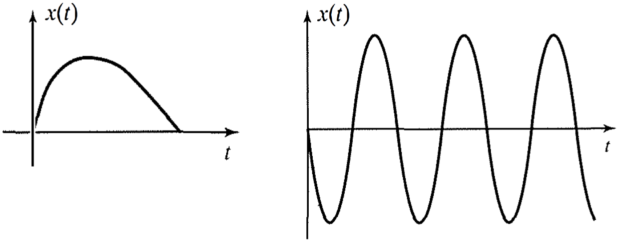
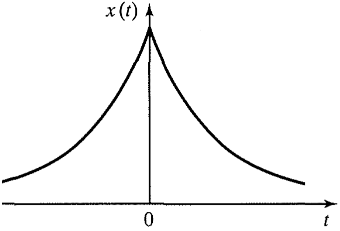


Figure: Energy and power signals.

$$0 < \mathcal{E}_x = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt < \infty; \quad 0 < \mathcal{P}_x = \lim_{T\to\infty} \frac{\int_{-T/2}^{T/2} |x(t)|^2 dt}{T} < \infty$$

Classification of Signals



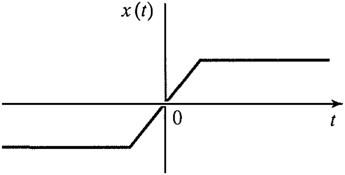


Figure: Even and odd signals.

$$x(t) = x(-t); \quad x(t) = -x(-t)$$

Statement (Even-Odd Decomposition)

Any signal $x(t)$ can be written as the sum of its even and odd parts as $x(t) = x_e(t) + x_o(t)$, where

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

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Classification of Signals

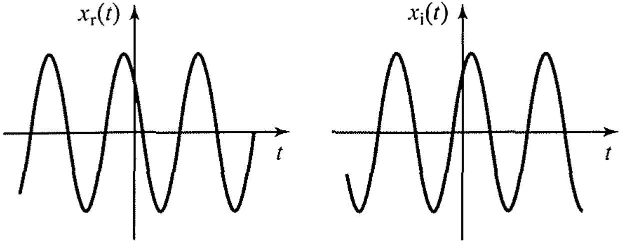


Figure: Real and complex signals.

$$x(t) \in \mathbb{R}; \quad x(t) \in \mathbb{C}$$

$$x_r(t) = A\cos(2\pi f_0 t + \theta); \quad x_i(t) = A\sin(2\pi f_0 t + \theta)$$

$$x(t) = \Re\{x(t)\} + j\Im\{x(t)\} = x_r(t) + jx_i(t)$$

Classification of Signals

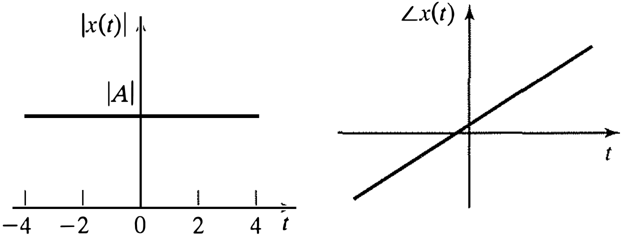


Figure: Real and complex signals.

$$x(t) \in \mathbb{R}; \quad x(t) \in \mathbb{C}$$

$$|x(t)| = |A|; \quad \angle x(t) = 2\pi f_0 t + \theta$$

$$x(t) = |x(t)| e^{j\angle x(t)}$$

Statement (Complex Signal Representation)

For the complex signal $x(t) = x_r(t) + jx_i(t) = \Re\{x(t)\} + j\Im\{x(t)\} = |x(t)|e^{j\angle x(t)}$,

$$x_r(t) = \Re\{x(t)\} = |x(t)|\cos(\angle x(t))$$

$$x_i(t) = \Im\{x(t)\} = |x(t)|\sin(\angle x(t))$$

$$|x(t)| = \sqrt{x_r^2(t) + x_i^2(t)}$$

$$\angle x(t) = \tan^{-1}\left(\frac{x_i(t)}{x_r(t)}\right)$$

.

Some Important Signals

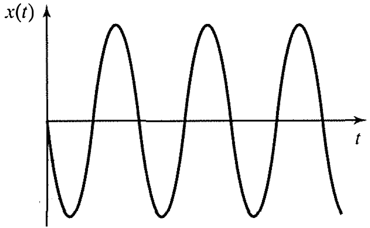


Figure: Sinusoidal signal.

$$x(t) = A\cos(2\pi f_0 t + \theta) = A\cos(2\pi t/T_0 + \theta)$$

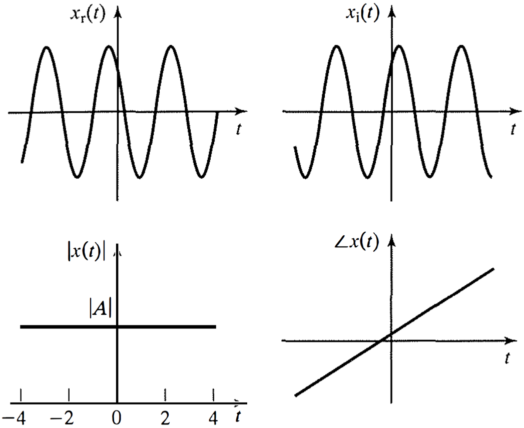


Figure: Complex exponential signal.

$$x(t) = A\cos(2\pi f_0 t + \theta) + jA\sin(2\pi f_0 t + \theta) = Ae^{j(2\pi f_0 t + \theta)}$$

Some Important Signals

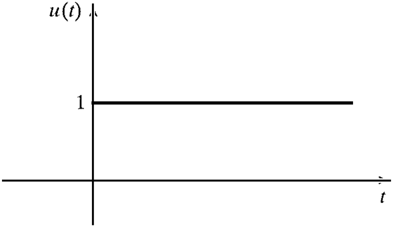


Figure: Unit step signal.

$$u(t) = \begin{cases} 1, & t \geqslant 0 \\ 0, & t < 0 \end{cases}$$

Some Important Signals

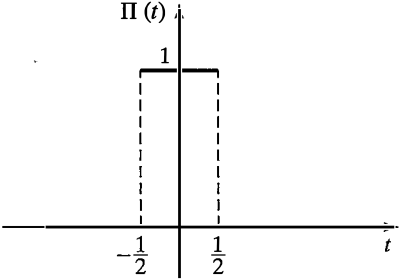


Figure: Rectangular signal.

$$\sqcap(t) = \text{rect}(t) = \begin{cases} 1, & |t| \leqslant 0.5 \\ 0, & |t| > 0.5 \end{cases}$$

Some Important Signals

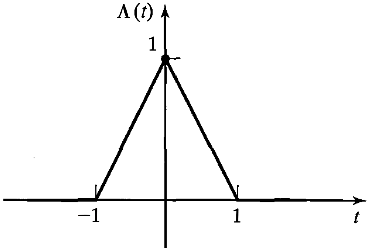


Figure: Triangle signal.

$$\Lambda(t) = \text{tri}(t) = \begin{cases} 1 - |t|, & |t| \leqslant 1 \\ 0, & |t| > 1 \end{cases}$$

Some Important Signals

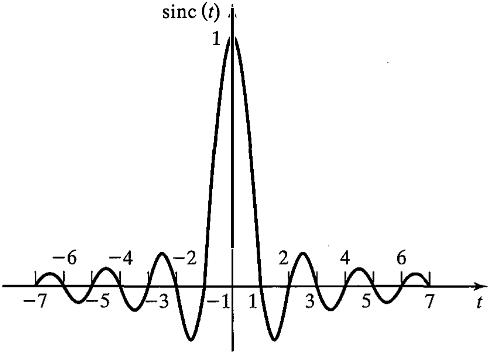


Figure: Sinc signal.

$$\mathrm{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Some Important Signals

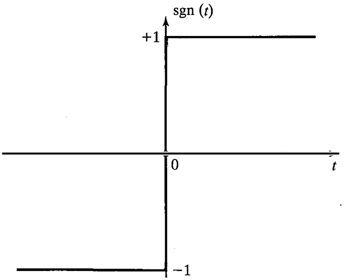
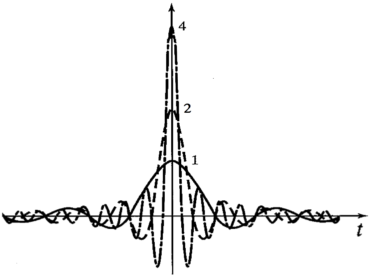


Figure: Sign signal.

$$\text{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

Some Important Signals



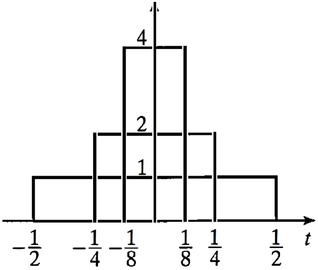


Figure: Unit impulse signal.

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \text{sinc}(\frac{t}{\epsilon}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sqcap (\frac{t}{\epsilon})$$

Some Important Signals

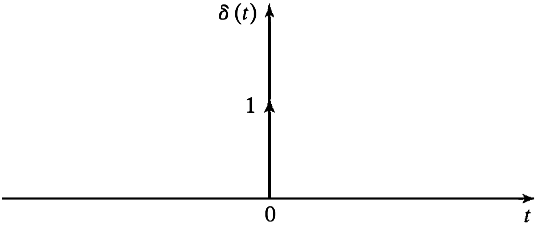


Figure: Unit impulse signal.

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

Singular Functions

Definition (Convolution)

The convolution of the functions $h(t)$ and $x(t)$ is defined as

$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Definition (Test Function)

$x(t)$ is called a test function if it is infinitely differentiable and is zero outside a finite interval.

Singular Functions

Definition (Unit Impulse Signal)

The unit impulse function $u_0(t) = \delta(t)$ is defined as the function satisfying

$$\int_{-\infty}^{+\infty} \delta(t)x(t)dt = x(0)$$

for any test function $x(t)$.

Singular Functions

Theorem (Properties of Unit Impulse Signal)

The unit impulse function satisfies the following identities

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$

$$x(t) = \delta(t) * x(t)$$

$$\delta(at) = \frac{1}{|a|}\delta(t), a \neq 0$$

$$x(t)\delta(t) = x(0)\delta(t)$$

$$t\delta(t) = 0$$

$$\delta(t) = 0, t \neq 0$$

Singular Functions

Example (Area under $\delta(t)$)

The area under the unit impulse function is 1.

For $x(t) = 1$,

$$\int_{-\infty}^{+\infty} \delta(t)x(t)dt = \int_{-\infty}^{+\infty} \delta(t)dt = x(0) = 1$$

Example (Convolution with $\delta(t)$)

$\delta(t)$ is the neutral function of the convolution operation, i.e. $x(t) = \delta(t) * x(t)$.

$$\delta(t) * x(t) = \int_{-\infty}^{+\infty} \delta(\tau)x(t-\tau)d\tau = x(t-0) = x(t)$$

Singular Functions

Definition (Unit Doublet Signal)

The unit doublet function $u_1(t) = \delta'(t)$ is defined as the function satisfying

$$\int_{-\infty}^{+\infty} \delta'(t)x(t)dt = -x'(0)$$

for any test function $x(t)$.

Definition (Higher-order Impulse Signals)

Generally, $u_n(t) = \delta^{(n)}(t), n \geq 0$ is defined as the function satisfying

$$\int_{-\infty}^{+\infty} \delta^{(n)}(t)x(t)dt = (-1)^n x^{(n)}(0)$$

for any test function $x(t)$.

Singular Functions

Theorem (Convolution with $u_n(t)$)

$u_n(t), n \geq 1$ satisfies $x^{(n)}(t) = u_n(t) * x(t)$.

For $n = 1$,

$$u_1(t) * x(t) = \int_{-\infty}^{+\infty} \delta'(\tau)x(t-\tau)d\tau = -\frac{dx(t-\tau)}{d\tau}|_{\tau=0} = x'(t)$$

Theorem (Relation of $\delta'(t)$ and $u_n(t)$)

$u_n(t), n \geq 2$ relates to $u_1(t) = \delta'(t)$ as $u_n(t) = \underbrace{u_1(t) * u_1(t) * \cdots * u_1(t)}_{n \text{ times}}$.

For $n = 2$,

$$\frac{d^2(t)}{dt^2} = \frac{d}{dt}(\frac{dx(t)}{dt}) = \frac{d}{dt}(x(t) * u_1(t)) = x(t) * u_1(t) * u_1(t)$$

Singular Functions

Definition (Unit Step Signal)

The unit step function $u_{-1}(t) = u(t)$ is defined as the function satisfying

$$\int_{-\infty}^{+\infty} u(t)x(t)dt = \int_{0}^{+\infty} x(t)dt$$

for any test function $x(t)$.

Definition (Higher-order Step Signals)

Generally, $u_{-n}(t), n \geq 2$ is defined as

$$u_{-n}(t) = \underbrace{u_{-1}(t) * u_{-1}(t) * \cdots * u_{-1}(t)}_{n \text{ times}}$$

Theorem (Explicit representation of $u_{-n}(t), n \geq 2$)

$u_{-n}(t), n \geq 2$ can be represented as

$$u_{-n}(t) = \frac{t^{n-1}}{(n-1)!} u_{-1}(t)$$

.

For $n = 2$,

$$u_{-2}(t) = u_{-1}(t) * u_{-1}(t) = u(t) * u(t) = tu(t) = r(t)$$

Singular Functions

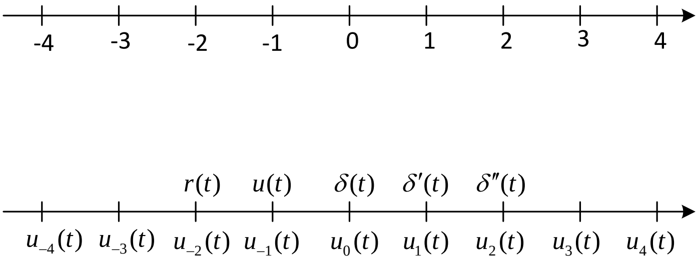


Figure: Singular functions.

Singular Functions

Example (Representation of other signals using the singular signals)

$x(t)$ can be represented by $u(t)$ and its shifted versions as

$$x(t) = u(t) + 2u(t-1) - u(t-2)$$

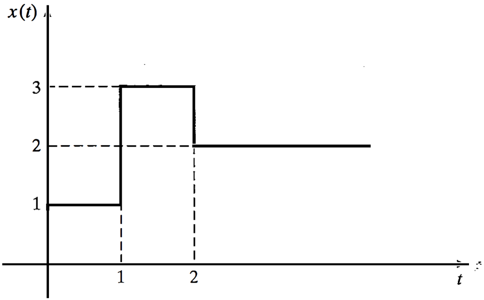


Figure: The signal $u(t) + 2u(t-1) - u(t-2)$.

Example (Simplification using the properties of the singular functions)

$$\cos(t)\delta(t) = \cos(0)\delta(t) = \delta(t)$$

$$\cos(t)\delta(2t-3) = \cos(t)\delta(2(t-\frac{3}{2})) = \frac{1}{2}\delta(t-\frac{3}{2})\cos(t) = \frac{\cos(\frac{3}{2})}{2}\delta(t-\frac{3}{2})$$

$$\int_{-\infty}^{\infty} e^{-t}\delta'(t-1)dt = \int_{-\infty}^{\infty} e^{-u-1}\delta'(u)du = e^{-1}(-1)\frac{de^{-u}}{du}|_{u=0} = e^{-1}$$

Systems

Definition (System)

A system is an entity that is excited by an input signal $x(t)$ and, as a result of this excitation, produces an output signal $y(t)$. The output is uniquely defined for any legitimate input by

$$y(t) = \mathcal{T}\{x(t)\}$$

.

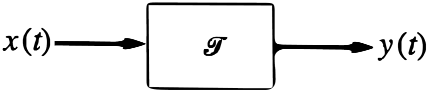


Figure: System block diagram.

Classification of Systems

Definition (Continuous-time System)

For a continuous-time system, both input and output signals are continuous-time signals.

Definition (Discrete-time System)

For a discrete-time system, both input and output signals are discrete-time signals.

Classification of Systems

Definition (Linear System)

A system $\mathcal{T}$ is linear if and only if, for any two input signals $x_1(t)$ and $x_2(t)$ and for any two scalars α and β, we have,

$$\mathcal{T}\{\alpha x_1(t) + \beta x_2(t)\} = \alpha \mathcal{T}\{x_1(t)\} + \beta \mathcal{T}\{x_2(t)\}$$

.

Definition (Nonlinear System)

A system is nonlinear if it is not linear.

Classification of Systems

Definition (Time-Invariant System)

A system is time-invariant if and only if, for all $x(t)$ and all values of t_0, its response to $x(t - t_0)$ is $y(t - t_0)$, where $y(t)$ is the response of the system to $x(t)$.

Definition (Time-variant System)

A system is time-variant if it is not time-invariant.

Classification of Systems

Definition (Causal System)

A system is causal if its output at any time t_0 depends on the input at times prior to t_0, i.e.,

$$y(t_0) = \mathcal{T}\{x(t) : t \leqslant t_0\}.$$

Definition (Noncausal System)

A system is noncausal if it is not causal.

Classification of Systems

Definition (Stable System)

A system is stable if its output is bounded for any bounded input, i.e.,

$$|x(t)| < B \Rightarrow |y(t)| < M.$$

Definition (Instable System)

A system is instable if it is not stable.

LTI Systems

Statement (Linear Time-Invariant System)

A system is Linear Time-Invariant (LTI) if it is simultaneously linear and time-invariant. An LTI system is completely characterized by its impulse response $h(t) = \mathcal{T}\{\delta(t)\}$.

$$\begin{aligned} y(t) &= \mathcal{T}\{x(t)\} \\ &= \mathcal{T}\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathcal{T}\{\delta(t-\tau)\}d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= x(t) * h(t) \end{aligned}$$

LTI System

Statement (Causality of LTI Systems)

An LTI system is causal if and only if $h(t) = 0, t < 0$.

Statement (Stability of LTI Systems)

An LTI system is stable if and only if $\int_{-\infty}^{+\infty} |h(t)| dt < \infty$.

LTI System

Example (Complex exponential response)

The response of an LTI system $h(t)$ to the exponential input $x(t) = Ae^{j(2\pi f_0 t+\theta)}$ can be obtained by

$$y(t) = AH(f_0)e^{j(2\pi f_0 t+\theta)} = A|H(f_0)|e^{j(2\pi f_0 t+\theta+\angle H(f_0))}$$

, where

$$H(f_0) = |H(f_0)|e^{j\angle H(f_0)} = \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f_0 \tau} d\tau$$

.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)Ae^{j(2\pi f_0(t-\tau)+\theta)} d\tau \\ &= Ae^{j(2\pi f_0 t+\theta)} \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f_0 \tau} d\tau \\ &= A|H(f_0)|e^{j(2\pi f_0 t+\theta+\angle H(f_0))} \end{aligned}$$

Fourier Series

Definition (Fourier Series)

The periodic signal $x(t+T_0) = x(t)$ can be expanded in terms of the complex exponential $\{e^{j2\pi nt/T_0}\}_{n=-\infty}^{\infty}$ as

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}$$

, where

$$x_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-j2\pi nt/T_0}dt$$

.

Fourier Series and Its Properties

Dirichlet sufficient conditions for existence of the Fourier series are:

1. $x(t)$ is absolutely integrable over its period, i.e., $\int_0^{T_0} |x(t)|dt < \infty$.
2. The number of maxima and minima of $x(t)$ in each period is finite.
3. The number of discontinuities of $x(t)$ in each period is finite.

1. The quantity $f_0 = 1/T_0$ is called the fundamental frequency of the signal $x(t)$.
2. The frequency of the nth complex exponential signal is nf_0, which is called the nth harmonic.
3. In general, $x_n = |x_n|e^{j\angle x_n}$, where $|x_n|$ gives the magnitude of the nth harmonic and $\angle x_n$ gives its phase.
4. For real signals $x(t) = x^*(t)$, $x_{-n} = x_n^*$.

Fourier Series and Its Properties

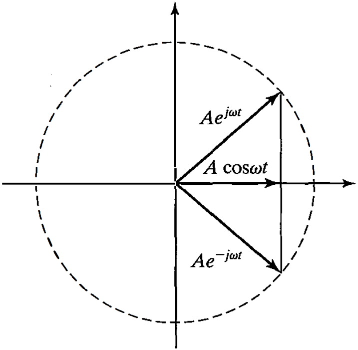


Figure: Positive and negative frequencies.

Example (Fourier series of rectangular-pulse train)

$$x(t) = \sum_{n=-\infty}^{\infty} \sqcap(\frac{t - nT_0}{\tau}) = \sum_{n=-\infty}^{\infty} \frac{\tau}{T_0}\text{sinc}(\frac{n\tau}{T_0})e^{jn2\pi t/T_0}$$

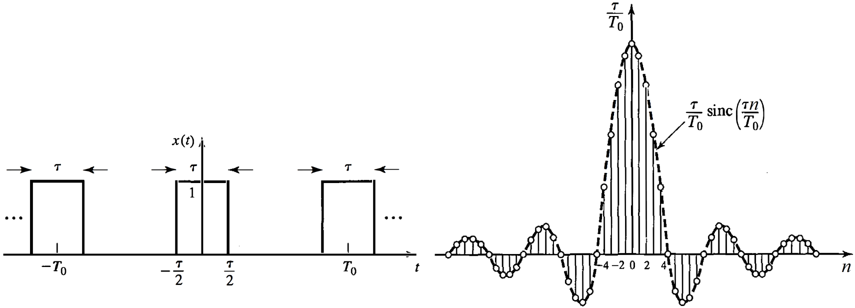


Figure: The discrete spectrum of the rectangular-pulse train.

Fourier Series and Its Properties

Definition (Trigonometric Fourier Series)

The real periodic signal $x(t+T_0) = x(t)$ can be expanded as

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nt/T_0) + \sum_{n=1}^{\infty} b_n \sin(2\pi nt/T_0)$$

, where

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(2\pi nt/T_0)dt$$

and

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(2\pi nt/T_0)dt$$

.

1. $x_n = \frac{a_n}{2} - j\frac{b_n}{2}$.
2. For even real periodic signals, $b_n = 0$.
3. For odd real periodic signals, $a_n = 0$.

Example (Response of LTI Systems to Periodic Signals)

The response of an LTI system $h(t)$ to the periodic input $x(t+T_0) = x(t)$ can be obtained by

$$y(t) = \sum_{n=-\infty}^{\infty} x_n H(n/T_0) e^{j2\pi nt/T_0}$$

, where

$$H(f) = |H(f)| e^{j\angle H(f)} = \int_{-\infty}^{+\infty} h(t) e^{-j2\pi ft} dt.$$

$$y(t) = \mathcal{T}\{x(t)\} = \mathcal{T}\{\sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}\}$$

$$= \sum_{n=-\infty}^{\infty} x_n \mathcal{T}\{e^{j2\pi nt/T_0}\} = \sum_{n=-\infty}^{\infty} x_n H(n/T_0) e^{j2\pi nt/T_0}$$

Fourier Series and Its Properties

1. If the input to an LTI system is periodic with period T_0, then the output is also periodic with period T_0.
2. The output has a Fourier-series expansion given by $y(t) = \sum_{n=-\infty}^{\infty} y_n e^{\frac{j2\pi nt}{T_0}}$, where $y_n = x_n H(n/T_0)$.
3. An LTI system cannot introduce new frequency components in the output.

Statement (Rayleigh's Relation)

For a periodic signal $x(t + T_0) = x(t)$,

$$\mathcal{P}_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x_n|^2$$

.

Fourier Transform

Fourier Transform

Definition (Fourier Transform)

If the Fourier transform of $x(t)$, defined by

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

exists, the original signal can be obtained from its Fourier transform by

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

.

Dirichlet sufficient conditions for existence of the Fourier transform are:

1. $x(t)$ is absolutely integrable over the real line, i.e., $\int_{-\infty}^{\infty} |x(t)|dt < \infty$.
2. The number of maxima and minima of $x(t)$ in any finite real interval is finite.
3. The number of discontinuities of $x(t)$ in any finite real interval is finite.

Fourier Transform and Its Properties

1. $X(f)$ is generally a complex function. Its magnitude $|X(f)|$ and phase $\angle X(f)$ represent the amplitude and phase of various frequency components in $x(t)$.
2. The function $X(f)$ is sometimes referred to as the spectrum of the signal $x(t)$.
3. To denote that $X(f)$ is the Fourier transform of $x(t)$, we frequently employ the notations $X(f) = \mathcal{F}\{x(t)\}$, $x(t) = \mathcal{F}^{-1}\{X(f)\}$, or $x(t) \leftrightarrow X(f)$.

Fourier Transform and Its Properties

1. For real signals $x(t) = x^*(t)$,

$$
\begin{aligned}
X(-f) &= X^*(f) \\
\Re[X(-f)] &= \Re[X(f)] \\
\Im[X(-f)] &= -\Im[X(f)] \\
|X(-f)| &= |X(f)| \\
\angle X(-f) &= -\angle X(f)
\end{aligned}
$$

2. If $x(t)$ is real and even, $X(f)$ will be real and even.
3. If $x(t)$ is real and odd, $X(f)$ will be imaginary and odd.

Statement (Signal Bandwidth)

We define the bandwidth of a real signal $x(t)$ as the range of positive frequencies contributing strongly in the spectrum of the signal.

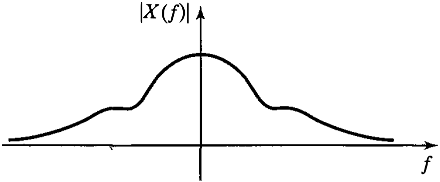


Figure: Bandwidth of a real signal.

Example (Fourier transform of $\sqcap(t)$)

$$\mathcal{F}\{\sqcap(t)\} = \int_{-\infty}^{+\infty} \sqcap(t) e^{-j2\pi ft} dt = \int_{-0.5}^{0.5} e^{-j2\pi ft} dt = \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f)$$

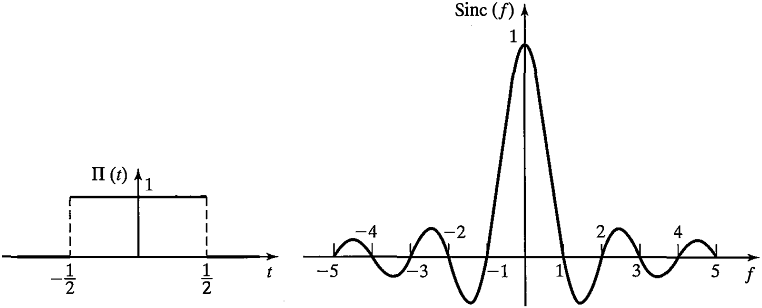


Figure: $\sqcap(t)$ and its Fourier transform.

Fourier Transform and Its Properties

Example (Modulation Property)

$$x(t)\cos(2\pi f_0 t) \leftrightarrow \frac{1}{2}[X(f-f_0)+X(f+f_0)]$$

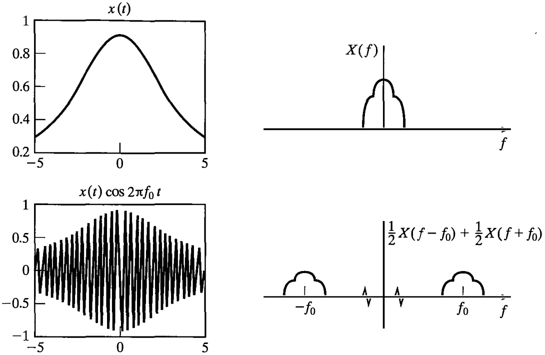


Figure: Effect of modulation in both the time and frequency domain.

Fourier Transform and Its Properties

Property	**Signal**	**Fourier**
Assumption	$x(t)$	$X(f)$
Assumption	$y(t)$	$Y(f)$
Linearity	$ax(t) + by(t)$	$aX(f) + bY(f)$
Time Shifting	$x(t - t_0)$	$e^{-j2\pi f t_0} X(f)$
Frequency Shifting	$e^{j2\pi f_0 t} x(t)$	$X(f - f_0)$
Time Scaling	$x(at)$	$\frac{1}{\|a\|} X(\frac{f}{a})$
Conjugation	$x^*(t)$	$X^*(-f)$
Convolution	$x(t) * y(t)$	$X(f)Y(f)$
Modulation	$x(t)y(t)$	$X(f) * Y(f)$
Sinusoidal Modulation	$x(t)\cos(2\pi f_0 t)$	$\frac{1}{2}[X(f - f_0) + X(f + f_0)]$
Auto-correlation	$x(t) * x^*(-t)$	$\|X(f)\|^2$
Time Differentiation	$\frac{dx(t)}{dt}$	$j2\pi f X(f)$
Time Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Frequency Differentiation	$t^n x(t)$	$(\frac{j}{2\pi})^n \frac{d^n X(f)}{df^n}$
Integration	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{X(f)}{j2\pi f} + \frac{1}{2} X(0)\delta(f)$
Duality	$X(t)$	$x(-f)$
Periodicity	$\sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}$	$\sum_{n=-\infty}^{\infty} x_n \delta(f - n/T_0)$

Table: Properties of the Fourier transform.

Fourier Transform and Its Properties

Signal	Fourier
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$\delta^n(t)$	$(j2\pi f)^n$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{t}$	$-j\pi\mathrm{sgn}(f)$
$u(t)$	$\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}[\delta(f-f_0)-\delta(f+f_0)]$
$\sqcap(t)$	$\mathrm{sinc}(f)$
$\mathrm{sinc}(t)$	$\sqcap(f)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$
$\mathrm{sinc}^2(t)$	$\Lambda(f)$
$e^{-at}u(t), a>0$	$\frac{1}{j2\pi f+a}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t), a>0$	$\frac{1}{(j2\pi f+a)^n}$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta(f-n/T_0)$

Table: Fourier transform of elementary functions.

Fourier Transform and Its Properties

Statement (Parseval's Relation)

If the Fourier transforms of the signals $x(t)$ and $y(t)$ are denoted by $X(f)$ and $Y(f)$, respectively, then

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

.

Statement (Rayleigh's Relation)

If the Fourier transforms of the signals $x(t)$ is denoted by $X(f)$, then

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

.

Example (LTI Systems)

The output of an LTI system is represented by the convolution integral

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

, where $h(t)$ is the impulse response of the LTI system. In the frequency domain,

$$Y(f) = H(f)X(f)$$

, where the frequency response $H(f)$ is the Fourier transform of the impulse response $h(t)$.

Fourier Transform and Its Properties

Example (Interconnection of LTI systems)

The overall frequency response $H(f)$ of the parallel, feedback, and series interconnection of the LTI systems $H_1(f)$ and $H_2(f)$ is $H_1(f) + H_2(f)$, $H_1(f)/(1 + H_1(f)H_2(f))$, and $H_1(f)H_2(f)$, respectively.

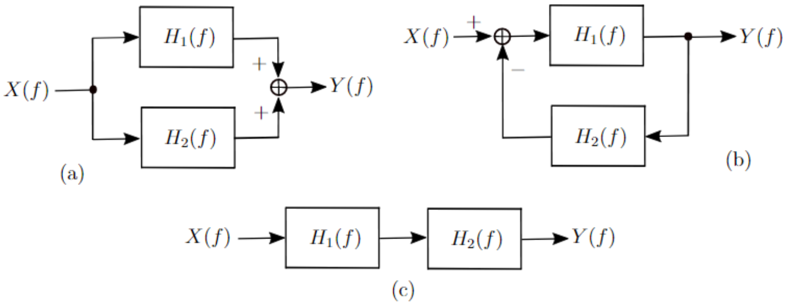


Figure: (a) Parallel, (b) Feedback, and (c) series interconnection of LTI systems.

Power and Energy

Power and Energy

Definition (Energy Signal)

The signal $x(t)$ is energy-type if its energy content is nonzero and limited, i.e.,

$$0 < \mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

.

Definition (Power Signal)

The signal $x(t)$ is power-type if its power content is nonzero and limited, i.e.,

$$0 < \mathcal{P}_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt < \infty$$

.

Power and Energy

1. A signal cannot be both power- and energy-type because $\mathcal{P}_x = 0$ for energy-type signals, and $\mathcal{E}_x = \infty$ for power-type signals.
2. A signal can be neither energy-type nor power-type.

Energy-Type Signals

Definition (Autocorrelation)

For an energy-type signal $x(t)$, we define the autocorrelation function

$$R_x(\tau) = x(\tau) * x^*(-\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt = \int_{-\infty}^{\infty} x(t+\tau)x^*(t)dt$$

.

Energy-Type Signals

1. $\mathcal{F}\{R_x(\tau)\} = |X(f)|^2 = \mathcal{E}_x(f)$, where $\mathcal{E}_x(f)$ is called the energy spectral density of a signal $x(t)$.
2. $\mathcal{E}_x = R_x(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \mathcal{E}_x(f) df$.
3. If we pass the signal $x(t)$ through an LTI system with the impulse response $h(t)$ and frequency response $H(f)$,

$$\begin{aligned} R_y(\tau) &= \mathcal{F}^{-1}\{|Y(f)|^2\} \\ &= \mathcal{F}^{-1}\{|X(f)|^2 |H(f)|^2\} \\ &= \mathcal{F}^{-1}\{|X(f)|^2\} * \mathcal{F}^{-1}\{|H(f)|^2\} = R_x(\tau) * R_h(\tau) \end{aligned}$$

Energy-Type Signals

Example (Energy of rectangular pulse)

The energy content of $x(t) = A \sqcap \left(\frac{t}{T}\right)$ is $\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-T/2}^{T/2} A^2 dt = A^2 T$.

Example (Energy spectral density of rectangular pulse)

The energy spectral density of $x(t) = A \sqcap \left(\frac{t}{T}\right)$ is $\mathcal{E}_x(f) = \left|\mathcal{F}\{A \sqcap \left(\frac{t}{T}\right)\}\right|^2 = T^2 A^2 \text{sinc}^2(Tf)$.

Example (Autocorrelation of rectangular pulse)

The autocorrelation of $x(t) = A \sqcap \left(\frac{t}{T}\right)$ is $\mathcal{R}_x(\tau) = \mathcal{F}^{-1}\{\mathcal{E}_x(f)\} = A^2 T \Lambda\left(\frac{\tau}{T}\right)$.

Power-Type Signals

Definition (Time-Average Autocorrelation)

For a power-type signal $x(t)$, we define the time-average autocorrelation function

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt$$

.

Power-Type Signals

1. $\mathcal{S}_x(f) = \mathcal{F}\{R_x(\tau)\}$ is called power-spectral density or the power spectrum of the signal $x(t)$.
2. $\mathcal{P}_x = R_x(0) = \int_{-\infty}^{\infty} \mathcal{S}_x(f) df$.
3. If we pass the signal $x(t)$ through an LTI system with the impulse response $h(t)$ and frequency response $H(f)$,
$R_y(\tau) = R_x(\tau) * h(\tau) * h^*(-\tau)$ and $S_y(f) = S_x(f)|H(f)|^2$.

Power-Type Signals

Example (Power of periodic signals)

Any periodic signal $x(t) = x(t + T_0)$ is a power-type signal and its power content equals the average power in one period as

$$\mathcal{P}_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{n\to\infty} \frac{1}{nT_0} \int_{-nT_0/2}^{nT_0/2} |x(t)|^2 dt$$
$$= \lim_{n\to\infty} \frac{n}{nT_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 dt$$

Example (Power of cosine)

The power content of $x(t) = A\cos(2\pi f_0 t + \theta)$ is

$$\mathcal{P}_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A^2 \cos^2(2\pi f_0 t + \theta) dt = \frac{A^2}{2}$$

Power-Type Signals

Example (Time-average autocorrelation of periodic signals)

Let the signal $x(t)$ be a periodic signal with the period T_0. Then,

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt$$

.

$$\begin{aligned}
R_x(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt \\
&= \lim_{k\to\infty}\frac{1}{kT_0}\int_{-kT_0/2}^{kT_0/2} x(t)x^*(t-\tau)dt \\
&= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt
\end{aligned}$$

Example (Time-average autocorrelation of periodic signals)

Let the signal $x(t)$ be a periodic signal with the period T_0 and have the Fourier-series coefficients x_n. Then, $R_x(\tau) = \sum_{n=-\infty}^{\infty} |x_n|^2 e^{j2\pi n\tau/T_0}$.

$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{j2\pi(n-m)t/T_0} dt = \delta_{nm}$, which is nonzeros when $n = m$. So

$$
\begin{aligned}
R_x(\tau) &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt \\
&= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n x_m^* e^{j2\pi m\tau/T_0} e^{j2\pi(n-m)t/T_0} dt \\
&= \sum_{n=-\infty}^{\infty} |x_n|^2 e^{j2\pi n\tau/T_0}
\end{aligned}
$$

Hilbert Transform

Hilbert Transform

Definition (Hilbert Transform)

The Hilbert transform of the signal $x(t)$ is a signal $\hat{x}(t)$ whose frequency components lag the frequency components of $x(t)$ by 90°.

1. A delay of $\pi/2$ for $e^{j2\pi f_0 t}$ results in $e^{j(2\pi f_0 t - \pi/2)} = -je^{j2\pi f_0 t}$.
2. A delay of $\pi/2$ for $e^{-j2\pi f_0 t}$ results in $e^{-j(2\pi f_0 t - \pi/2)} = je^{-j2\pi f_0 t}$.

Hilbert Transform

Statement (Hilbert Transform)

Assume that $x(t)$ *is real and has no DC component, i.e.,* $X(0) = 0$. *Then,*

$$\mathcal{F}\{\hat{x}(t)\} = -jsgn(f)X(f)$$

and

$$\hat{x}(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

Hilbert Transform

1. The Hilbert transform of an even real signal is odd, and the Hilbert transform of an odd real signal is even.
2. Applying the Hilbert-transform operation to a signal twice causes a sign reversal of the signal, i.e., $\hat{\hat{x}}(t) = -x(t)$.
3. Energy content of a signal is equal to the energy content of its Hilbert transform, i.e., $\mathcal{E}_x = \mathcal{E}_{\hat{x}}$.
4. The signal $x(t)$ and its Hilbert transform are orthogonal, i.e.,

$$\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$$

.

Hilbert Transform

Example (Hilbert transform of a cosine)

$$x(t) = A\cos(2\pi f_0 t + \theta) \leftrightarrow \frac{A}{2}e^{j\theta}\delta(f - f_0) + \frac{A}{2}e^{-j\theta}\delta(f + f_0)$$

$$\hat{x}(t) \leftrightarrow -j\text{sgn}(f)\big[\frac{A}{2}e^{j\theta}\delta(f - f_0) + \frac{A}{2}e^{-j\theta}\delta(f + f_0)\big]$$

$$\hat{x}(t) \leftrightarrow \frac{A}{2j}e^{j\theta}\delta(f - f_0) - \frac{A}{2j}e^{-j\theta}\delta(f + f_0)$$

$$\hat{x}(t) = A\sin(2\pi f_0 t + \theta) \leftrightarrow \frac{A}{2j}e^{j\theta}\delta(f - f_0) - \frac{A}{2j}e^{-j\theta}\delta(f + f_0)$$

Hilbert Transform

Example (Energy of a signal and its Hilbert transform)

$$\mathcal{E}_{\hat{x}} = \int_{-\infty}^{\infty} |\hat{x}(t)|^2 dt = \int_{-\infty}^{\infty} |\mathcal{F}\{\hat{x}(t)\}|^2 df$$
$$= \int_{-\infty}^{\infty} |-j\text{sgn}(f)X(f)|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} |x(t)|^2 dt = \mathcal{E}_x$$

Example (Orthogonality of a signal and its Hilbert transform)

$$\int_{-\infty}^{\infty} \hat{x}(t)x(t)dt \int_{-\infty}^{\infty} \hat{x}(t)\left[x^*(t)\right]^* dt =$$
$$\int_{-\infty}^{\infty} -j\text{sgn}(f)X(f)\left[X^*(-f)\right]^* df = \int_{-\infty}^{\infty} -j\text{sgn}(f)X(f)X(-f)df = 0$$

Lowpass and Bandpass Signals

Lowpass and Bandpass Signals

Definition (Lowpass Signal)

A lowpass signal is a signal, whose spectrum is located around the zero frequency.

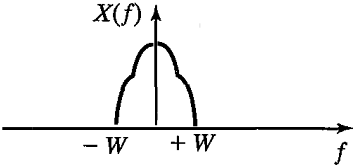


Figure: Spectrum of a lowpass signal.

Lowpass and Bandpass Signals

Definition (Bandpass Signal)

A bandpass signal is a signal with a spectrum far from the zero frequency.

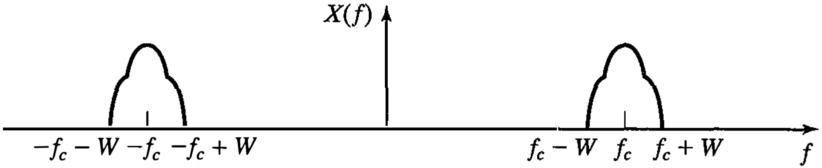


Figure: Spectrum of a bandpass signal.

Lowpass and Bandpass Signals

1. The spectrum of a bandpass signal is usually located around a center frequency f_c, which is much higher than the bandwidth of the signal.
2. The extreme case of a bandpass signal is $x(t) = A\cos(2\pi f_c t + \theta)$, which can be represented by a phasor $x_l = Ae^{j\theta} = x_c + jx_s$, where A, θ, x_c, and x_s are called envelope, phase, in-phase component, and quadrature component, respectively.
3. The original signal $x(t)$ can be reconstructed from its phasor as $x(t) = A\cos(2\pi f_c t + \theta) = x_c \cos(2\pi f_c t) - x_s \sin(2\pi f_c t)$.

Lowpass and Bandpass Signals

Statement (Slowly-varying Lowpass Phasor)

Assume that we have a slowly-varying lowpass phasor $x_l(t) = A(t)e^{j\theta(t)} = x_c(t) + jx_s(t)$, *where* $A(t) \geq 0$, $\theta(t)$, $x_s(t)$, *and* $x_c(t)$ *are slowly-varying signals compared to* f_c. *The real bandpass signal* $x(t) = A(t)\cos(2\pi f_c t + \theta(t))$ *relates to the complex time-varying phasor* $x_l(t)$ *as*

$$\begin{aligned} x(t) &= \Re\{x_l(t)e^{j2\pi f_c t}\} = \Re\{A(t)e^{j(2\pi f_c t + \theta(t))}\} \\ &= x_c(t)\cos(2\pi f_c t) - x_s(t)\sin(2\pi f_c t) \end{aligned}$$

.

Lowpass and Bandpass Signals

1. $x_l(t) = A(t)e^{j\theta(t)} = x_c(t) + jx_s(t)$ is is called the lowpass equivalent of the bandpass signal $x(t) = A(t)\cos(2\pi f_c t + \theta(t))$.
2. The envelope $|x_l(t)|$ and the phase $\angle x_l(t)$ of the bandpass signal are defined as

$$|x_l(t)| = A(t) = \sqrt{x_c^2(t) + x_s^2(t)}$$

and

$$\angle x_l(t) = \theta(t) = \tan^{-1}\left(\frac{x_s(t)}{x_c(t)}\right)$$

3. Obviously, the in-phase and quadrature components satisfy

$$x_c(t) = A(t)\cos(\theta(t))$$

and

$$x_s(t) = A(t)\sin(\theta(t))$$

Example (Spectrum of the bandpass signal)

$$x(t) = \Re\{x_l(t)e^{j2\pi f_c t}\} = \frac{1}{2}\left[x_l(t)e^{j2\pi f_c t} + x_l^*(t)e^{-j2\pi f_c t}\right]$$

So,

$$X(f) = \frac{1}{2}X_l(f - f_c) + \frac{1}{2}X_l^*(-(f + f_c))$$

Lowpass and Bandpass Signals

Example (Spectrum of the bandpass signal)

$$X(f) = \frac{1}{2}X_l(f - f_c) + \frac{1}{2}X_l^*(-(f + f_c))$$

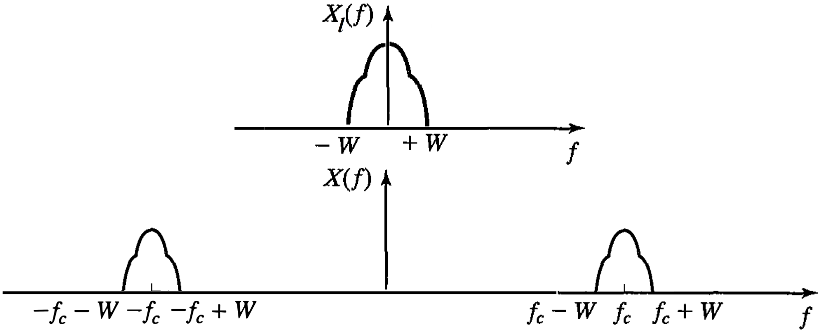


Figure: Spectrum of the lowpass signal and its associated bandpass signal.

Lowpass and Bandpass Signals

Example (Spectrum of the lowpass signal)

If the bandwidth of the bandpass signal W is much less than the central frequency f_c, then

$$
\begin{aligned}
X(f) &= \frac{1}{2}X_l(f - f_c) + \frac{1}{2}X_l^*(-(f + f_c)) \\
X(f + f_c) &= \frac{1}{2}X_l(f) + \frac{1}{2}X_l^*(-(f + 2f_c)) \\
X(f + f_c)u(f + f_c) &= \frac{1}{2}X_l(f)u(f + f_c) + \frac{1}{2}X_l^*(-(f + 2f_c))u(f + f_c) \\
X(f + f_c)u(f + f_c) &= \frac{1}{2}X_l(f) \\
2X(f + f_c)u(f + f_c) &= X_l(f)
\end{aligned}
$$

Example (Spectrum of the lowpass signal)

If the bandwidth of the bandpass signal W is much less than the central frequency f_c, then

$$X_l(f) = 2X(f + f_c)u(f + f_c)$$

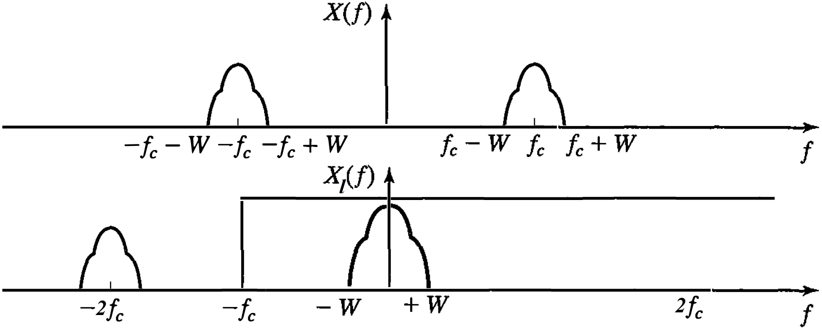


Figure: Spectrum of the bandpass signal and its associated lowpass signal.

Example (Lowpass equivalent of a bandpass signal)

$$
\begin{aligned}
X_l(f) &= 2X(f+f_c)u(f+f_c) \\
&= 2X(f+f_c)\frac{1+\text{sgn}(f+f_c)}{2} \\
&= 2X(f+f_c)\frac{1-j^2\text{sgn}(f+f_c)}{2} \\
&= X(f+f_c)+j\big[-j\text{sgn}(f+f_c)X(f+f_c)\big]
\end{aligned}
$$

So,

$$x_l(t) = \big[x(t) + j\hat{x}(t)\big]e^{-j2\pi f_c t}$$

Lowpass and Bandpass Signals

Example (In-phase component of a bandpass signal)

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right] e^{-j2\pi f_c t}$$

So,

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right]\left[\cos(2\pi f_c t) - j\sin(2\pi f_c t)\right]$$

$$x_l(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t) + j\left[\hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)\right]$$

and,

$$\Re\{x_l(t)\} = x_c(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t)$$

Example (Quadrature component of a bandpass signal)

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right] e^{-j2\pi f_c t}$$

So,

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right]\left[\cos(2\pi f_c t) - j\sin(2\pi f_c t)\right]$$

$$x_l(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t) + j\left[\hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)\right]$$

and,

$$\Im\{x_l(t)\} = x_s(t) = \hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)$$

Example (Envelope of a bandpass signal)

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right] e^{-j2\pi f_c t}$$

So,

$$|x_l(t)| = A(t) = \sqrt{x^2(t) + \hat{x}^2(t)}$$

Example (Phase of a bandpass signal)

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right] e^{-j2\pi f_c t}$$

So,

$$x_l(t) = \left[x(t) + j\hat{x}(t)\right]\left[\cos(2\pi f_c t) - j\sin(2\pi f_c t)\right]$$

$$x_l(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t) + j\left[\hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)\right]$$

and,

$$\angle x_l(t) = \theta(t) = \tan^{-1}\left[\frac{\hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t)}{x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t)}\right]$$

Example (Lowpass equivalent of sinusoidal signal)

Lowpass equivalent of the bandpass signal $x(t) = A\cos(2\pi f_c t + \theta)$ is

$$\begin{aligned} x_l(t) &= \left[x(t) + j\hat{x}(t)\right] e^{-j2\pi f_c t} \\ &= \left[A\cos(2\pi f_c t + \theta) + jA\sin(2\pi f_c t + \theta)\right] e^{-j2\pi f_c t} \\ &= Ae^{j(2\pi f_c t + \theta)} e^{-j2\pi f_c t} = Ae^{j\theta} \end{aligned}$$

So, $A(t) = |A|$, $\theta(t) = \theta + u(-A)\pi$, $x_s(t) = A\cos(\theta)$, and $x_s(t) = A\sin(\theta)$.

Lowpass and Bandpass Signals

Example (Lowpass equivalent of sinusoidal signal)

Lowpass equivalent of the bandpass signal $x(t) = \text{sinc}(t)\cos(2\pi f_c t + \frac{\pi}{4})$ can be obtained as

$$x(t) = \text{sinc}(t)\cos(\frac{\pi}{4})\cos(2\pi f_c t) - \text{sinc}(t)\sin(\frac{\pi}{4})\sin(2\pi f_c t)$$

$$x_c(t) = \frac{\sqrt{2}}{2}\text{sinc}(t), \quad x_s(t) = \frac{\sqrt{2}}{2}\text{sinc}(t)$$

$$x_l(t) = x_c(t) + jx_s(t) = \frac{\sqrt{2}}{2}\text{sinc}(t)(1+j) = \text{sinc}(t)e^{j\frac{\pi}{4}}$$

Filters

Lowpass Filter

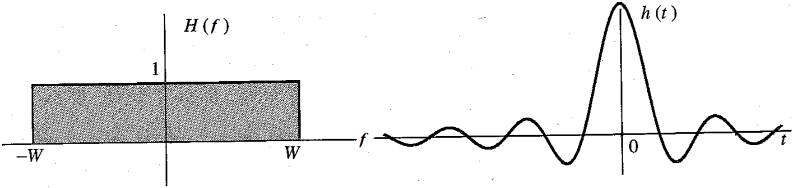


Figure: Ideal LPF frequency response and its impulse response.

$$H(f) = \sqcap(\frac{f}{2W}) \leftrightarrow h(t) = 2W\,\text{sinc}(2Wt)$$

Lowpass Filter

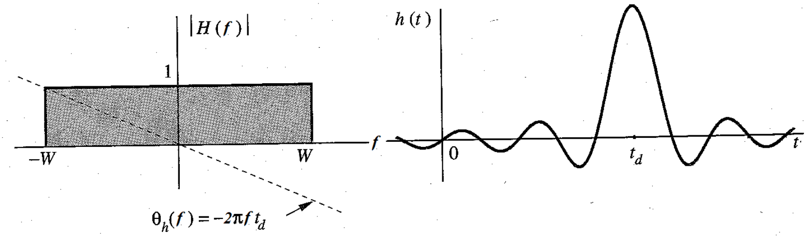


Figure: Linear-phase ideal LPF frequency response and its impulse response.

$$H(f) = \sqcap(\frac{f}{2W})e^{-j2\pi f t_d} \leftrightarrow h(t) = 2W\,\text{sinc}(2W(t-t_d))$$

Lowpass Filter

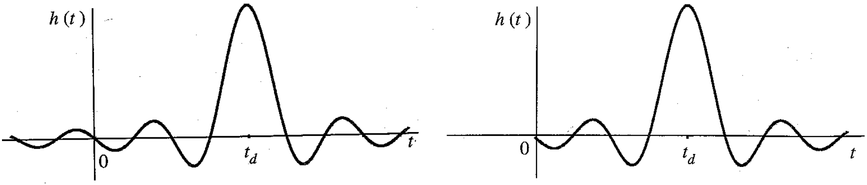


Figure: Truncated LPF impulse response.

$$h(t) = 2W\,\mathrm{sinc}(2W(t - t_d)) \qquad h(t) = 2W\,\mathrm{sinc}(2W(t - t_d))u(t)$$

Lowpass Filter

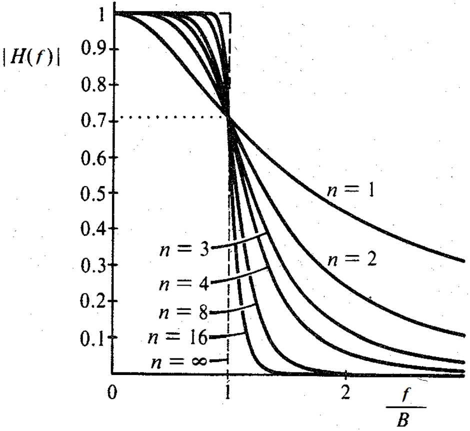


Figure: Butterworth LPF frequency characteristic.

$$|H(f)| = \frac{1}{\sqrt{1 + (\frac{f}{B})^{2n}}}$$

Lowpass Filter

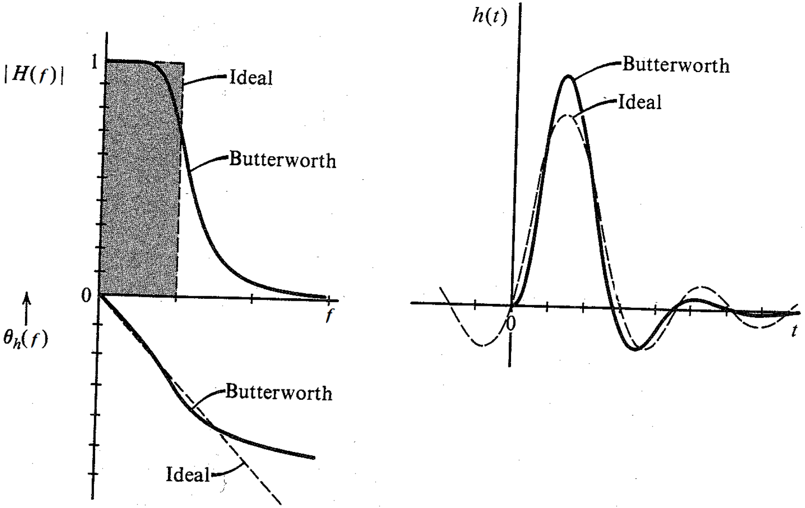


Figure: Comparison of butterworth and ideal filters.

Basic Filters

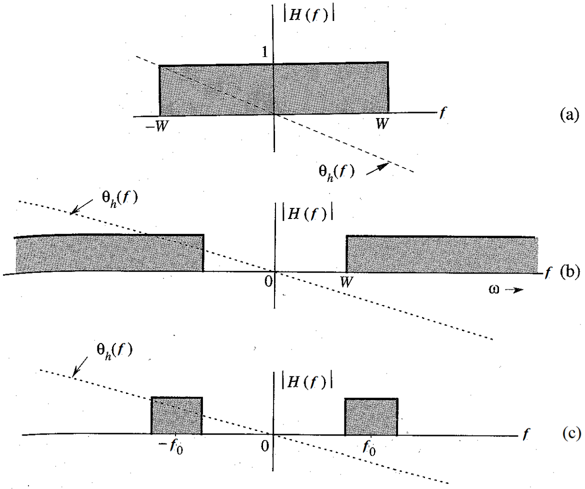


Figure: Basic filters. (a) LPF (b) HPF (c) BPF.

Filter Design

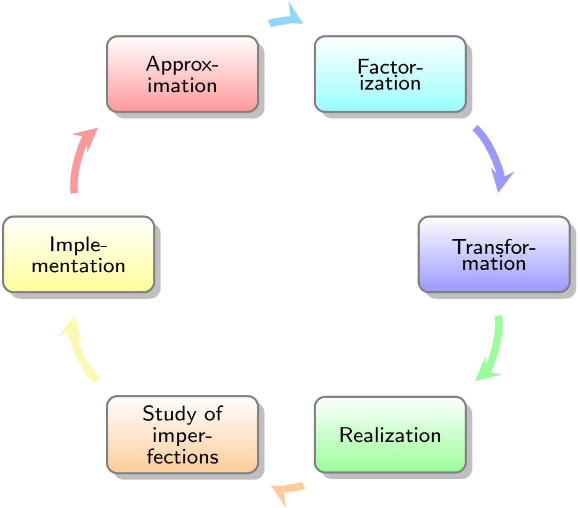


Figure: Design process.

The End