# MATHEMATICAL QUESTIONS

## Question 1

**Use the definitions of the unit step, unit impulse, and unit doublet function to prove the following identities.**
**Hint: Obviously, if $\int_{-\infty}^{+\infty} f(t)x(t)dt = \int_{-\infty}^{+\infty} g(t)x(t)dt$ for any test function $x(t)$, the singular functions $f(t)$ and $g(t)$ are equal.**

*(a)* $u'_{-1}(t) = u_0(t)$.

Integration by parts yields

$$\int_{-\infty}^{+\infty} u'_{-1}(t)x(t)dt = u_{-1}(t)x(t)\big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u_{-1}(t)x'(t)dt$$
$$= -\int_{-\infty}^{+\infty} u_{-1}(t)x'(t)dt = -\int_{0}^{+\infty} x'(t)dt = -x(t)\big|_{0}^{+\infty} = x(0) \tag{1}$$

. On the other hand,

$$x(0) = \int_{-\infty}^{+\infty} u_0(t)x(t)dt \tag{2}$$

Equating (1) and (2)

$$\int_{-\infty}^{+\infty} u_0(t)x(t)dt = \int_{-\infty}^{+\infty} u'_{-1}(t)x(t)dt \tag{3}$$

results in $u'_{-1}(t) = u_0(t)$ by the definition of the equality of singular functions.

*(b)* $u'_0(t) = u_1(t)$.

Integration by parts yields

$$\int_{-\infty}^{+\infty} u'_0(t)x(t)dt = u_0(t)x(t)\big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u_0(t)x'(t)dt = -\int_{-\infty}^{+\infty} u_0(t)x'(t)dt = -x'(0) \tag{4}$$

. On the other hand,

$$-x'(0) = \int_{-\infty}^{+\infty} u_1(t)x(t)dt \tag{5}$$

Equating (4) and (5)

$$\int_{-\infty}^{+\infty} u_1(t)x(t)dt = \int_{-\infty}^{+\infty} u'_0(t)x(t)dt \tag{6}$$

leads to $u_1(t) = u'_0(t)$ by the definition of the equality of singular functions.

*Question 1 continued on next page...*

*(c)* $\delta(at) = \frac{1}{|a|}\delta(t), a \neq 0$.

Assume that $a > 0$. We have

$$\int_{-\infty}^{+\infty} u_0(at)x(t)dt = \frac{1}{a}\int_{-\infty}^{+\infty} u_0(v)x(\frac{v}{a})dv = \frac{1}{a}x(\frac{0}{a}) = \frac{1}{a}x(0) \tag{7}$$

. On the other hand,

$$\frac{1}{a}x(0) = \int_{-\infty}^{+\infty} u_0(t)\frac{1}{a}x(t)dt \tag{8}$$

Equating (7) and (8)

$$\int_{-\infty}^{+\infty} u_0(t)\frac{1}{a}x(t)dt = \int_{-\infty}^{+\infty} u_0(at)x(t)dt \tag{9}$$

results in $\delta(at) = \frac{1}{|a|}\delta(t), a > 0$ by the definition of the equality of singular functions. The same method can be used to prove $\delta(at) = \frac{1}{-a}\delta(t), a < 0$.

## Question 2

**Take the Fourier transform of $x(t) = Ae^{-\frac{t^2}{\sigma^2}}$, where $A$ and $\sigma$ are given real values.**

$$\mathcal{F}\{x(t)\} = X(f) = \int_{-\infty}^{\infty} Ae^{-\frac{t^2}{\sigma^2}}e^{-j2\pi ft}dt = A\int_{-\infty}^{\infty} e^{-(\frac{t^2}{\sigma^2}+j2\pi ft)}dt$$

$$\Rightarrow \mathcal{F}\{x(t)\} = A\int_{-\infty}^{\infty} e^{-\frac{1}{\sigma^2}(t^2+j2\pi\sigma^2 ft+\pi^2f^2\sigma^4-\pi^2f^2\sigma^4)}dt = Ae^{-\pi^2f^2\sigma^2}\int_{-\infty}^{\infty} e^{-(\frac{t+j\pi f\sigma^2}{\sigma})^2}dt$$

Assuming $\frac{t+j\pi f\sigma^2}{\sigma} = s$, we have $ds = \frac{dt}{\sigma}$. Thus,

$$X(f) = Ae^{-\pi^2f^2\sigma^2}\int_{-\infty}^{\infty} e^{-s^2}\sigma ds = A\sigma e^{-\pi^2f^2\sigma^2}\int_{-\infty}^{\infty} e^{-s^2}ds$$

To compute $I = \int_{-\infty}^{\infty} e^{-s^2}ds$, we note that

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)}dudv$$

Using the rectangular-polar variable change, $u^2+v^2 = r^2$, $dudv = rdrd\theta$, and

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta = 2\pi(-\frac{1}{2}e^{r^2})\Big|_0^{\infty} = \pi$$

$$\Rightarrow F\{x(t)\} = X(f) = A\sigma\sqrt{\pi}e^{-\pi^2f^2\sigma^2}$$

## Question 3

**The analytic signal $x_a(t)$ of the real signal $x(t)$ is a signal with the spectrum $2X(f)u(f)$, where $X(f)$ is the Fourier transform of $x(t)$.**

*Question 3 continued on next page...* 

*(a) Show that the real and imaginary parts of $x_a(t)$ relates to $x(t)$ and its Hilbert transform $\hat{x}(t)$.*

$$x_a(t) \leftrightarrow 2X(f)u(f)$$
$$x_a(t) \leftrightarrow X(f)(1+\text{sgn}(f))$$
$$x_a(t) \leftrightarrow X(f)(1-jj\,\text{sgn}(f))$$
$$x_a(t) \leftrightarrow X(f)+j\big[-j\,\text{sgn}(f)X(f)\big]$$

So,

$$x_a(t) = x(t) + j\hat{x}(t)$$

*(b) Find the analytic signal of $x(t) = A\cos(2\pi f_0 t + \theta)$.*

We know that $\hat{x}(t) = A\sin(2\pi f_0 t + \theta)$. So,

$$x_a(t) = x(t) + j\hat{x}(t) = A\cos(2\pi f_0 t + \theta) + jA\sin(2\pi f_0 t + \theta) = Ae^{j(2\pi f_0 t + \theta)} = Ae^{j\theta}e^{j2\pi f_0 t}$$

*(c) How does the analytic signal generalize the concept of phasors?*

Clearly, for $x(t) = A\cos(2\pi f_0 t + \theta)$, $x_a(t)e^{-j2\pi f_0 t}$ equals the equivalent phasor of $x(t)$, i.e., $x_l = Ae^{j\theta}$. This can be simply generalized to the real signal $x(t) = A(t)\cos(2\pi f_0 t + \theta(t))$ with a time-varying amplitude and phase. In fact, the time-varying phasor of $x(t)$ is defined as $x_l(t) = x_a(t)e^{-j2\pi f_0 t}$.

## Question 4

**Let $\{\phi_i(t)\}_{i=1}^N$ be an orthogonal set of $N$ signals, i.e.,**

$$\int_{-\infty}^{\infty} \phi_i(t)\phi_j^*(t)dt = 0, \quad 1 \le i, j \le N, \quad i \ne j$$

**and**

$$\int_{-\infty}^{\infty} |\phi_i(t)|^2 = 1, \quad 1 \le i \le N$$

**. Let $\hat{x}(t) = \sum_{i=1}^N \alpha_i \phi_i(t)$ be the linear approximation of an arbitrary signal $x(t)$ in terms of $\{\phi_i(t)\}_{i=1}^N$, where $\alpha_i$'s are chosen such that**

$$\epsilon^2 = \int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 dt$$

**is minimized.**

*(a) Show that the minimizing $\alpha_i$'s satisfy*

$$\alpha_i = \int_{-\infty}^{\infty} x(t)\phi_i^*(t)dt$$

*Question 4 continued on next page...*

$$\epsilon^2 = \int_{-\infty}^{\infty} |x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t)|^2 dt = \int_{-\infty}^{\infty} (x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t))(x^*(t) - \sum_{j=1}^{N} \alpha_j^* \phi_j^*(t) dt)$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} x(t) \phi_j^*(t) dt$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j^* \int_{-\infty}^{\infty} \phi_i(t) \phi_j^*(t) dt$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt + \sum_{i=1}^{N} |\alpha_i|^2 - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} x(t) \phi_j^*(t) dt$$

Completing the square in terms of $\alpha_i$, we obtain

$$\epsilon^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \Big| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \Big|^2 + \sum_{i=1}^{N} \Big| \alpha_i - \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \Big|^2 \tag{10}$$

The first two terms are independent of $\alpha_i$ and the last term is always positive. Therefore the minimum is achieved for

$$\alpha_i = \int_{-\infty}^{\infty} x(t) \phi_i^*(t) dt$$

*(b) Show that*

$$\epsilon_{min}^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} |\alpha_i|^2$$

With this choice of $\alpha_i$, the last term of (10) vanishes and we get

$$\epsilon_{min}^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \Big| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \Big|^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} |\alpha_i|^2$$

*(c) How does this general linear approximation relate to the Fourier series expansion?*

Taking $\phi_i(t) = e^{j2\pi i t/T_0}$, $\hat{x}(t)$ roughly takes the form of the Fourier series expansion while the minimizing $\alpha_i$'s are very similar to the coefficients of the Fourier series expansion.

*Question 5 continued on next page...*

# Question 5

**The generalized Fourier transform of the singular function $y(t)$ is defined as the function $Y(f)$ satisfying the integral equation**

$$\int_{-\infty}^{\infty} Y(\alpha)x(\alpha)d\alpha = \int_{-\infty}^{\infty} y(\beta)X(\beta)d\beta$$

**, where $x(t)$ is any test function such that the existence of its Fourier transform $X(f)$ is guaranteed under Dirichlet sufficient conditions.**
**Hint: It can be shown that the properties of the normal Fourier transform remain valid for the generalized Fourier transform.**

*(a) Discuss the reasons behind the definition.*

Assume that $X(f)$ and $Y(f)$, the Fourier transform of $x(t)$ and $y(t)$, exist. We have

$$\begin{aligned}
&\int_{-\infty}^{\infty} Y(\alpha)x(\alpha)d\alpha \\
&= \int_{\alpha=-\infty}^{\infty} Y(\alpha) \int_{\beta=-\infty}^{\infty} X(\beta)e^{j2\pi\beta\alpha}d\beta d\alpha \\
&= \int_{\alpha=-\infty}^{\infty} \int_{\beta=-\infty}^{\infty} Y(\alpha)X(\beta)e^{j2\pi\beta\alpha}d\beta d\alpha \\
&= \int_{\beta=-\infty}^{\infty} X(\beta) \int_{\alpha=-\infty}^{\infty} Y(\alpha)e^{j2\pi\beta\alpha}d\alpha d\beta \\
&= \int_{-\infty}^{\infty} X(\beta)y(\beta)d\beta \\
&= \int_{-\infty}^{\infty} y(\beta)X(\beta)d\beta
\end{aligned}$$

, which is another form of the Parseval's theorem.
Now, let $y(t)$ be a singular function, which does not satisfy Dirichlet sufficient conditions. Further, assume that $x(t)$ is an arbitrary signal, whose Fourier transform exists under Dirichlet sufficient conditions. Obviously, if this integral equation holds for all pairs of $x(t) \leftrightarrow X(f)$, $Y(f)$ can be considered as the generalized Fourier transform of $y(t)$.

*(b) Use the definition to find the Fourier transform of $\delta(t)$.*

$$\begin{aligned}
&\int_{-\infty}^{\infty} Y(\alpha)x(\alpha)d\alpha = \int_{-\infty}^{\infty} y(\beta)X(\beta)d\beta \\
&= \int_{-\infty}^{\infty} \delta(\beta)X(\beta)d\beta = X(0) = \int_{-\infty}^{\infty} x(\alpha)d\alpha
\end{aligned}$$

So, $Y(f) = \mathcal{F}\{\delta(t)\} = 1$ by the definition of the equality of singular functions. Using the duality property, we conclude that $\mathcal{F}\{1\} = \delta(-f) = \delta(f)$.

*Question 5 continued on next page...* 

*(c) Use the definition to find the Fourier transform of $u(t)$.*

We know that

$$u(t) + u(-t) = 1 \Rightarrow U(f) + U(-f) = \mathcal{F}\{1\} = \delta(f)$$

. Let $U(f) = B(f) + k\delta(f)$. We have

$$\delta(f) = U(f) + U(-f) = B(f) + B(-f) + k\delta(f) + k\delta(-f) = B(f) + B(-f) + 2k\delta(f)$$

Therefore,

$$k = \frac{1}{2}, \quad B(f) = -B(-f)$$

. To find $B(f)$,

$$1 = \mathcal{F}\{\delta(t)\} = \mathcal{F}\{u'(t)\} = j2\pi f\mathcal{F}\{u(t)\} = j2\pi f(B(f) + \frac{1}{2}\delta(f)) = j2\pi fB(f)$$

So, $B(f) = \frac{1}{j2\pi f}$ and

$$U(f) = B(f) + k\delta(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

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## SOFTWARE QUESTIONS

# Question 6

**Validate the performance of the tapped delay-line microwave equalizer using MATLAB simulation. To do this,**

*(a) Develop a function, which simulates the point-to-point microwave radio channel.*

Here is a sample time-domain implementation of the channel.

```
function [s_out, t_out] = p2pmrc_chn(s_in, t_in, A1, D1, A2, D2)
% time step
Dt = t_in(2)-t_in(1);
% shifted time axis
t_out = t_in(1):Dt:t_in(end)+(ceil(max([D1 D2])/Dt)+1)*Dt;
% line of sight signal
s_los = zeros(size(t_out));
s_los(ceil(D1/Dt)+1:ceil(D1/Dt)+length(s_in)) = A1*s_in;
%reflect signal
s_ref = zeros(size(t_out));
s_ref(ceil(D2/Dt)+1:ceil(D2/Dt)+length(s_in)) = A2*s_in;
% received signal
s_out = s_los+s_ref;
end
```

Question 6 continued on next page...

*(b) Develop a function, which simulates the taped delay line microwave equalizer.*

Here is a sample time-domain implementation of the equalizer.

```
function [s_out, t_out] = p2pmrc_eql(s_in, t_in, A1, D1, A2, D2, N)
% equalizer parameters
A= A2/A1;
D=D2-D1;
% time step
Dt = t_in(2)-t_in(1);
% shifted time axis
t_out = t_in(1):Dt:t_in(end)+(ceil(N*D/Dt)+N)*Dt;
% tap signals
s_tap=zeros(N+1,length(t_out));
for i=0:N
    s_tap(i+1,i*ceil(D/Dt)+1:i*ceil(D/Dt)+length(s_in))=(-1)^i*A^i*s_in;
end
% equalized signal
s_out = sum(s_tap,1);
end
```

*(c) Observe the output of the channel before and after the equalizer and discuss the observations for different number of taps.*

To validate the performance, the mfile below can be used.

```
clear all
close all

% parameters
A1=1;
D1=1;
D2=1.7;
A2=0.8;
N=5;

% channel input
t_in=0:0.001:10;
s_in=5*sinc(2*(t_in-0.5));
% channel output
[chn_s, chn_t]=p2pmrc_chn(s_in, t_in, A1, D1, A2, D2);
% equalizer output
[eql_s, eql_t] = p2pmrc_eql(chn_s, chn_t, A1, D1, A2, D2, N);

% plot
subplot(3,1,1);
plot(t_in,s_in, 'b', 'LineWidth', 1.5)
title('channel input','Interpreter','latex');
xlim([min(eql_t) max(eql_t)])
box on
grid on

subplot(3,1,2);
plot(chn_t,chn_s, 'r', 'LineWidth', 1.5)
title('channel output','Interpreter','latex')
xlim([min(eql_t) max(eql_t)])
box on
grid on

subplot(3,1,3);
plot(eql_t,eql_s, 'black', 'LineWidth', 1.5)
title('equalizer output','Interpreter','latex')
xlim([min(eql_t) max(eql_t)])
box on
grid on
```

*Question 6 continued on next page...*

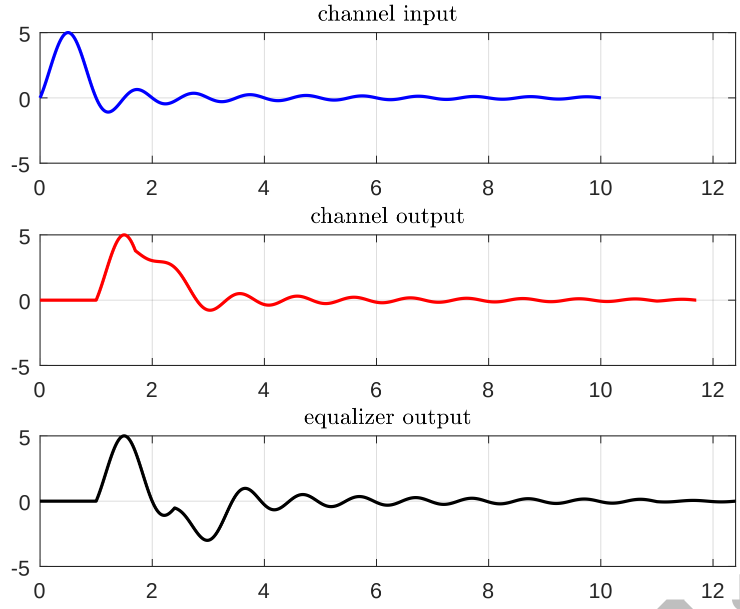


Figure 1: Simulation results for $N = 1$ delay elements.

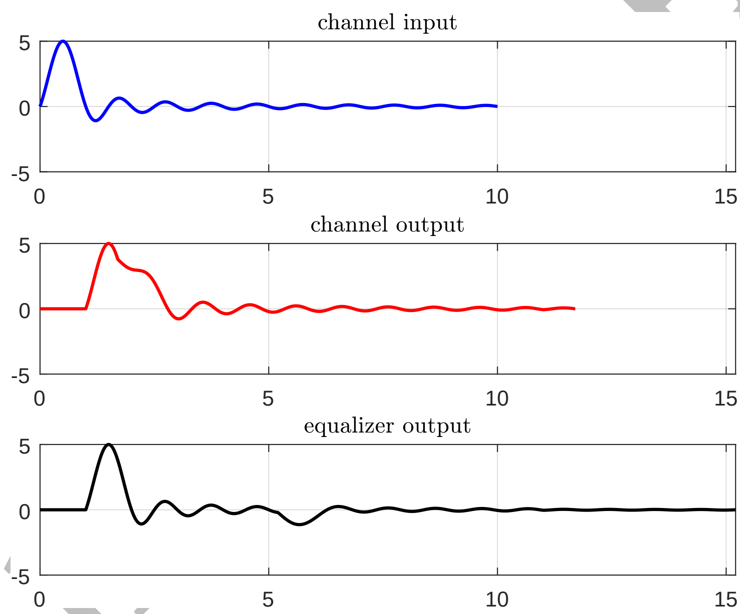


Figure 2: Simulation results for $N = 5$ delay elements.

. Let $A_1 = 1$, $D_1 = 1$, $D_2 = 1.7$, and $A_2 = 0.8$. Fig. 1 shows the involved signals for $N = 1$ delay element. As you can see, the equalizer could not mitigate the distortion. However, for $N = 5$ the performance seems acceptable, as shown in Fig. 2.

Question 6 continued on next page. . .

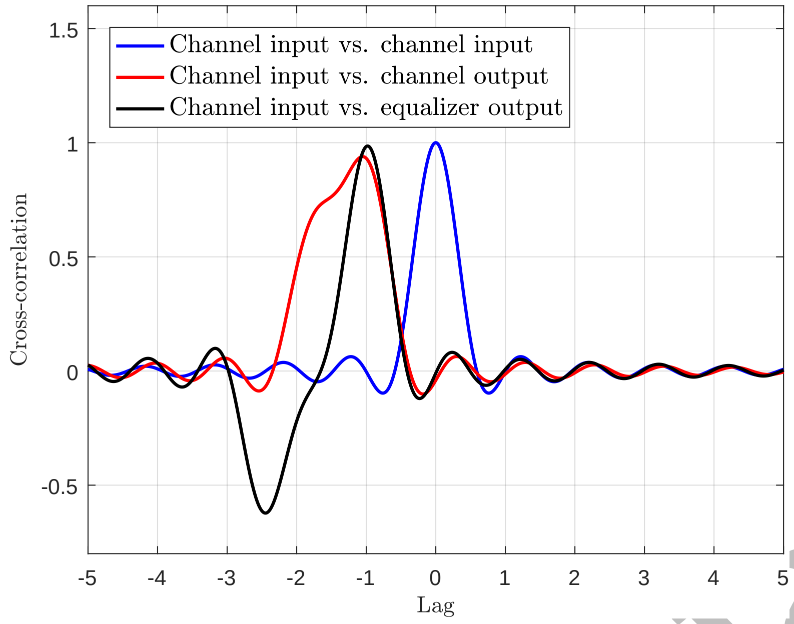


Figure 3: Cross-correlation curves for $N = 1$.

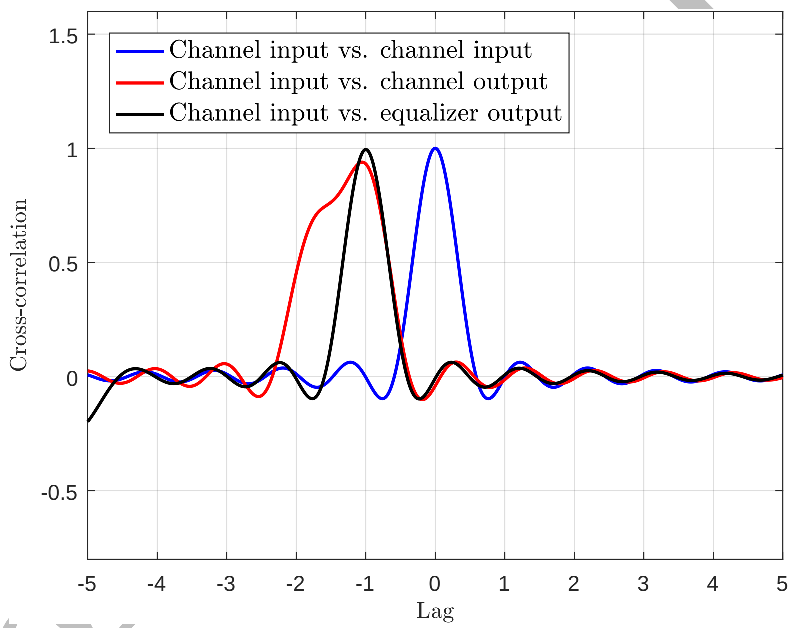


Figure 4: Cross-correlation curves for $N = 5$.

*(d) How can we measure the distortion before and after the equalizer. Do you know any suitable metric?*

Cross-correlation, $R_{xy}(\tau) = x(\tau) * y^*(-\tau) = \int_{-\infty}^{+\infty} x(t)y^*(t-\tau)dt$ measures the similarity between $x(t)$ and shifted (lagged) copies of $y(t)$ as a function of the lag $\tau$. Cross-correlation might be used to measure the (phase) distortion. When distortion is mitigated, the cross-correlation achieves a higher and narrower peak value. Figs. 3 and 4 show the cross-correlation of the channel input with respect to the channel output and equalizer output for $N = 1$ and $N = 5$. Clearly, the cross-correlation gets a lower and wider peak after the channel. The peak increases and tapers after the equalization, where a more acceptable curve is obtained for $N = 5$.

# BONUS QUESTIONS

## Question 7

**Return your answers by filling the LaTeXtemplate of the assignment.**