MATHEMATICAL QUESTIONS

Question 1

A vestigial-sideband modulation system is shown in Fig. 1. The bandwidth of the message signal $m(t)$ is W, and the transfer function of the bandpass filter is shown in the figure.

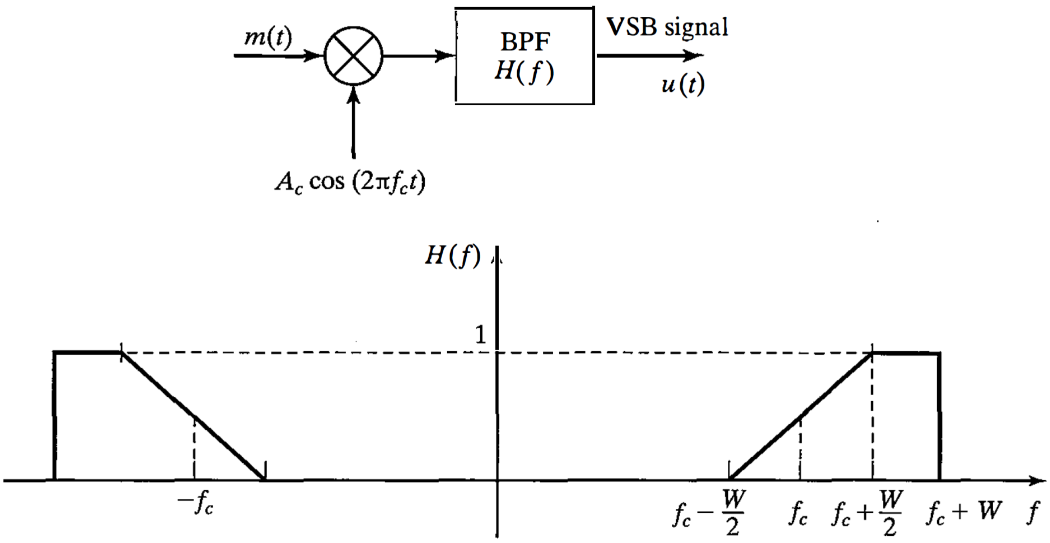

Figure 1: A VSB modulation system.

(a) Determine $h_l(t)$, which is the lowpass equivalent of $h(t)$, where $h(t)$ represents the impulse response of the bandpass filter.

The lowpass equivalent transfer function of the system is

$$H_l(f) = 2u\left(f+f_c\right)H\left(f+f_c\right) = 2\begin{cases} \frac{1}{W}f+\frac{1}{2}, & |f| \leq \frac{W}{2} \\ 1, & \frac{W}{2} < f \leq W \end{cases}.$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned}
h_l(t) &= \mathcal{F}^{-1}\left[H_l(f)\right] = \int_{-\frac{W}{2}}^{W} H_l(f) e^{j2\pi ft} df \\
&= 2\int_{-\frac{W}{2}}^{\frac{W}{2}} \left(\frac{1}{W}f + \frac{1}{2}\right) e^{j2\pi ft} df + 2\int_{\frac{W}{2}}^{W} e^{j2\pi ft} df \\
&= \frac{2}{W}\left(\frac{1}{j2\pi t} f e^{j2\pi ft} + \frac{1}{4\pi^2 t^2} e^{j2\pi ft}\right)\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{1}{j2\pi t} e^{j2\pi ft}\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{2}{j2\pi t} e^{j2\pi ft}\Bigg|_{\frac{W}{2}}^{W} \\
&= \frac{1}{j\pi t} e^{j2\pi Wt} + \frac{j}{\pi^2 t^2 W}\sin(\pi Wt) \\
&= \frac{j}{\pi t}\left[\operatorname{sinc}(Wt) - e^{j2\pi Wt}\right]
\end{aligned}$$

(b) Derive an expression for the modulated signal $u(t)$.

Since $M(f)$ is nonzero in $f \in [-W, W]$, it is clear that

$$\mathcal{F}[m(t) * \frac{1}{j\pi t} e^{j2\pi Wt}] = -M(f)\operatorname{sgn}(f - W) = M(f) \Rightarrow m(t) * \frac{1}{j\pi t} e^{j2\pi Wt} = m(t)$$

. The modulated signal can be represented as

$$u(t) = \Re\{[A_c m(t) * h_l(t)] e^{j2\pi f_c t}\}$$

. As shown in (a), $h_l(t) = \frac{j}{\pi t}[\operatorname{sinc}(Wt) - e^{j2\pi Wt}]$. So,

$$\Re\Big\{\big[A_c m(t) * \frac{j}{\pi t}(\operatorname{sinc}(Wt) - e^{j2\pi Wt})\big] e^{j2\pi f_c t}\Big\}$$

$$= \Re\big\{\big[(A_c m(t) * (\frac{j}{\pi t}\operatorname{sinc}(Wt))\big] e^{j2\pi f_c t} + \big[A_c m(t) * \frac{1}{j\pi t} e^{j2\pi Wt}\big] e^{j2\pi f_c t}\Big\}$$

Therefore,

$$u(t) = \Re\big\{\big[A_c m(t) * (\frac{j}{\pi t}\operatorname{sinc}(Wt))\big] e^{j2\pi f_c t} + A_c m(t) e^{j2\pi f_c t}\big]$$

$$= A_c m(t)\cos(2\pi f_c t) - A_c\big[m(t) * \frac{1}{\pi t}\operatorname{sinc}(Wt)\big]\sin(2\pi f_c t)$$

Question 2

Follow the steps below to show the power of the FM signal $u(t) = A_c\cos(2\pi f_c t + \phi(t))$ is $\frac{A_c^2}{2}$.

(a) Write the power expression for the FM signal and show that the power equals $P = \frac{A_c^2}{2} + I$, where

$$I = \lim_{T\to\infty} \frac{A_c^2}{2T}\int_{-T/2}^{T/2} \cos(4\pi f_c t + 2\phi(t)) dt$$

Question 2 continued on next page...

$$
\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A_c^2 \cos^2(2\pi f_c t + \phi(t)) dt \\
&= \lim_{T\to\infty} \frac{A_c^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1+\cos(4\pi f_c t + 2\phi(t))}{2} dt \\
&= \lim_{T\to\infty} \frac{A_c^2 T}{2T} + \lim_{T\to\infty} \frac{A_c^2}{2T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(4\pi f_c t + 2\phi(t)) dt \\
&= \frac{A_c^2}{2} + I
\end{aligned}
$$

(b) Show that

$$
I_\infty = \int_{-\infty}^{\infty} \cos(4\pi f_c t + 2\phi(t)) dt
$$

relates to the Fourier transforms $\mathcal{F}\{e^{j2\phi(t)}\}$ and $\mathcal{F}\{e^{-j2\phi(t)}\}$ at the frequencies $-2f_c$ and $2f_c$, respectively.

$$
\begin{aligned}
I_\infty &= \int_{-\infty}^{\infty} \cos(4\pi f_c t + 2\phi(t)) dt \\
&= \int_{-\infty}^{\infty} \frac{e^{j(2\phi(t)+4\pi f_c t)} + e^{-j(2\phi(t)+4\pi f_c t)}}{2} dt \\
&= \frac{1}{2} \int_{-\infty}^{\infty} e^{j2\phi(t)} e^{-j2\pi(-2f_c)t} dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j2\phi(t)} e^{-j2\pi(2f_c)t} dt \\
&= \frac{1}{2}\mathcal{F}\{e^{j2\phi(t)}\}|_{f\to -2f_c} + \frac{1}{2}\mathcal{F}\{e^{-j2\phi(t)}\}|_{f\to 2f_c}
\end{aligned}
$$

(c) Use Taylor series expansion to show that I_∞ depends to the Fourier transforms $\mathcal{F}\{\phi^n(t)\}, n \in \mathbb{W}$ at the frequency $\pm 2f_c$.

$$
\begin{aligned}
I_\infty &= \int_{-\infty}^{\infty} \cos(4\pi f_c t + 2\phi(t)) dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} e^{j2\phi(t)} e^{-j2\pi(-2f_c)t} dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\phi(t)} e^{-j2\pi(2f_c)t} dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty} [1 + \frac{j2\phi(t)}{1!} + \frac{(j2\phi(t))^2}{2!} + \cdots] e^{-j2\pi(-2f_c)t} dt \\
&+ \frac{1}{2}\int_{-\infty}^{\infty} [1 + \frac{-j2\phi(t)}{1!} + \frac{(-j2\phi(t))^2}{2!} + \cdots] e^{-j2\pi(2f_c)t} dt \\
&= \frac{1}{2}[\mathcal{F}\{1\}|_{f\to -2f_c} + \mathcal{F}\{\frac{j2\phi(t)}{1!}\}|_{f\to -2f_c} + \mathcal{F}\{\frac{(j2\phi(t))^2}{2!}\}|_{f\to -2f_c} + \cdots] \\
&+ \frac{1}{2}[\mathcal{F}\{1\}|_{f\to 2f_c} + \mathcal{F}\{\frac{-j2\phi(t)}{1!}\}|_{f\to 2f_c} + \mathcal{F}\{\frac{(-j2\phi(t))^2}{2!}\}|_{f\to 2f_c} + \cdots] \\
&= \frac{1}{2}[\mathcal{F}\{1\}|_{f\to -2f_c} + \frac{j2}{1!}\mathcal{F}\{\phi(t)\}|_{f\to -2f_c} + \frac{(j2)^2}{2!}\mathcal{F}\{\phi^2(t)\}|_{f\to -2f_c} + \cdots] \\
&+ \frac{1}{2}[\mathcal{F}\{1\}|_{f\to 2f_c} + \frac{-j2}{1!}\mathcal{F}\{\phi(t)\}|_{f\to 2f_c} + \frac{(-j2)^2}{2!}\mathcal{F}\{\phi^2(t)\}|_{f\to 2f_c} + \cdots] \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(j2)^n}{n!}\mathcal{F}\{\phi^n(t)\}|_{f\to -2f_c} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-j2)^n}{n!}\mathcal{F}\{\phi^n(t)\}|_{f\to 2f_c}
\end{aligned}
$$

(d) *Show that if $f_c \gg W$, where W is the bandwidth of the message-related phase $\phi(t)$, $I_\infty \approx 0$.*

Assuming a bandwidth of W for the message-related phase, $\phi(t)$, the bandwidth of signal $\phi^n(t)$ is smaller than nW. Therefore,

For small n, we have $nW < 2f_c$, so $\mathcal{F}\{\phi^n(t)\}_{f\to 2f_c} = \mathcal{F}\{\phi^n(t)\}_{f\to -2f_c} = 0$.
For large n, we have $\frac{2^n}{n!} \approx 0$, so $\frac{2^n}{n!}\mathcal{F}\{\phi^n(t)\}_{f\to 2f_c} = \frac{2^n}{n!}\mathcal{F}\{\phi^n(t)\}_{f\to -2f_c} \approx 0$.

Therefore, we conclude that $I_\infty \approx 0$.

(e) *Show that the power is approximately equal to $\frac{A_c^2}{2}$.*

Since $I_\infty \approx 0$, we have

$$I = \lim_{T\to\infty} \frac{A_c^2}{2T}\int_{-T/2}^{T/2} \cos(4\pi f_c t + 2\phi(t)) dt = \lim_{T\to\infty} \frac{A_c^2 I_\infty}{2T} \approx 0$$

Finally,

$$P = \frac{A_c^2}{2} + I \approx \frac{A_c^2}{2}$$

.

Question 3 continued on next page. . .

Question 3

Find the spectrum of the narrowband FM signal

$$u(t) = A_c \cos(2\pi f_c t) - A_c\big[2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\big] \sin(2\pi f_c t)$$

in terms of the message spectrum $M(f)$.

We know that

$$\mathcal{F}\{2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\} = \frac{2\pi k_f M(f)}{j2\pi f} + \frac{2\pi k_f}{2} M(0)\delta(f) = \frac{k_f M(f)}{jf}$$

by the integral property of the Fourier transform. Note that the message is assumed to have no DC component so $M(0) = 0$. Now,

$$\begin{aligned} U(f) &= \mathcal{F}\{A_c \cos(2\pi f_c t)\} - \mathcal{F}\{2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\} * \mathcal{F}\{A_c \sin(2\pi f_c t)\} \\ &= \frac{A_c}{2}(\delta(f-f_c) + \delta(f+f_c)) - \frac{k_f M(f)}{jf} * \big[\frac{A_c}{2j}(\delta(f-f_c) - \delta(f+f_c))\big] \\ &= \frac{A_c}{2}(\delta(f-f_c) + \delta(f+f_c)) - \frac{A_c k_f M(f-f_c)}{2j^2(f-f_c)} + \frac{A_c k_f M(f+f_c)}{2j^2(f+f_c)} \\ &= \frac{A_c}{2}(\delta(f-f_c) + \delta(f+f_c)) + \frac{A_c k_f M(f-f_c)}{2(f-f_c)} - \frac{A_c k_f M(f+f_c)}{2(f+f_c)} \end{aligned}$$

Question 4

The signal $m(t)$, whose Fourier transform $M(f)$ is shown in Fig. 2, is to be communicated. We know that the signal is normalized, meaning that $-1 \leq m(t) \leq 1$.

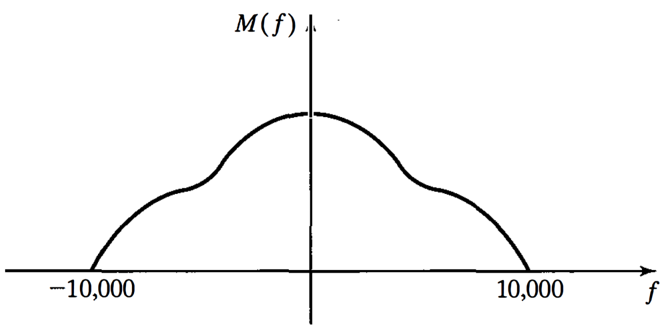

Figure 2: A sample message signal.

(a) If USSB is employed, what is the bandwidth of the modulated signal?.

$$W_{\text{USSB}} = W = 10^4 \text{ Hz}$$

Question 4 continued on next page...

(b) If DSB is employed, what is the bandwidth of the modulated signal?

$$W_{\text{DSB}} = 2W = 2 \times 10^4 \text{ Hz}$$

(c) If an AM scheme with the index $a = 0.8$ is used, what is the bandwidth of the modulated signal?

$$W_{\text{AM}} = 2W = 2 \times 10^4 \text{ Hz}$$

(d) If an FM scheme with the deviation $k_f = 50000$ is used, what is the bandwidth of the modulated signal?

$$W_{\text{FM}} = 2(\beta_f + 1)W = 2(\frac{k_f \max\{|m(t)|\}}{W} + 1) = 2 \times (\frac{50000}{10000} + 1) \times 10000 = 120000 \text{ Hz}$$

Question 5

Determine the in-phase and quadrature components, $x_c(t)$ and $x_s(t)$, as well as the envelope and the phase, $V(t)$ and $\Theta(t)$ of an FM-modulated signal.

For an angle modulated signal, we have $x(t) = A_c \cos(2\pi f_c t + \phi(t))$. Therefore, the low-pass equivalent of the signal is $x_l(t) = A_c e^{j\phi(t)}$ with the envelope $V(t) = A_c$ and the phase $\Theta(t) = \phi(t)$. Obviously,

$$x(t) = A_c \cos(\phi(t)) \cos(2\pi f_c t) - A_c \sin(\phi(t)) \sin(2\pi f_c t)$$

. So,

$$x_c(t) = A_c \cos(\phi(t))$$

$$x_s(t) = A_c \sin(\phi(t))$$

Now, for the FM case, $\phi(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau$ and

$$V(t) = A_c$$

$$\Theta(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau$$

$$x_c(t) = A_c \cos(2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau)$$

$$x_s(t) = A_c \sin(2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau)$$

.

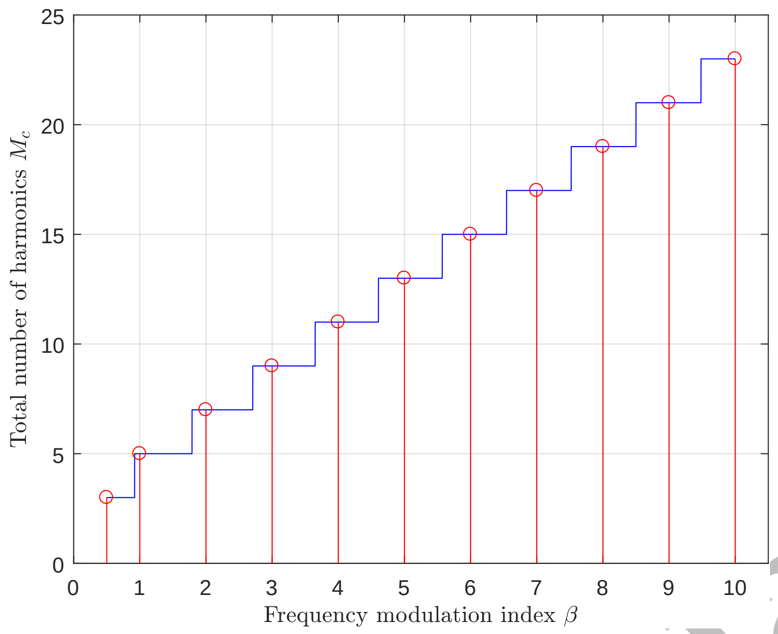

Figure 3: Number of required harmonics M_c versus frequency modulation index β for 98% of power content.

SOFTWARE QUESTIONS

Question 6

Develop a MATLAB mfile that returns the required number of harmonics M_c that includes $x\%$ of the total power content of a sinusoidally-modulated FM signal with the modulation index β. Use the mfile to obtain the number of required harmonics M_c for $x = 98$ and $\beta = 0.5, 1, 2, \cdots, 10$.

Here is a sample implementation.

```
function Mc = num_harmonics_fm(beta, x)

har_id = 0;
pow = besselj(har_id,beta)^2;

while (pow < x/100)
    har_id = har_id + 1;
    pow = pow+2*besselj(har_id,beta)^2;
end

Mc = 2*har_id+1;

end
```

You may use the following mfile to plot Fig. 3 showing the number of required harmonics for $x = 98$ and $\beta = 0.5, 1, 2, \cdots, 10$.

```
close all
clear all
```

Question 6 continued on next page. . .

```

% continuous plot
num_har = [];
for beta=0.5:0.001:10
    num_har = [num_har num_harmonics_fm(beta, 98)];
end
plot(0.5:0.001:10, num_har, 'blue')

% samples plot
num_har = [];
for beta=[0.5 1:1:10]
    num_har = [num_har num_harmonics_fm(beta, 98)];
end
hold on
stem([0.5 1:1:10], num_har, 'red')

% label and save figures
xlim([0 10.5])
xlabel('Frequency modulation index $\beta$', 'Interpreter', 'latex')
ylabel('Total number of harmonics $M_c$', 'Interpreter', 'latex')
box on
grid on
fig = gcf;
saveas(fig, 'Mc', 'fig')
saveas(fig, 'Mc', 'pdf')
system('pdfcrop Mc.pdf Mc.pdf');
```

BONUS QUESTIONS

Question 7

Return your answers by filling the LaTeX template of the assignment.

EXTRA QUESTIONS

Question 8

Feel free to solve the following questions from the book *Fundamentals of Communication Systems* by J. Proakis and M. Salehi.

1. **Chapter 3, question 7.**
2. **Chapter 3, question 14.**
3. **Chapter 3, question 18.**
4. **Chapter 4, question 5.**

Question 8 continued on next page...

5. **Chapter 4, question 8.**
6. **Chapter 6, question 5.**