MATHEMATICAL QUESTIONS

Question 1

Show that if the periodic signal $x(t + T_0) = x(t)$ passes through an LTI system with the frequency response $H(f)$, then the power spectral density of the output signal equals

$$S_y(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 |H(n/T_0)|^2 \delta(f - n/T_0)$$

.

Note: Consult the slides on the calculation of the autocorrelation of the periodic signals.

As we know from the course lecture, the autocorrelation of the periodic signal $x(t + T_0) = x(t)$ equals

$$R_x(\tau) = \sum_{n=-\infty}^{\infty} |x_n|^2 e^{j2\pi n\tau/T_0}$$

, where x_n's are the coefficient of the Fourier series expansion of $x(t)$. Since $S_x(f) = \mathcal{F}\{R_x(\tau)\}$ and $S_y(f) = S_x(f)|H(f)|^2$. So,

$$S_y(f) = S_x(f)|H(f)|^2 = \mathcal{F}\{R_x(\tau)\}|H(f)|^2$$

$$= |H(f)|^2 \sum_{n=-\infty}^{\infty} |x_n|^2 \delta(f - \frac{n}{T_0})$$

$$= \sum_{n=-\infty}^{\infty} |x_n|^2 |H(\frac{n}{T_0})|^2 \delta(f - \frac{n}{T_0})$$

.

Question 2

Although the solution of this question has been already posted to the course Telegram channel, it deserves to be repeated again due to its involved intuition. The real bandpass signal $x(t)$, whose frequency components are in the neighborhood of some f_0 (and $-f_0$), is passed through a filter with the transfer function $H(f)$, and the output is denoted by $y(t)$. The magnitude of the transfer function is denoted by $A(f)$, and its phase is denoted by $\theta(f)$. Assume that the transfer function of the filter is so smooth that in the bandwidth of the input signal, the magnitude of the transfer function is essentially constant and its phase can be approximated by its first-order Taylor-series expansion, i.e.,

$$A(f) \approx A(f_0)$$

and

$$\theta(f) \approx \theta(f_0) + (f - f_0)\theta'(f_0)$$

(a) Show that $Y_l(f)$, the Fourier transform of the lowpass equivalent of the output, can be written as

$$Y_l(f) \approx X_l(f)A(f_0)e^{j(\theta(f_0)+f\theta'(f_0))}$$

, where $X_l(f)$ is the Fourier transform of the lowpass equivalent of the input $x(t)$.

The spectrum of the output signal $y(t)$ is the product of $X(f)$ and $H(f)$. Thus,

$$Y(f) = H(f)X(f) \approx X(f)A(f_0)e^{j(\theta(f_0)+(f-f_0)\theta'(f_0))}$$

. $y(t)$ is a bandpass signal centered at frequencies $f = \pm f_0$. To obtain the lowpass equivalent signal, we have to shift the right side spectrum of $y(t)$ to the left by f_0 and multiply it by 2. Hence,

$$Y_l(f) \approx 2u(f+f_0)X(f+f_0)A(f_0)e^{j(\theta(f_0)+f\theta'(f_0))} = X_l(f)A(f_0)e^{j(\theta(f_0)+f\theta'(f_0))}$$

(b) Calculate that

$$y(t) \approx A(f_0)V_x(t-t_g)\cos(2\pi f_0(t-t_p) + \Theta_x(t-t_g))$$

, where $V_x(t)$ and $\Theta_x(t)$ are the envelop and phase of the input $x(t)$, $t_g = -\theta'(f_0)/(2\pi)$, and $t_p = -\theta(f_0)/(2\pi f_0)$.

Taking the inverse Fourier transform of $Y_l(f)$ obtained in the previous part,

$$y_l(t) \approx \mathcal{F}^{-1}[X_l(f)A(f_0)e^{j\theta(f_0)}e^{jf\theta'(f_0)}] = A(f_0)e^{j\theta(f_0)}x_l(t + \frac{1}{2\pi}\theta'(f_0))$$

. With $y(t) = \mathrm{Re}[y_l(t)e^{j2\pi f_0 t}]$ and $x_l(t) = V_x(t)e^{j\Theta_x(t)}$,

$$\begin{aligned}
y(t) &= \mathrm{Re}[y_l(t)e^{j2\pi f_0 t}] \\
&\approx \mathrm{Re}[A(f_0)x_l(t + \frac{1}{2\pi}\theta'(f_0))\,e^{j\theta(f_0)}\,e^{j2\pi f_0 t}] \\
&\approx \mathrm{Re}[A(f_0)V_x(t + \frac{1}{2\pi}\theta'(f_0))\,e^{j\theta(f_0)}\,e^{j2\pi f_0 t}\,e^{j\Theta_x(t+\frac{1}{2\pi}\theta'(f_0))}] \\
&= A(f_0)V_x(t-tg)\,\cos[2\pi f_0 t + \theta(f_0) + \Theta_x(t-t_g)] \\
&= A(f_0)V_x(t-tg)\,\cos[2\pi f_0(t + \frac{\theta(f_0)}{2\pi f_0}) + \Theta_x(t-t_g)] \\
&= A(f_0)V_x(t-tg)\,\cos[2\pi f_0(t-t_p) + \Theta_x(t-t_g)]
\end{aligned}$$

, where $\quad t_g = -\frac{\theta'(f_0)}{2\pi}$ and $t_p- = \frac{\theta(f_0)}{2\pi f_0}$.

(c) The quantities t_g and t_p are called envelope delay (or group delay) and phase delay, respectively. Can you interpret their role and justify this nomenclature?

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tg can be considered as the time lag of the envelope of the signal, whereas tp is the time corresponding to a phase delay of $\frac{\theta(f_0)}{2\pi f_0}$.

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Question 3

Intuitively, the bandwidth measures for the extent of significant spectral content of the real signal for positive frequencies. Different mathematical definitions might be used to calculate the bandwidth of the signal $|X(f)|$. Here are some of the common definitions:

1. **Absolute bandwidth is the smallest positive frequency band, where, for frequencies outside it, $|X(f)|$ is zero.**

2. **3-dB or half-power bandwidth is the positive frequency band, where, for frequencies outside it, $|X(f)|$ is never greater than $1/\sqrt{2}$ times its maximum value.**

3. **Null-to-null or zero-crossing bandwidth is the frequency band, where the band edge frequencies create the first spectrum nulls. For the lowpass signals, the right side edge frequency only creates the null.**

4. **Power bandwidth is the positive frequency band in which 49.5% of the total power (or energy) resides.**

5. **The Root Mean Square (RMS) bandwidth is defined as $\sqrt{\int_0^{+\infty} f^2 |X(f)|^2 df / \int_0^{+\infty} |X(f)|^2 df}$ for a lowpass signal and $2\sqrt{\int_0^{+\infty} (f-f_0)^2 |X(f)|^2 df / \int_0^{+\infty} |X(f)|^2 df}$ for a bandpass signal centered around f_0.**

Fig. 1 shows a typical illustration of these definitions.
Note: If you need to numerically evaluate an integral or a function while answering the ques-
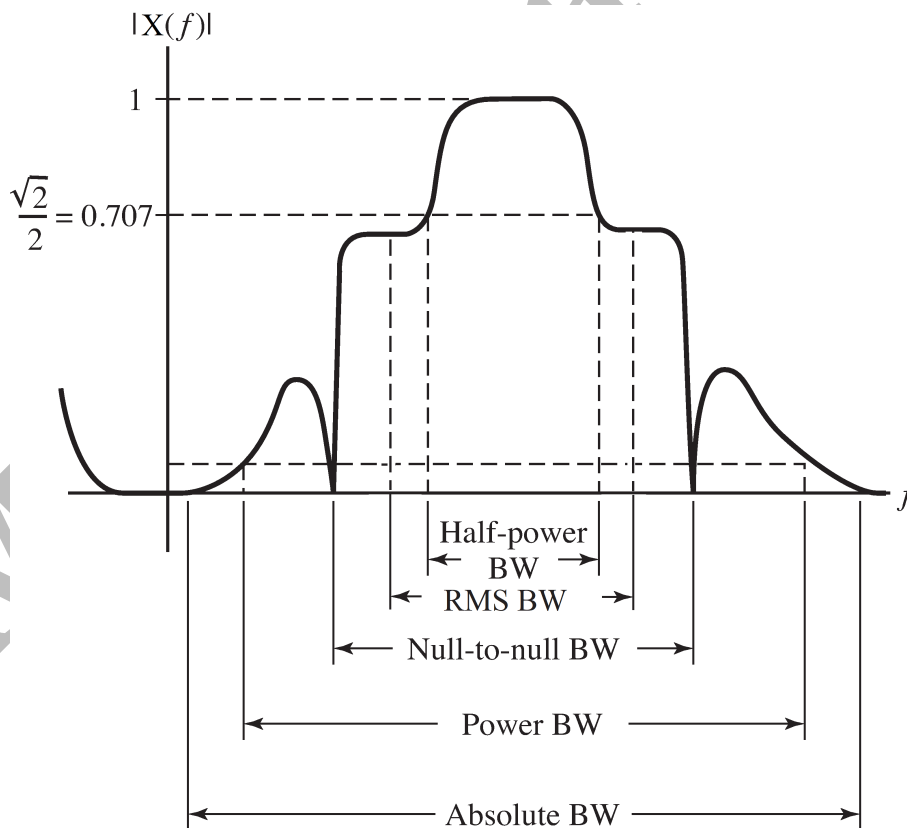

Figure 1: Various ways to define bandwidth for a real-valued signal.

tions below, feel free to consult online calculators such as <u>Symbolab.</u>

(a) Find the defined bandwidths for the signal $X(f) = A \sqcap (\frac{f}{B})$.

1. $X(f)$ is zero for the frequencies outside $[-\frac{B}{2}, +\frac{B}{2}]$. Therefore, $W_{abs} = \frac{B}{2}$.

2. $|X(f)| = A$ when $|f| < \frac{B}{2}$ while for the frequencies outside $[-\frac{B}{2}, +\frac{B}{2}]$, we have $|X(f)| = 0 < 1/\sqrt{2}\max\{|X(f)|\} = A$. So, $W_{3db} = \frac{B}{2}$.

3. The first spectrum null occurs at $f = \frac{B}{2}$. Thus, $W_{n2n} = \frac{B}{2}$.

4. The energy content of $X(f)$ is

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X(f)|^2 \, df = \int_{-\frac{B}{2}}^{+\frac{B}{2}} A^2 \, df = BA^2$$

Therefore, the power bandwidth W_{pow} is

$$\int_{0}^{W_{pow}} A^2 \, df = W_{pow} A^2 = \frac{49.5}{100} BA^2 \Rightarrow W_{pow} = 0.495B$$

5. The RMS bandwidth is calculated as

$$W_{rms} = \sqrt{\frac{\int_0^{+\infty} f^2 |X(f)|^2 df}{\int_0^{+\infty} |X(f)|^2 df}} = \sqrt{\frac{\int_0^{\frac{B}{2}} f^2 A^2 df}{\int_0^{\frac{B}{2}} A^2 df}} = \sqrt{\frac{\frac{B^3}{24} A^2}{\frac{B}{2} A^2}} = \frac{B}{\sqrt{12}} = 0.287B$$

(b) Find the defined bandwidths for the signal $X(f) = A\operatorname{sinc}(\frac{f}{B})$.

1. $X(f)$ has nonzero values even for $f \to \infty$. So, $B_{abs} = \infty$.

2. The 3-dB bandwidth of $X(f)$ is calculated as

$$|X(f)| = A|\operatorname{sinc}(\frac{W_{3db}}{B})| = \frac{\max\{|X(f)|\}}{\sqrt{2}} = \frac{A}{\sqrt{2}}$$

Let $u = \frac{W_{3db}}{B}$. We should find the positive root of $|\operatorname{sinc}(u)| = 1/\sqrt{2}$. Numerical solution of the equation gives $u = 0.443$, and therefore, $W_{3db} = 0.443B$.

3. The first positive spectrum null of $X(f) = A\operatorname{sinc}(\frac{f}{B})$ occurs at $f = B$ and consequently, $W_{n2n} = B$.

4. The energy content of $X(f)$ is

$$\int_{-\infty}^{+\infty} |X(f)|^2 \, df = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} A^2 B^2 \sqcap^2 (Bt) dt = \int_{-\frac{1}{2B}}^{+\frac{1}{2B}} A^2 B^2 dt = A^2 B$$

Therefore, the power bandwidth is obtained as

$$\int_0^{W_{pow}} A^2 \operatorname{sinc}^2(\frac{f}{B})\, df = \frac{49.5}{100} A^2 B \Rightarrow \int_0^{\frac{W_{pow}}{B}} BA^2 \operatorname{sinc}^2(v)\, dv = \frac{49.5}{100} A^2 B$$

Now, we should find the value of $u = \frac{W_{pow}}{B}$ such that $\int_0^u \operatorname{sinc}^2(v)dv = 0.495$. This equation can be numerically solved to obtain $u = 10.2$, or equivalently $W_{pow} = 10.2B$.

5. The RMS bandwidth is calculated as

$$W_{rms} = \sqrt{\frac{\int_0^{+\infty} f^2 |X(f)|^2 df}{\int_0^{+\infty} |X(f)|^2 df}} = \sqrt{\frac{\int_0^{+\infty} f^2 A^2 \operatorname{sinc}^2(\frac{f}{B})\, df}{\int_0^{+\infty} A^2 \operatorname{sinc}^2(\frac{f}{B})\, df}} = \sqrt{\frac{\int_0^{+\infty} \frac{B^2 A^2}{\pi^2} \sin^2(\frac{\pi f}{B})\, df}{A^2 B/2}}$$

$$= \sqrt{\frac{2B}{\pi^2} \int_0^{+\infty} \sin^2(\frac{\pi f}{B})\, df} = \sqrt{\frac{2B^2}{\pi^3} \int_0^{+\infty} \sin^2(u)du} = \infty$$

(c) Which definition is better and more applicable? Why?

The applicability of a definition depends on the type of the involved signal. For example, for $X(f) = A \sqcap(\frac{f}{B})$, the absolute bandwidth seems acceptable while the absolute bandwidth is not a suitable choice for $X(f) = A \operatorname{sinc}(\frac{f}{B})$. For real practical signals such as audio signals, 3-dB bandwidth is usually used.

Question 4

Let Y be a positive-valued random variable, i.e., $f_Y(y) = 0$ for $y < 0$. You might consult Google to prove the following useful inequalities.

(a) Let α be any positive constant, and prove the Markov inequality $P(Y > \alpha) \leq \frac{E(Y)}{\alpha}$.

$$E[Y] = \int_0^\infty y f_Y(y)dy \geq \int_\alpha^\infty y f_Y(y)dy \geq \alpha \int_\alpha^\infty f_Y(y)dy = \alpha P(Y \geq \alpha)$$

. Thus, $P(Y \geq \alpha) \leq E[Y]/\alpha$.

(b) Let X be any random variable with variance σ^2, and define $Y = (X - E[X])^2$ and $\alpha = \epsilon^2$ for some positive ϵ. Derive the Chebychev inequality $P(|X - E(X)| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$.

We have

$$P(|X - E[X]| > \epsilon) = P\left((X - E[X])^2 > \epsilon^2\right)$$

. Now, $Y = (X - E[X])^2$ and $\alpha = \epsilon^2$ satisfy the requirements of the Markov inequality. Therefore,

$$P(Y > \alpha) = P(|X - E[X]| > \epsilon) = P\left((X - E[X])^2 > \epsilon^2\right) \leq \frac{E\left[(X - E[X])^2\right]}{\epsilon^2} = \frac{\sigma^2}{\epsilon^2}$$

Question 5

The characteristic function provides an alternative way for describing a random variable, similar to the CDF, which completely determines the behavior and properties of a random variable. The characteristic function of the random variable X is defined as

$$\phi_X(t) = E\{e^{jtX}\}$$

. Take a look at the free encyclopedia of wikipedia to answer the following questions.

(a) How does the characteristic function relate to the Fourier transform of the PDF $f_X(x)$?

From the definition of the Fourier Transform, the Fourier Transform of the PDF $f_X(x)$ equals

$$X(t) = \int_{-\infty}^{\infty} f_X(x)e^{-j2\pi tx}\, dx$$

Also we have,

$$\phi_X(t) = E\{e^{jtX}\} = \int_{-\infty}^{\infty} f_X(x)e^{jtx}\, dx$$

Comparing $X(t)$ and $\phi_X(t)$ we conclude that

$$\phi_X(t) = X(\frac{-t}{2\pi})$$

(b) How can we find the mean and variance using the characteristic function?

Using the Taylor series expansion of the exponential function, we can rewrite the characteristic function as

$$\phi_X(t) = E\{e^{jtX}\} = E\{\sum_{k=0}^{\infty} \frac{(jtX)^k}{k!}\}$$

$$= \sum_{k=0}^{\infty} j^k \frac{E(X^k)}{k!}t^k = 1 + jE(X)t + j^2\frac{E(X^2)}{2!}t^2 + \cdots + j^k\frac{E(X^k)}{k!}t^k + \cdots$$

Taking the first derivative of the expression above at $t = 0$, we get

$$\frac{\partial \phi_X(t)}{\partial t}\bigg|_{t=0} = jE(X) \Rightarrow \mu = E(X) = \frac{1}{j}\frac{\partial \phi_X(t)}{\partial t}\bigg|_{t=0}$$

Similarly, the second derivative of the aforesaid expression gives

$$E(X^2) = \frac{1}{j^2}\frac{\partial^2 \phi_X(t)}{\partial t^2}\bigg|_{t=0}$$

So, we can calculate the variance as

$$\sigma^2 = E(X^2) - (E(X))^2 = \frac{1}{j^2}\frac{\partial^2 \phi_X(t)}{\partial t^2}\Big|_{t=0} - \left(\frac{1}{j}\frac{\partial \phi_X(t)}{\partial t}\Big|_{t=0}\right)^2$$

(c) Calculate the characteristic function of the normal distribution $\mathcal{N}(\mu, \sigma^2)$ and use it to derive the mean and variance of the normal distribution.

The PDF of the normal distribution is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

To derive the Fourier Transform of the PDF, we use the following trick. Let $g(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$. So,

$$g'(x) = -\frac{x-\mu}{\sigma}\frac{1}{\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = -\frac{x-\mu}{\sigma^2}g(x) = -\frac{1}{\sigma^2}xg(x) + \frac{\mu}{\sigma^2}g(x)$$

Assuming that $G(t)$ is the Fourier Transform of $g(x)$, we have

$$j2\pi t G(t) = -\frac{1}{\sigma^2}\frac{j}{2\pi}G'(t) + \frac{\mu}{\sigma^2}G(t) \Rightarrow \frac{dG(t)}{dt} = \frac{2\pi\sigma^2}{j}\left(\frac{\mu}{\sigma^2} - j2\pi t\right)G(t) = -(2j\pi\mu + 4\pi^2\sigma^2 t)G(t)$$

To solve the obtained first-order differential equation,

$$\frac{dG(t)}{G(t)} = -(2j\pi\mu + 4\pi^2\sigma^2 t)dt \Rightarrow \ln(G(t)) = -(2j\pi\mu t + 2\pi^2\sigma^2 t^2) + C_1$$

$$\Rightarrow G(t) = C_2 e^{-(2j\pi\mu t + 2\pi^2\sigma^2 t^2)}$$

Since $G(0) = \int_{-\infty}^{\infty}g(x)dx = \sigma\sqrt{2\pi}\int_{-\infty}^{\infty}f_X(x)dx = \sigma\sqrt{2\pi}$, we have $C_2 = \sigma\sqrt{2\pi}$ and consequently,

$$G(t) = \sigma\sqrt{2\pi}e^{-2\pi j\mu t - 2\pi^2\sigma^2 t^2}$$

. Now,

$$\phi_X(t) = X\left(\frac{-t}{2\pi}\right) = \frac{1}{\sigma\sqrt{2\pi}}G\left(\frac{-t}{2\pi}\right) = e^{-\frac{1}{2}\sigma^2 t^2 + j\mu t}$$

and therefore,

$$E(X) = \frac{1}{j}\frac{\partial \phi_X(t)}{\partial t}\Big|_{t=0} = \frac{1}{j}\left((-\sigma^2 t + j\mu)e^{-\frac{1}{2}\sigma^2 t^2 + j\mu t}\right)\Big|_{t=0} = \mu$$

$$E(X^2) - (E(X))^2 = \frac{1}{j^2}\frac{\partial^2 \phi_X(t)}{\partial t^2}\Big|_{t=0} - \mu^2 = \frac{1}{j^2}\left(-\sigma^2 - \mu^2\right) - \mu^2 = \sigma^2$$

SOFTWARE QUESTIONS

Question 6

MATLAB provides a useful function named *hilbert()*. Refer to the help page of the MATLAB software to learn more about *hilbert()* command and answer the questions below.

(a) How does the command of hibert() differ from the Hilbert transform introduced in the course lectures?

> x = hilbert(xr) returns a complex helical sequence, sometimes called the analytic signal, from a real data sequence. The analytic signal x = xr + j*xi has a real part, xr, which is the original data, and an imaginary part, xi, which contains the Hilbert transform. The imaginary part is a version of the original real sequence with a 90° phase shift.

(b) What is Fast Fourier Transform and how does MATLAB use it to calculate the hilbert() command?

> For the vectors X and Y of length n, Discrete Fourier Transform (DFT) and inverse DFT are defined as
>
> $$Y(k) = \sum_{j=1}^{n} X(j) W_n^{(j-1)(k-1)}$$
>
> $$X(j) = \frac{1}{n} \sum_{k=1}^{n} Y(k) W_n^{-(j-1)(k-1)}$$
>
> , where $W_n = e^{-j2\pi/n}$. When n is a power of 2, the DFT is referred to as Fast Fourier Transform (FFT) because the DFT can be computed efficiently when n is a power of 2. FFT has many applications in communication systems and mathematical computation. For instance, FFT is a main block of Orthogonal Frequency division Multiplexing (OFDM) modulation in 5G mobile systems.
> As another example, FFT (or DFT) can be used to approximate the Fourier transform of a signal. If so, we can obtain the spectrum $2X(f)u(f)$ for a given real signal $x(t)$ using the FFT (or DFT) and then take the inverse FFT (or inverse DFT) from $2X(f)u(f)$ to obtain the analytic signal $x_a(t) = x(t) + j\hat{x}(t)$, whose imaginary part is the Hilbert transform of $x(t)$.

(c) Code an mfile, which takes an arbitrary signal as its input and plots the time- and frequency-domain graphs of the Hilbert transform of its input signal. Feel free to use MATLAB internal functions such as hilbert().

> Here is a sample implementation.
> ```matlab
> 1  function hilbert_plot(t,x)
> 2
> 3  close all
> 4
> 5  % take the hilbert transform
> 6  x_hat = imag(hilbert(x));
> 7
> 8  % take the DFT and approximate the spectrum using the DFT
> 9  Fs = 1/abs(t(2)-t(1));
> 10 L = length(t);
> 11 f = Fs*((1:L)-L/2)/L;
> ```

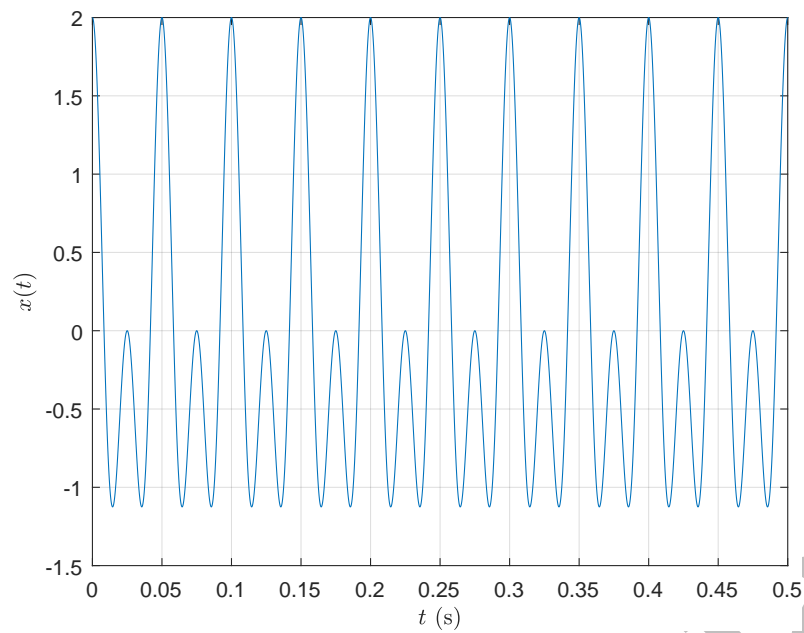*Question 6 continued on next page. . .*   Page 8 of 11
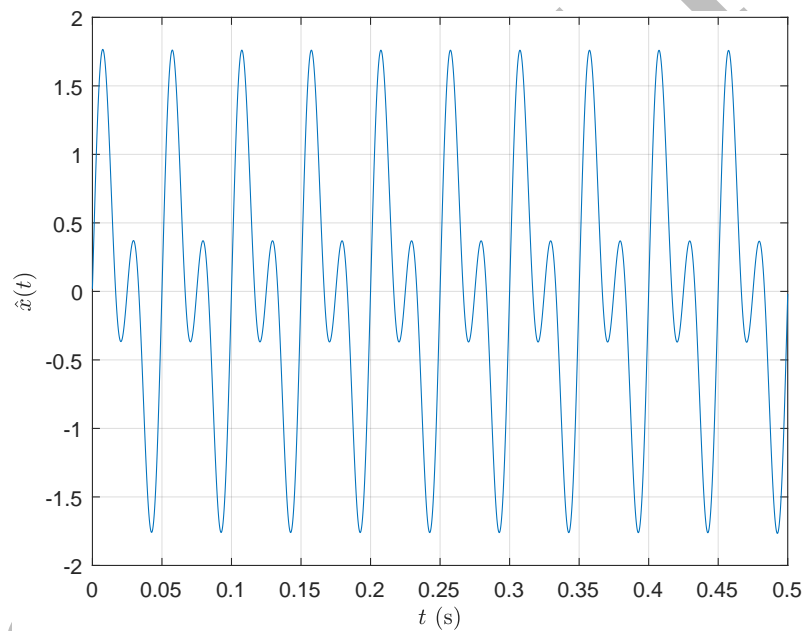
Figure 2: Sample time-domain plot for $x(t)$.

Figure 3: Sample time-domain plot for $\hat{x}(t)$.

```
12  X_hat = fftshift(fft(x_hat))/L;
13  X = fftshift(fft(x))/L;
14
15  % plot time-domin curves
16  figure
17  plot(t,x)
18  box on
19  grid on
20  ylabel('$|x(t)|$','Interpreter','latex')
21  xlabel('t (s)','Interpreter','latex')
```
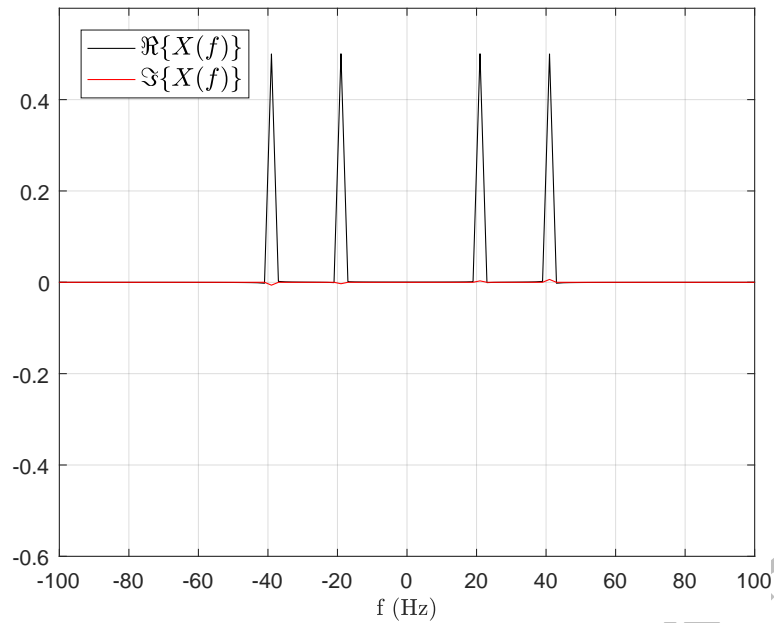
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Figure 4: Sample frequency-domain plot for $X(f)$.

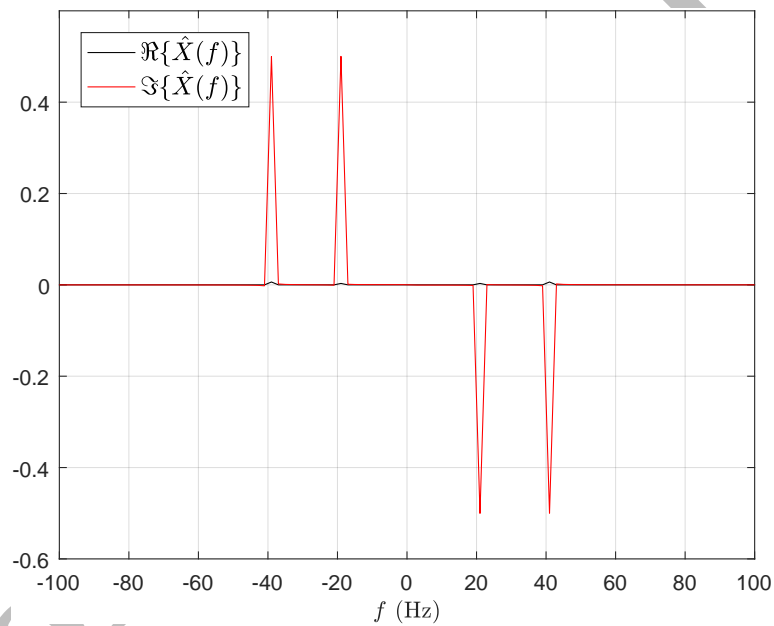Figure 5: Sample frequency-domain plot for $\hat{X}(f)$.

```
22  figure
23  plot(t,x_hat)
24  box on
25  grid on
26  ylabel('$|\hat{x}(t)|$','Interpreter','latex')
27  xlabel('t (s)','Interpreter','latex')
28
29  % plot frequency-domin curves
30  figure
31  hold on
32  plot(f,abs(X))
```

```
33  plot(f,imag(X),'r')
34  box on
35  grid on
36  ylabel('$|X(f)|$','Interpreter','latex')
37  xlabel('f (Hz)','Interpreter','latex')
38  figure
39  hold on
40  plot(f,real(X_hat))
41  plot(f,imag(X_hat),'r')
42  box on
43  grid on
44  ylabel('$|\hat{X}(f)|$','Interpreter','latex')
45  xlabel('f (Hz)','Interpreter','latex')
```

We have used the coded function to plot time-domain and frequency-domain curves of $x(t) = \cos(40\pi t) + \cos(80\pi t)$ in Figs. 2-5.

BONUS QUESTIONS

Question 7

Return your answers by filling the LATEXtemplate of the assignment.