MATHEMATICAL QUESTIONS

Question 1

Let $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ be a jointly Gaussian random vector. Show that a linear combination of $X_1, \cdots, X_n$ has a univariate normal distribution.
Hint: You may use the characteristic function of the jointly Gaussian random vector $\mathbf{X}$ to prove the statement.
Hint: You became familiar with the univariate characteristic function in the previous assignment. Look at Wikipedia to know how the characteristic function is defined for a random vector, especially, for a jointly Gaussian random vector.

The multivariate characteristic function of the jointly Gaussian random vector $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ is

$$\Phi_{\boldsymbol{X}}(\boldsymbol{t}) = E\{e^{j\boldsymbol{X}\boldsymbol{t}^T}\} = E\{e^{j\sum_{i=1}^{n} X_i t_i}\} = \exp(j\boldsymbol{t}\boldsymbol{\mu}^T - \frac{1}{2}\boldsymbol{t}\boldsymbol{\Sigma}\boldsymbol{t}^T)$$

where $\boldsymbol{\mu}$ $\boldsymbol{\Sigma}$ are the mean and covariance matrices, respectively. Now, define $Y = \sum_{i=1}^{n} a_i X_i = \boldsymbol{X}\boldsymbol{a}^T$ as a linear combination of $X_1, \cdots, X_n$. The univariate characteristic function of Y is

$$\Phi_Y(t) = E\{e^{jYt}\} = E\{e^{jt\sum_{i=1}^{n} a_i X_i}\} = \Phi_{\boldsymbol{X}}(t\boldsymbol{a}) = \exp\left(jt(\boldsymbol{a}\boldsymbol{\mu}^T) - \frac{1}{2}t^2(\boldsymbol{a}\boldsymbol{\Sigma}\boldsymbol{a}^T)\right)$$

, which is the characteristic function of the univariate normal distribution $Y \sim \mathcal{N}(\boldsymbol{a}\boldsymbol{\mu}^T, \boldsymbol{a}\boldsymbol{\Sigma}\boldsymbol{a}^T)$.

Question 2

The random variable Y is defined as

$$Y = \frac{1}{n}\sum_{i=1}^{n} X_i$$

where $X_i, i = 1, 2, \cdots, n$ are statistically independent and identically distributed random variables.

(a) Determine the characteristic function of Y?

We have

$$\phi_Y(t) = E\{e^{jtY}\} = E\{e^{jt(\frac{1}{n}\sum_{i=1}^{n} X_i)}\} = E\{\prod_{i=1}^{n} e^{j\frac{t}{n}X_i}\}$$

Question 2 continued on next page...

Since the random variables are i.i.d,

$$\phi_Y(t) = \prod_{i=1}^{n} E\{e^{j\frac{t}{n}X_i}\} = \big(E\{e^{\frac{jt}{n}X_1}\}\big)^n = \big(\phi_{X_1}(\frac{t}{n})\big)^n$$

(b) Determine the PDF of Y.

We know that $\phi_{X_1}(t) = X_1(-\frac{t}{2\pi})$, where $X_1(f)$ is the Fourier transform of the PDF $f_{X_1}(x_1)$. Therefore,

$$\phi_Y(t) = Y(-\frac{t}{2\pi}) = \big(\phi_{X_1}(\frac{t}{n})\big)^n = \big(X_1(-\frac{t}{2n\pi})\big)^n = \underbrace{X_1(-\frac{t}{2n\pi})X_1(-\frac{t}{2n\pi})\cdots X_1(-\frac{t}{2n\pi})}_{n \text{ times}}$$

, or equivalently,

$$Y(nt) = \big(X_1(t)\big)^n = \underbrace{X_1(t)X_1(t)\cdots X_1(t)}_{n \text{ times}}$$

, where $Y(f)$ is the Fourier transform of the PDF $f_Y(y)$. Taking the inverse Fourier transform,

$$\frac{1}{n}f_Y(\frac{y}{n}) = \underbrace{f_{X_1}(y) * f_{X_1}(y) * \cdots * f_{X_1}(y)}_{n \text{ times}}$$

or,

$$f_Y(y) = n\Big[\underbrace{f_{X_1}(y) * f_{X_1}(y) * \cdots * f_{X_1}(y)}_{n \text{ times}}\big|_{y\to ny}\Big]$$

.

(c) Assume that X_i's have Cauchy PDF given by

$$f_X(x) = \frac{a}{\pi(a^2 + x^2)}, \quad -\infty < x < \infty$$

. Does the central limit theorem hold as $n \to \infty$?
Hint: Find the characteristic function of Y for any given n and then, determine its distribution type.

Let find the Fourier transform of $f_X(x) = \frac{a}{\pi(a^2+x^2)}$. We have,

$$
\begin{aligned}
e^{-ax}u(x) &\leftrightarrow \frac{1}{a+j2\pi t} \\
e^{-ax}u(x) + e^{ax}u(-x) &\leftrightarrow \frac{1}{a+j2\pi t} + \frac{1}{a-j2\pi t} \\
e^{-a|x|}u(x) + e^{-a|x|}u(-x) &\leftrightarrow \frac{2a}{a^2+4\pi^2 t^2} \\
e^{-a|x|} &\leftrightarrow \frac{2}{1+4\pi^2 t^2} \\
\frac{2a}{a^2+4\pi^2 x^2} &\leftrightarrow e^{-a|-t|} \\
\frac{2a}{a^2+4\pi^2 (\frac{x}{2\sqrt{\pi}})^2} &\leftrightarrow 2\sqrt{\pi}e^{-2a\sqrt{\pi}|t|} \\
\frac{2a}{a^2+\pi x^2} &\leftrightarrow 2\sqrt{\pi}e^{-2a\sqrt{\pi}|t|} \\
\frac{2a\sqrt{\pi}}{\pi a^2+\pi x^2} &\leftrightarrow 2\sqrt{\pi}e^{-2a\pi|t|} \\
\frac{a}{\pi(a^2+x^2)} &\leftrightarrow e^{-2a\pi|t|} = X(t)
\end{aligned}
$$

As a result, the characteristic function of X is

$$\phi_X(t) = X(-\frac{t}{2\pi}) = e^{-a|t|}$$

. According to Part (a),

$$\phi_Y(t) = \left(\phi_X(\frac{t}{n})\right)^n = \left(e^{-a|\frac{t}{n}|}\right)^n = e^{-a|t|}$$

, which implies that Y has Cauchy PDF independent of n. Hence, the central limit theorem does not hold. The reason is that the Cauchy distribution does not have a finite variance, which is implicitly required by the central limit theorem. In fact,

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx = \int_{-\infty}^{+\infty} \frac{ax}{\pi(a^2+x^2)} dx = 0$$

due to evenness of $\frac{ax}{\pi(a^2+x^2)}$. Further,

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x) dx = \int_{-\infty}^{+\infty} \frac{ax^2}{\pi(a^2+x^2)} dx = \infty$$

, which yields Var$(X) = \infty$.

Question 3 continued on next page. . .

Question 3

Prove Schwarz's inequality $E^2[XY] \leq E[X^2]E[Y^2]$ for two random variables X and Y. Then, use it to show that for two jointly stationary processes $X(t)$ and $Y(t)$, we have

$$|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)} \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

Schwartz's inequality can be proven as follows. Assume that $W = (X - \alpha Y)^2$ is a function of two random variables X and Y, where α is a real number. We have,

$$0 \leq E[W] = E[X^2 + \alpha^2 Y^2 - 2\alpha XY] = E[X^2] - 2\alpha E[XY] + \alpha^2 E[Y^2]$$

. Now, we have a positive second-order polynomial in terms α. So,

$$\Delta = 4E^2[XY] - 4E[X^2]E[Y^2] \leq 0 \Rightarrow E^2[XY] \leq E[X^2]E[Y^2]$$

Employing Schwartz's inequality

$$R_{XY}^2(\tau) = E^2[X(t+\tau)Y(t)] \leq E[X^2(t+\tau)]E[Y^2(t)] = R_X(0)R_Y(0)$$

Using the above expression and arithmetic inequality $2ab \leq a^2 + b^2$ yields

$$|R_{XY}(\tau)| = \sqrt{E^2[X(t+\tau)Y(t)]} \leq \sqrt{R_X(0)R_Y(0)} \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

.

Question 4

Prove that the power spectral density of the quadrature component of the bandpass random process $X(t)$ equals

$$S_{X_s}(f) = \left[S_X(f+f_c) + S_x(f-f_c)\right] \sqcap (\frac{f}{2f_c})$$

We know that $X_s(t) = \hat{X}(t)\cos(2\pi f_c t) - X(t)\sin(2\pi f_c t)$. As a result,

$$\begin{aligned}
R_{X_s}(t+\tau, t) =& E\big\{X_s(t+\tau)X_s(t)\big\}\\
=& E\big\{[\hat{X}(t+\tau)\cos(2\pi f_c(t+\tau)) - X(t+\tau)\sin(2\pi f_c(t+\tau))]\\
&\times [\hat{X}(t)\cos(2\pi f_c t) - X(t)\sin(2\pi f_c t)]\big\}\\
=& R_{\hat{X}\hat{X}}(t+\tau, t)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t)\\
&- R_{\hat{X}X}(t+\tau, t)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t)\\
&- R_{X\hat{X}}(t+\tau, t)\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t)\\
&+ R_X(t+\tau, t)\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t)\\
=& R_X(\tau)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t)\\
&- \widehat{R_X}(\tau)(\tau)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t)\\
&+ \widehat{R_X}(\tau)\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t)\\
&+ R_X(\tau)\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t)\\
=& R_X(\tau)\big[\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) + \sin(2\pi f_c(t+\tau))\sin(2\pi f_c t)\big]\\
&+ \widehat{R_X}(\tau)\big[\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) - \cos(2\pi f_c(t+\tau))\sin(2\pi f_c t)\big]\\
=& R_X(\tau)\cos(2\pi f_c \tau) + \widehat{R_X}(\tau)\sin(2\pi f_c\tau) = R_{X_s}(\tau)
\end{aligned}$$

Taking the Fourier transform of $R_{X_s}(\tau)$,

$$\begin{aligned}
S_{X_s}(f) =& S_X(f) * \frac{1}{2}\big[\delta(f-f_c) + \delta(f+f_c)\big] - j\text{sgn}(f)S_X(f) * \frac{1}{2j}\big[\delta(f-f_c) - \delta(f+f_c)\big]\\
=& S_X(f-f_c)\frac{1-\text{sgn}(f-f_c)}{2} + S_X(f+f_c)\frac{1+\text{sgn}(f+f_c)}{2}\\
=& S_X(f-f_c)u(-f+f_c) + S_X(f+f_c)u(f+f_c)\\
=& S_X(f-f_c)\sqcap(\frac{f}{2f_c}) + S_X(f+f_c)\sqcap(\frac{f}{2f_c})\\
=& \big[S_X(f+f_c) + S_x(f-f_c)\big]\sqcap(\frac{f}{2f_c})
\end{aligned}$$

Question 5

In the block diagram shown in Fig. 1, $X(t)$ denotes a zero-mean Gaussian white WSS noise process with the power spectral density $S_X(f) = \frac{N_0}{2}$.

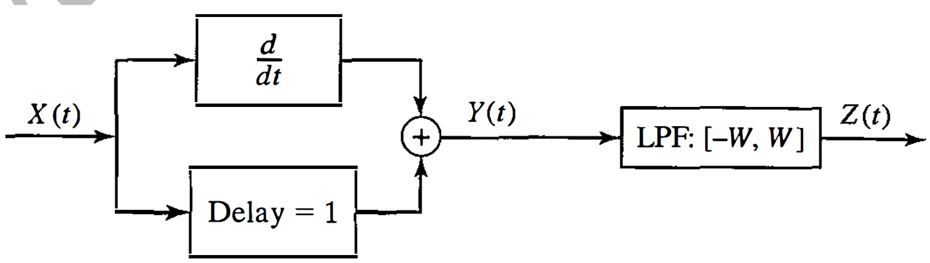

Figure 1: Block diagram of a system with random input.

Question 5 continued on next page. . .

(a) Is $Z(t)$ a WSS random process? Why?

Since the WSS noise passes the LTI system with the frequency response

$$H(f) = [j2\pi f + e^{-j2\pi f}] \sqcap (\frac{f}{2W}) = [\cos(2\pi f) + j(2\pi f - \sin(2\pi f))] \sqcap (\frac{f}{2W})$$

, the output process $Z(t)$ is WSS.

(b) What is the power spectral density and the mean of $Z(t)$?

$$m_Z = H(0)m_X = m_X = 0$$
$$S_Z(f) = S_X(f)|H(f)|^2 = \frac{N_0}{2}[(2\pi f - \sin(2\pi f))^2 + \cos^2(2\pi f)] \sqcap (\frac{f}{2W})$$
$$S_Z(f) = \frac{N_0}{2}[4\pi^2 f^2 - 4\pi f \sin(2\pi f) + 1] \sqcap (\frac{f}{2W})$$

(c) What is the power in $Z(t)$?

$$P_Z = \int_{-\infty}^{\infty} S_Z(f)df = \frac{N_0}{2}\int_{-W}^{W}[4\pi^2 f^2 - 4\pi f \sin(2\pi f) + 1]df$$
$$P_Z = \frac{N_0}{2}2\int_0^W [4\pi^2 f^2 - 4\pi f \sin(2\pi f) + 1]df = N_0\big[\frac{4\pi^2}{3}f^3 + 2f\cos(2\pi f) - \frac{1}{\pi}\sin(2\pi f) + f\big]_0^W$$
$$P_Z = N_0\big[\frac{4\pi^2}{3}W^3 + 2W\cos(2\pi W) - \frac{1}{\pi}\sin(2\pi W) + W\big]$$

(d) What is the variance of $Z(t)$?

$$V_Z = E\{Z^2(t)\} - m_Z^2 = E\{Z^2(t)\} = R_Z(0) = \int_{-\infty}^{\infty} S_Z(f)df = P_Z$$
$$V_Z = P_Z = N_0\big[\frac{4\pi^2}{3}W^3 + 2W\cos(2\pi W) - \frac{1}{\pi}\sin(2\pi W) + W\big]$$

(e) What is the pdf of $Z(t_0)$?

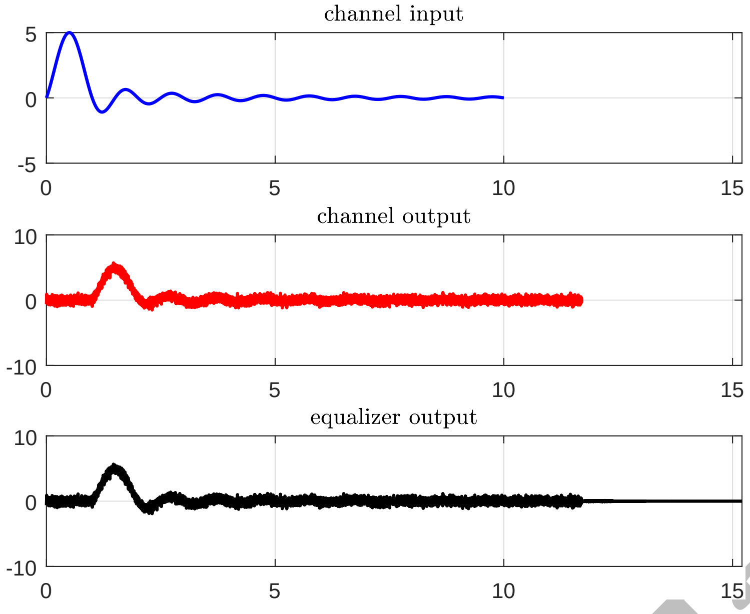

Figure 2: Simulation results for $A_2 = 0.1$ and $\text{SNR} = 10$ dB.

Since the Gaussian noise passes the LTI system with the frequency response $H(f) = [j2\pi f + e^{-j2\pi f}] \sqcap (\frac{f}{2W})$, the output process $Z(t)$ is Gaussian. So, $Z(t_0) \sim \mathcal{N}(m_Z, V_Z) = \mathcal{N}(0, N_0\left[\frac{4\pi^2}{3}W^3 + 2W\cos(2\pi W) - \frac{1}{\pi}\sin(2\pi W) + W\right])$ and

$$f_{Z(t_0)}(z) = f_Z(z) = \frac{\exp\left(-\frac{z^2}{2N_0\left[\frac{4\pi^2}{3}W^3 + 2W\cos(2\pi W) - \frac{1}{\pi}\sin(2\pi W) + W\right]}\right)}{\sqrt{2\pi N_0\left[\frac{4\pi^2}{3}W^3 + 2W\cos(2\pi W) - \frac{1}{\pi}\sin(2\pi W) + W\right]}}$$

.

SOFTWARE QUESTIONS

Question 6

MATLAB provides a function named awgn(), which can add white Gaussian noise to a given signal.

(a) Extend the function that you developed for modeling the point to point microwave radio channel such that the output signal of the channel is polluted by additive while Gaussian noise.

Here is a sample time-domain implementation of the channel.

```
function [s_out, t_out] = p2pmrc_chn(s_in, t_in, A1, D1, A2, D2, snr)
% time step
```


Question 6 continued on next page...

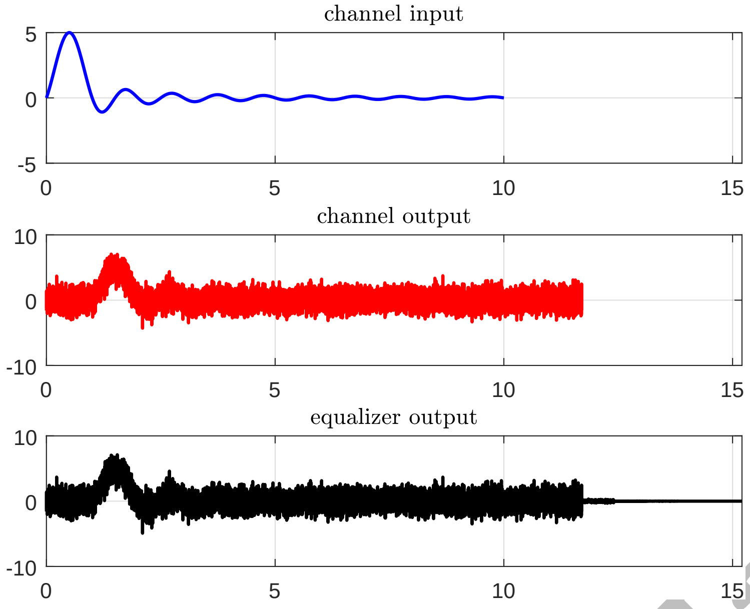

Figure 3: Simulation results for $A_2 = 0.1$ and SNR $= 0$ dB.

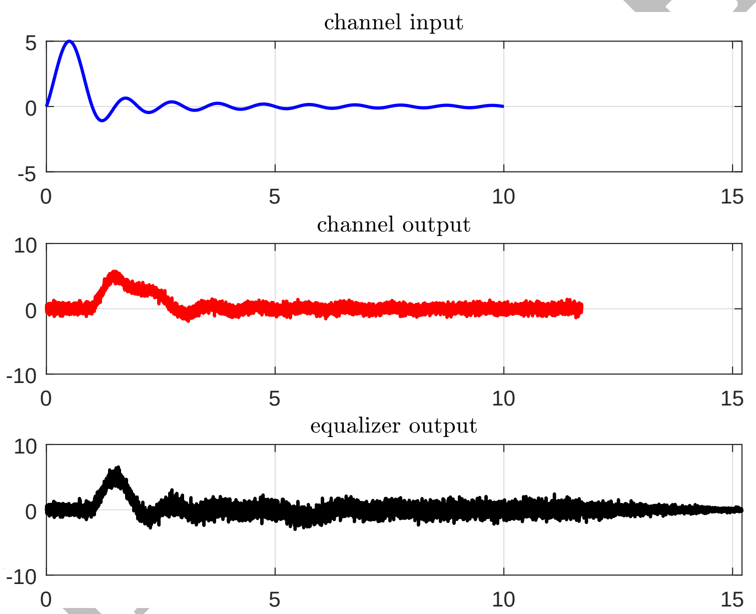

Figure 4: Simulation results for $A_2 = 0.8$ and SNR $= 10$ dB.

```
Dt = t_in(2)-t_in(1);
% shifted time axis
t_out = t_in(1):Dt:t_in(end)+(ceil(max([D1 D2])/Dt)+1)*Dt;
% line of sight signal
s_los = zeros(size(t_out));
s_los(ceil(D1/Dt)+1:ceil(D1/Dt)+length(s_in)) = A1*s_in;
% reflect signal
s_ref = zeros(size(t_out));
s_ref(ceil(D2/Dt)+1:ceil(D2/Dt)+length(s_in)) = A2*s_in;
% merged signal
s_tot = s_los+s_ref;
% noisy signal
```

Question 6 continued on next page. . .

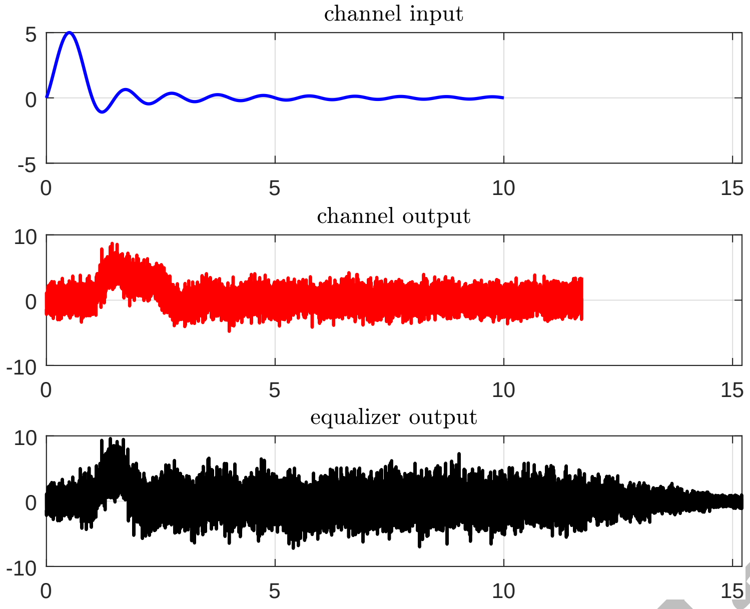

Figure 5: Simulation results for $A_2 = 0.8$ and SNR $= 0$ dB.

```
s_out = awgn(s_tot,snr,'measured');
end
```

(b) Observe the output of the channel for different levels of the distortion and noise, and discuss the results.

To validate the performance, the mfile below can be used.

```
clear all
close all

% parameters
A1=1;
D1=1;
D2=1.7;
A2=0.8;
N=5;
snr = 10;

% channel input
t_in=0:0.001:10;
s_in=5*sinc(2*(t_in-0.5));
% channel output
[chn_s, chn_t]=p2pmrc_chn_noise(s_in, t_in, A1, D1, A2, D2, snr);
% equalizer output
[eql_s, eql_t] = p2pmrc_eql(chn_s, chn_t, A1, D1, A2, D2, N);

% plot
subplot(3,1,1);
plot(t_in,s_in, 'b', 'LineWidth', 1.5)
title('channel input','Interpreter','latex');
xlim([min(eql_t) max(eql_t)])
box on
grid on

subplot(3,1,2);
plot(chn_t,chn_s, 'r', 'LineWidth', 1.5)
```

Question 6 continued on next page. . .

```
title('channel output','Interpreter','latex')
xlim([min(eql_t) max(eql_t)])
box on
grid on

subplot(3,1,3);
plot(eql_t,eql_s, 'black', 'LineWidth', 1.5)
title('equalizer output','Interpreter','latex')
xlim([min(eql_t) max(eql_t)])
box on
grid on
```

Let $A_1 = 1$, $D_1 = 1$, $D_2 = 1.7$, and $n = 5$. Figs. 2-5 show the involved signals for different values of distortion level A_2 and noise strength SNR. Clearly, as the noise and distortion levels increase, the resemblance of the received or equalized signal to the original signal degrades.

BONUS QUESTIONS

Question 7

Let $X(t)$ be a stationary real normal process with zero mean. Determine the autocorrelation function of the random process $Y(t) = X^2(t)$ in terms of the autocorrelation function of $X(t)$.

Let $\boldsymbol{X} = (X_1, X_2, X_3, X_4)$ be zero-mean jointly Gaussian random variables with covariance $\boldsymbol{\Sigma} = [c_{ij}], c_{ij} = E(X_i X_j)$ and characteristic function $\Phi_{\boldsymbol{X}}(\boldsymbol{t})$. We know that

$$\Phi_{\boldsymbol{X}}(\boldsymbol{t}) = \phi_{\boldsymbol{X}}(t_1, t_2, t_3, t_4) = E\{e^{j\boldsymbol{X}\boldsymbol{t}^T}\} = E\{e^{j(X_1t_1+X_2t_2+X_3t_3+X_4t_4)}\}$$

, which simplifies to

$$\Phi_{\boldsymbol{X}}(\boldsymbol{t}) = \exp(j\boldsymbol{t}\boldsymbol{\mu}^T - \frac{1}{2}\boldsymbol{t}\boldsymbol{\Sigma}\boldsymbol{t}^T) = \exp(-\frac{1}{2}\boldsymbol{t}\boldsymbol{\Sigma}\boldsymbol{t}^T)$$

by the zero-mean assumption. We have

$$E\{X_1X_2X_3X_4\} = \frac{\partial^4 \phi_{\boldsymbol{X}}(t_1, t_2, t_3, t_4)}{\partial t_1 \partial t_2 \partial t_3 \partial t_4}|_{t_1=t_2=t_3=t_4=0}$$

. After some algebraic simplifications and noting that $c_{ij} = c_{ji}$

$$E\{X_1X_2X_3X_4\} = c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23}$$

Clearly, every sample vector of a normal process has jointly Gaussian distribution, especially the vector $(X(t+\tau), X(t+\tau), X(t), X(t))$. Therefore,

$$R_Y(\tau) = E\{Y(t+\tau)Y(t)\} = E\{X^2(t+\tau)X^2(t)\} = E\{X(t+\tau)X(t+\tau)X(t)X(t)\}$$

. Using the proven property,

$$\begin{aligned} R_Y(\tau) =& E\{X(t+\tau)X(t+\tau)\}E\{X(t)X(t)\} \\ &+ E\{X(t+\tau)X(t)\}E\{X(t+\tau)X(t)\} \\ &+ E\{X(t+\tau)X(t)\}E\{X(t+\tau)X(t)\} \\ =& R_X^2(0) + 2R_X^2(\tau) \end{aligned}$$

Question 8

Return your answers by filling the LaTeXtemplate of the assignment.

EXTRA QUESTIONS

Question 9

Feel free to solve the following questions from the book *Fundamentals of Communication Systems* by J. Proakis and M. Salehi.

1. **Chapter 5, question 38.**
2. **Chapter 5, question 40.**
3. **Chapter 5, question 41.**
4. **Chapter 5, question 45.**
5. **Chapter 5, question 57.**
6. **Chapter 5, question 59.**
7. **Chapter 5, question 61.**