

## MATHEMATICAL QUESTIONS

### Question 1

Calculate the power content of the SSB signal  $u(t) = A_c m(t) \cos(2\pi f_c t) \mp A_c \hat{m}(t) \sin(2\pi f_c t)$ .

$$\begin{aligned}
 P_u &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u(t)|^2 dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} [A_c m(t) \cos(2\pi f_c t) \mp A_c \hat{m}(t) \sin(2\pi f_c t)]^2 dt \\
 &= \lim_{T \rightarrow \infty} \frac{A_c^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} [m^2(t) \cos^2(2\pi f_c t) + \hat{m}^2(t) \sin^2(2\pi f_c t) \mp 2m(t)\hat{m}(t) \cos(2\pi f_c t) \sin(2\pi f_c t)] dt \\
 &= \lim_{T \rightarrow \infty} \frac{A_c^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} [m(t)^2 \frac{1 + \cos(4\pi f_c t)}{2} + \hat{m}(t)^2 \frac{1 - \cos(4\pi f_c t)}{2} \mp m(t)\hat{m}(t) \sin(4\pi f_c t)] dt \\
 &= \lim_{T \rightarrow \infty} \frac{A_c^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2} m(t)^2 dt + \lim_{T \rightarrow \infty} \frac{A_c^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2} \hat{m}(t)^2 dt \\
 &\quad + \lim_{T \rightarrow \infty} \frac{A_c^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} [\frac{1}{2} m(t)^2 \cos(4\pi f_c t) - \frac{1}{2} \hat{m}(t)^2 \cos(4\pi f_c t) \mp m(t)\hat{m}(t) \sin(4\pi f_c t)] dt \quad (1)
 \end{aligned}$$

Clearly,

$$\lim_{T \rightarrow \infty} \frac{A_c^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2} m(t)^2 dt = \frac{A_c^2}{2} P_m \quad (2)$$

Further,

$$\lim_{T \rightarrow \infty} \frac{A_c^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2} \hat{m}(t)^2 dt = \frac{A_c^2}{2} P_m \quad (3)$$

, where we have used the fact that the power of a signal and its Hilbert transform is equal. Now, we show that the remaining three terms in (1) are equal to zero. We have

$$\lim_{T \rightarrow \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} m(t)^2 \cos(4\pi f_c t) dt = \int_{-\infty}^{\infty} m(t) \cdot [m(t) \cos(4\pi f_c t)]^* dt \quad (4)$$

Let  $M(f) = \mathcal{F}\{m(t)\}$ . According to Parseval's theorem,

$$\int_{-\infty}^{\infty} m(t) \cdot [m(t) \cos(4\pi f_c t)]^* dt = \int_{-\infty}^{\infty} M(f) \times \frac{1}{2} [M(f - 2f_c) + M(f + 2f_c)]^* df \quad (5)$$

The message signal  $m(t)$  has a bandwidth of  $W \ll f_c$ . So, there is not overlap between  $M(f)$  and its shifted versions  $M(f \pm 2f_c)$  and therefore,  $M(f) \times M^*(f \pm 2f_c) = 0$  and (5) equals zero. As a result,

$$\lim_{T \rightarrow \infty} \frac{A_c^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2} m(t)^2 \cos(4\pi f_c t) dt = 0 \quad (6)$$

Using a similar approach, we can show that

$$\lim_{T \rightarrow \infty} \frac{A_c^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2} \hat{m}(t)^2 \cos(4\pi f_c t) dt = 0 \quad (7)$$

$$\lim_{T \rightarrow \infty} \frac{A_c^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2} m(t) \hat{m}(t) \sin(4\pi f_c t) dt = 0 \quad (8)$$

Noting (2), (3), 6, 7, and 8, we conclude that

$$P_u = \frac{A_c^2}{2} P_m + \frac{A_c^2}{2} P_m + 0 + 0 + 0 = A_c^2 P_m \quad (9)$$

## Question 2

Show that if the SSB signal

$$u(t) = A_c m(t) \cos(2\pi f_c t) \mp A_c \hat{m}(t) \sin(2\pi f_c t)$$

passes an AWGN channel, as shown in Fig. 1, the SNR at the output of the demodulator is

$$\left(\frac{S}{N}\right)_o = \frac{A_c^2 P_m}{N_0 W} = \frac{P_R}{N_0 W}$$

, where  $N_0/2$  and  $W$  are noise power spectral density and message bandwidth, respectively, while  $P_M$  and  $P_R$  denote message and received power, respectively.

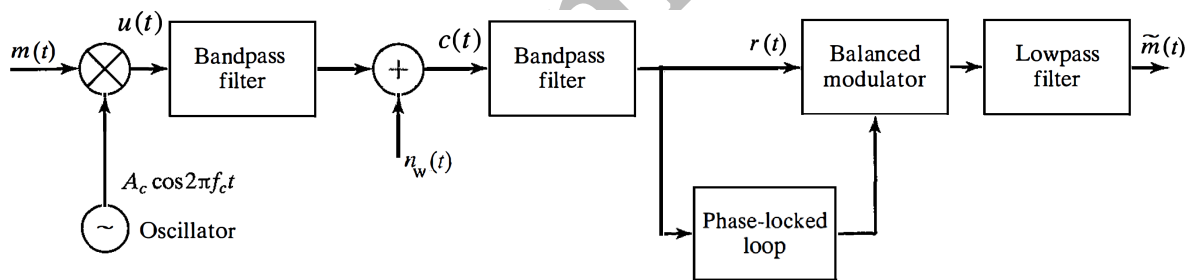


Figure 1: SNR analysis of SSB modulation in an AWGN channel.

The noisy SSB signal at the output of the channel is

$$c(t) = u(t) + n_w(t) = A_c m(t) \cos(2\pi f_c t) \mp A_c \hat{m}(t) \sin(2\pi f_c t) + n_w(t)$$

After the input BPF of the receiver,

$$r(t) = \text{BPF}\{c(t)\} = A_c m(t) \cos(2\pi f_c t) \mp A_c \hat{m}(t) \sin(2\pi f_c t) + n(t)$$

In terms of the in-phase and quadrature noise components,

$$r(t) = A_c m(t) \cos(2\pi f_c t) \mp A_c \hat{m}(t) \sin(2\pi f_c t) + n_c(t) \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t)$$

Now, we have

$$\begin{aligned}
 r(t) \cos(2\pi f_c t) &= A_c m(t) \cos^2(2\pi f_c t) \mp A_c \hat{m}(t) \sin(2\pi f_c t) \cos(2\pi f_c t) \\
 &\quad + n_c(t) \cos^2(2\pi f_c t) - n_s(t) \sin(2\pi f_c t) \cos(2\pi f_c t) \\
 &= \frac{1}{2} A_c m(t) + \frac{1}{2} n_c(t) + \frac{1}{2} A_c m(t) \cos(4\pi f_c t) \\
 &\quad + \frac{1}{2} n_c(t) \cos(4\pi f_c t) \mp \frac{1}{2} A_c \hat{m}(t) \sin(4\pi f_c t) - \frac{1}{2} n_s(t) \sin(4\pi f_c t) \\
 &= \frac{1}{2} A_c m(t) + \frac{1}{2} n_c(t) + \text{Double Frequency Terms}
 \end{aligned}$$

Applying the lowpass filtering,

$$\text{LPF}\{r(t) \cos(2\pi f_c t)\} = \frac{1}{2} A_c m(t) + \frac{1}{2} n_c(t)$$

The power of signal part is

$$P_o = \frac{A_c^2 P_m}{4}$$

while the power of noise part equals

$$P_{n_o} = \frac{P_{n_c}}{4} = \frac{1}{4} \frac{N_0}{2} W \times 2 = \frac{N_0 W}{4}$$

Note that the bandwidth of the BFP equals to  $W$ . Now, the SNR is

$$\left(\frac{S}{N}\right)_o = \frac{P_o}{P_{n_o}} = \frac{A_c^2 P_m}{N_0 W} = \frac{P_R}{N_0 W}$$

, where the received power

$$P_R = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} [A_c m(t) \cos(2\pi f_c t) \mp A_c \hat{m}(t) \sin(2\pi f_c t)]^2 dt = A_c^2 P_m$$

, as shown in question 1.

### Question 3

The DSB signal  $u(t) = A_c m(t) \cos(2\pi f_c t)$  passes the block diagram of Fig. 2, where  $N_0/2$  and  $W$  are noise power spectral density and message bandwidth, respectively. The frequency response of the bandpass filter at the input of the demodulator is

$$H(f) = \begin{cases} \sqrt{B^2 - (f - f_c)^2}, & |f - f_c| < W \\ \sqrt{B^2 - (f + f_c)^2}, & |f + f_c| < W, \quad B > W \\ 0, & \text{otherwise} \end{cases}$$

while the bandpass filter at the output of the modulator is described by

$$\begin{cases} \frac{1}{H(f)}, & H(f) \neq 0 \\ 0, & H(f) = 0 \end{cases}$$

**. Calculate the SNR at the output of the demodulator.**

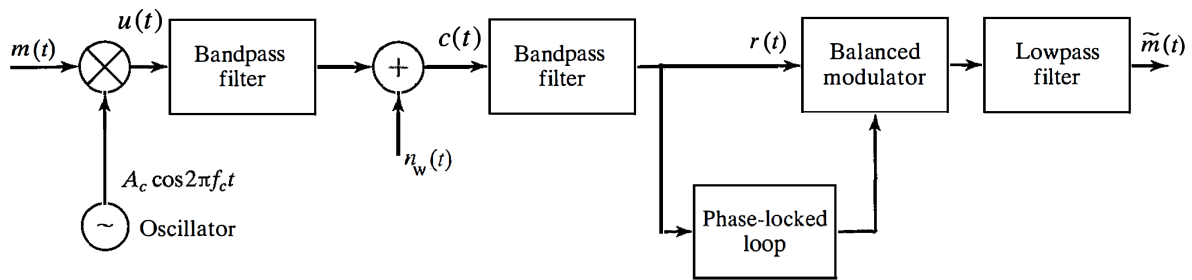


Figure 2: SNR analysis of DSB modulation in a linear filtered AWGN channel.

The calculations is almost similar to ordinary DSB system. The noisy signal at the output of the channel is

$$c(t) = u(t) * h_I(t) + n_w(t) = A_c m(t) \cos(2\pi f_c t) * h_I(t) + n_w(t)$$

where  $h_I(t)$  is the impulse response of the BPF at the output of the modulator. After the input BPF of the receiver,

$$r(t) = \text{BPF}\{c(t)\} = A_c m(t) \cos(2\pi f_c t) * h_I(t) * h(t) + n(t)$$

In terms of the in-phase and quadrature noise components,

$$r(t) = A_c m(t) \cos(2\pi f_c t) * h_I(t) * h(t) + n_c(t) \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t)$$

Clearly,  $H_I(f) = 1/H(f)$  in the bandwidth of the modulated signal. So,  $H_I(t)H(t)$ , or equivalently  $h_I(t) * h(t)$  acts like an ideal bandpass filter for  $u(t)$ . So,

$$r(t) = A_c m(t) \cos(2\pi f_c t) + n_c(t) \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t)$$

After the mixing at the demodulator,

$$\begin{aligned} r(t) \cos(2\pi f_c t) &= A_c m(t) \cos^2(2\pi f_c t) + n_c(t) \cos^2(2\pi f_c t) - n_s(t) \sin(2\pi f_c t) \cos(2\pi f_c t) \\ &= \frac{1}{2} A_c m(t) + \frac{1}{2} n_c(t) + \frac{1}{2} A_c m(t) \cos(4\pi f_c t) \\ &\quad + \frac{1}{2} n_c(t) \cos(4\pi f_c t) - \frac{1}{2} n_s(t) \sin(4\pi f_c t) \\ &= \frac{1}{2} A_c m(t) + \frac{1}{2} n_c(t) + \text{Double Frequency Terms} \end{aligned}$$

Applying the lowpass filtering,

$$\text{LPF}\{r(t) \cos(2\pi f_c t)\} = \frac{1}{2} A_c m(t) + \frac{1}{2} n_c(t)$$

The power of signal part is

$$P_o = \frac{A_c^2 P_m}{4}$$

while the power of noise part equals

$$\begin{aligned}
 P_{n_o} &= \frac{P_{n_c}}{4} = \frac{P_n}{4} = \frac{N_0}{8} \int_{-\infty}^{\infty} |H(f)|^2 df \\
 &= \frac{2N_0}{8} \int_{f_c-W}^{f_c+W} [B^2 - (f - f_c)^2] df \\
 &= \frac{N_0}{4} \int_{-W}^W (B^2 - u^2) du \\
 &= \frac{2N_0}{4} \int_0^W (B^2 - u^2) du \\
 &= \frac{N_0}{2} [B^2u - \frac{u^3}{3}]_0^W \\
 &= \frac{N_0}{2} (B^2W - \frac{W^3}{3})
 \end{aligned}$$

Now, the SNR is

$$\left(\frac{S}{N}\right)_o = \frac{P_o}{P_{n_o}} = \frac{A_c^2 P_m}{2N_0(B^2W - \frac{W^3}{3})} = \frac{P_R}{N_0(B^2W - \frac{W^3}{3})}$$

, where the received power

$$P_R = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} [A_c m(t) \cos^2(2\pi f_c t)]^2 dt = \frac{A_c^2 P_m}{2}$$

The maximum SNR is achieved for  $B = W$  as

$$\left(\frac{S}{N}\right)_o = \frac{3P_R}{2N_0W^3}$$

## Question 4

Find expressions for the in-phase and quadrature components,  $x_c(t)$  and  $x_s(t)$ , as well as the envelope and phase,  $V(t)$  and  $\Theta(t)$  for DSB and SSB signals.

1. A DSB modulated signal is expressed as  $u(t) = Am(t) \cos(2\pi f_0 t)$ . Hence,

$$\begin{aligned}
 x_c(t) &= Am(t) \\
 x_s(t) &= 0 \\
 V(t) &= A|m(t)| \\
 \Theta(t) &= 0
 \end{aligned}$$

2. An SSB signal is written as  $u_{SSB}(t) = Am(t) \cos(2\pi f_0 t) \mp A\hat{m}(t) \sin(2\pi f_0 t)$ . For the

USSB signal (minus sign)

$$x_c(t) = Am(t)$$

$$x_s(t) = A\hat{m}(t)$$

$$V(t) = \sqrt{A^2(m^2(t) + \hat{m}^2(t))} = A\sqrt{m^2(t) + \hat{m}^2(t)}$$

$$\Theta(t) = \arctan\left(\frac{\hat{m}(t)}{m(t)}\right)$$

. For the LSSB signal (plus sign)

$$x_c(t) = Am(t)$$

$$x_s(t) = -A\hat{m}(t)$$

$$V(t) = \sqrt{A^2(m^2(t) + \hat{m}^2(t))} = A\sqrt{m^2(t) + \hat{m}^2(t)}$$

$$\Theta(t) = -\arctan\left(\frac{\hat{m}(t)}{m(t)}\right)$$

### Question 5

The message signal  $m(t)$  is applied to the system shown in Fig. 3 to generate the signal  $y(t)$ .

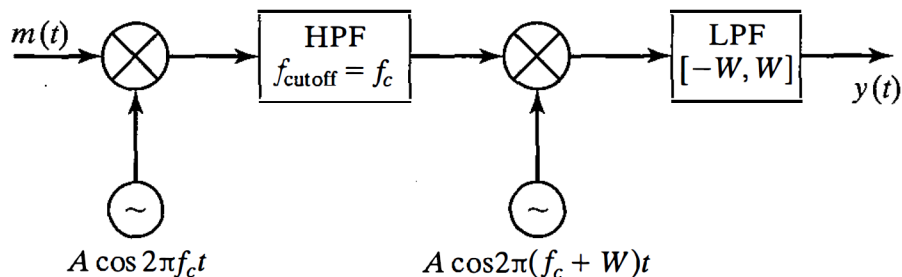


Figure 3: A sample scrambler.

(a) Find the spectrum of  $y(t)$ , i.e.  $Y(f)$ .

The spectrum of the modulated signal  $Am(t) \cos(2\pi f_c t)$  is

$$V(f) = \frac{A}{2}[M(f - f_c) + M(f + f_c)]$$

. The spectrum of the signal at the output of the highpass filter is

$$U(f) = \frac{A}{2}[M(f + f_c)u(-f - f_c) + M(f - f_c)u(f - f_c)]$$

. Multiplying the output of the HPF with  $A \cos(2\pi(f_c + W)t)$  results in the signal  $z(t)$  with spectrum

$$\begin{aligned}
 Z(f) &= \\
 &= \frac{A}{2} [M(f + f_c)u(-f - f_c) + M(f - f_c)u(f - f_c)] * \frac{A}{2} [\delta(f - (f_c + W)) + \delta(f + f_c + W)] = \\
 &= \frac{A^2}{4} [M(f + f_c - f_c - W)u(-f + f_c + W - f_c) + M(f + f_c - f_c + W)u(f + f_c + W - f_c) \\
 &+ M(f - 2f_c - W)u(f - 2f_c - W) + M(f + 2f_c + W)u(-f - 2f_c - W)] = \\
 &= \frac{A^2}{4} [M(f - W)u(-f + W) + M(f + W)u(f + W) + M(f - 2f_c - W)u(f - 2f_c - W) \\
 &+ M(f + 2f_c + W)u(-f - 2f_c - W)]
 \end{aligned}$$

The lowpass filter will cut the double frequency components, leaving the spectrum:

$$Y(f) = \frac{A^2}{4} [M(f - W)u(-f + W) + M(f + W)u(f + W)]$$

(b) Show that if  $y(t)$  is transmitted, the receiver can pass it through a replica of the system shown in Fig. 3 to obtain  $m(t)$  back.

As it is observed from the spectrum  $Y(f)$ , the system shifts the negative frequency components to the positive frequency axis and the positive frequency components to the negative frequency axis. If we transmit the signal  $y(t)$  through the system, we will get a scaled version of the original spectrum  $M(f)$ .

(c) How can the system be used as a simple scrambler to enhance communication privacy?

Clearly, the system reorders the frequency components of the message spectrum such that a positive component is located in the negative axis and vice versa. Now, if an intruder wants to eavesdrop the message, the reordering mechanism needs to be known. Otherwise, the intruder cannot extract the message.

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## SOFTWARE QUESTIONS

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### Question 6

**Validate the noise immunity of the DSB modulation using simulation results.**

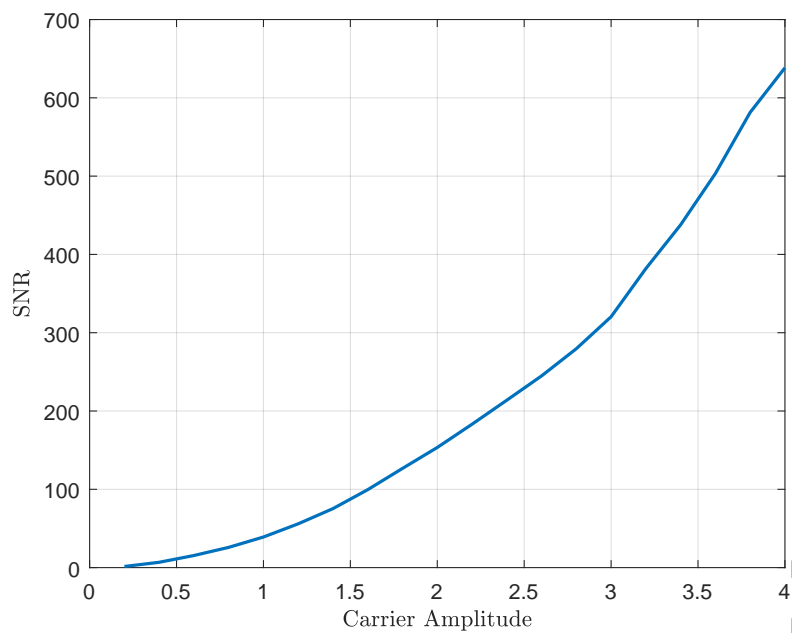


Figure 4: SNR versus carrier amplitude.

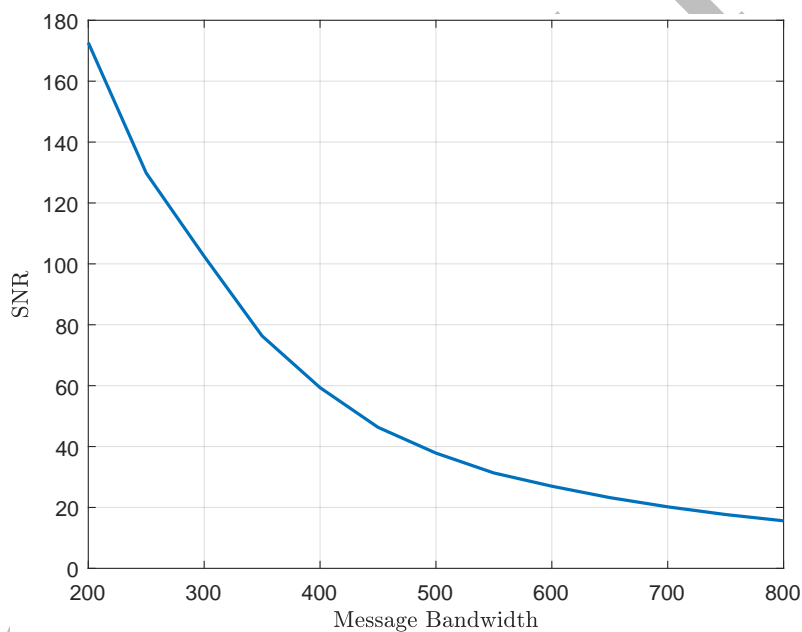


Figure 5: SNR versus message bandwidth.

(a) Develop three MATLAB functions simulating a DSB modulator, an AWGN channel, and a DSB demodulator. The input arguments of the functions may include message, modulation, and noise parameters such as noise spectral density, carrier frequency, and so on.

Here is a sample implementation for the DSB modulator.

```
1 % DSB modulator
2 function [mod, t] = dsb_mod(msg, t, W, fc, Ac)
3
4 mixed = Ac*msg.*cos(2*pi*fc*t);
```



```

5 B = 2*W;
6 bpf = abs(t(2)-t(1))*2*B*sinc(B*t).*cos(2*pi*fc*t);
7 mod = conv(mixed, bpf);
8 mod = mod(1:numel(t));
9
10 end
    
```

A sample implementation for the channel is as follows.

```

1 % ACGN channel
2
3 function [chn, t] = acgn_ch(mod, t, L, W, N0)
4
5 mod = L*mod;
6 B = 2*W;
7 n_pow = 2*N0/2*B;
8 s_pow = mean(mod.^2);
9 snr_ch = 10*log10(s_pow/n_pow);
10 chn = awgn(mod, snr_ch, 'measured');
11
12 end
    
```

Finally, the DSB demodulator can be implemented as

```

1 % DSB demodulator
2 function [demod, t] = dsb_demod(chn, t, W, fc)
3
4 mixed = chn.*cos(2*pi*fc*t);
5 BL=1*W;
6 lpf = abs(t(2)-t(1))*BL*sinc(BL*t);
7 demod = conv(mixed, lpf);
8 demod = demod(1:numel(t));
9
10 end
    
```

(b) Use your developed functions to plot the SNR of the demodulator versus carrier amplitude and message bandwidth. How do you obtain the SNR?

We can use the following mfile to calculate the SNR for a given set of parameters channel attenuation  $L$ , carrier amplitude  $A_c$ , carrier frequency  $f_c$ , message bandwidth  $W$ , and noise power spectral density  $N_0$ .

```

1 % parameters
2 L = 0.1;
3 Ac = 1;
4 fc = 400;
5 W = 10;
6 T = 300;
7 K = 1.38e-23;
8 N0 = K*T;
9
10 % signal
11 t = 0:0.0005:20;
12 msg = cos(2*pi*W*t);
13
14 % demodulated signal power
15 mod = dsb_mod(msg, t, W, fc, Ac);
16 chn = acgn_ch(mod, t, L, W, 0);
17 demod = dsb_demod(chn, t, W, fc);
18 s_pow = mean(demod.^2);
19
20 % demodulated noise power
21 mod = dsb_mod(msg, t, W, fc, Ac);
22 chn = acgn_ch(mod, t, L, W, N0);
23 demod = dsb_demod(chn, t, W, fc);
    
```

```
24 n_pow = mean(demod.^2)-s_pow;  
25  
26 % snr  
27 snr = s_pow/n_pow
```

As roughly shown in Figs. 4 and 5, the SNR is directly proportional to  $A_c^2$  and inversely proportional to  $W$ , which coincide to what is expected from the analytical analysis.

To calculate the SNR, we first run the simulation in a high SNR condition with  $N_0 \rightarrow 0$  to calculate the power of signal at the output of the demodulator. Then, for a given noise power spectral density  $N_0$  and the same set of parameters and message signal, the power content of the output of the demodulator is calculated. Since noise and signal are independent and the signal at the output of the demodulator is the sum of message and noise parts, the calculated power equals the sum of signal and noise power. Having the message power calculated, the noise power and consequently, the SNR can be computed.

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## BONUS QUESTIONS

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### Question 7

Return your answers by filling the  $\LaTeX$  template of the assignment.

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## EXTRA QUESTIONS

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### Question 8

Feel free to solve the following questions from the book *Fundamentals of Communication Systems* by J. Proakis and M. Salehi.

1. Chapter 3, question 3.
2. Chapter 3, question 4.
3. Chapter 3, question 8.
4. Chapter 3, question 9.
5. Chapter 3, question 17.
6. Chapter 6, question 1.
7. Chapter 6, question 3.