MATHEMATICAL QUESTIONS

Question 1

Show that at high SNR conditions, the SNR at the output of a PM demodulator in Fig. 1 is

$$(\frac{S}{N})_o = P_R(\frac{\beta_p}{\max\{|m(t)|\}})^2\frac{P_m}{N_0W}$$

, where P_R is the received power after the demodulator bandpass filter, β_p is the modulation index, $N_0/2$ is the power spectral density of the AWGN noise, and W is the bandwidth of the message signal $m(t)$.

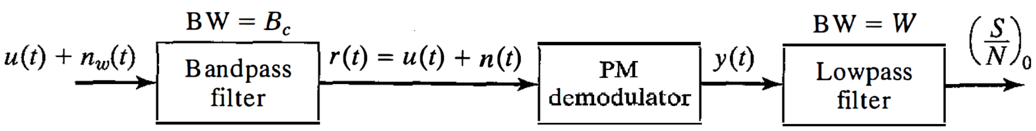

Figure 1: The block diagram of a PM demodulator.

The SNR calculation is almost similar to FM except some minor differences in calculation of the output noise power. Applying the phasor approximation, the received signal is

$$r(t) = [A_c + V_n(t)\cos(\Phi_n(t) - k_pm(t))]\cos[2\pi f_ct + k_pm(t) + \frac{V_n(t)}{A_c}\sin(\Phi_n(t) - k_pm(t))]$$

After the demodulation, the message-dependent part of the phase is detected as

$$y(t) = k_pm(t) + \frac{V_n(t)}{A_c}\sin(\Phi_n(t) - k_pm(t)) = k_pm(t) + Y_n(t)$$

Neglecting the variation of $m(t)$ compared to $n_s(t)$ and $n_c(t)$,

$$Y_n(t) = \frac{V_n(t)}{A_c}\sin(\Phi_n(t) - k_pm(t)) \approx \frac{1}{A_c}[n_s(t)\cos(k_pm) - n_c(t)\sin(k_pm)]$$

Since $n_s(t)$ and $n_c(t)$ are independent and zero-mean processes with the same power spectral density,

$$S_{Y_n}(f) = S_{n_c}(f)\frac{\cos^2(k_pm) + \sin^2(k_pm)}{A_c^2} = \frac{N_0}{A_c^2}\sqcap(\frac{f}{B_c})$$

Note that unlike FM, the noise power spectral density is constant over $[-B_c/2, B_c/2]$. The output noise power is

$$P_{n_o} = \int_{-W}^{W}\frac{N_0}{A_c^2}\sqcap(\frac{f}{B_c})df = \frac{2N_0W}{A_c^2}$$

. Clearly, the signal power equals

$$P_{s_o} = k_p^2P_m = (\frac{\beta_p}{\max\{|m(t)|\}})^2P_m$$

Question 1 continued on next page...

Finally,

$$(\frac{S}{N})_o = \frac{P_{s_o}}{P_{n_o}} = \frac{k_p^2 A_c^2 P_m}{2N_0 W} = P_R(\frac{\beta_p}{\max\{|m(t)|\}})^2 \frac{P_m}{N_0 W}$$

, where $P_r = \frac{A_c^2}{2}$ is the power of received signal $r(t)$.

Question 2

As shown in Fig. 2, preemphasis and deemphasis filters may accompany an FM modulation system to improves its achievable SNR.

Figure 2: FM with preemphasis and deemphasis fillers.

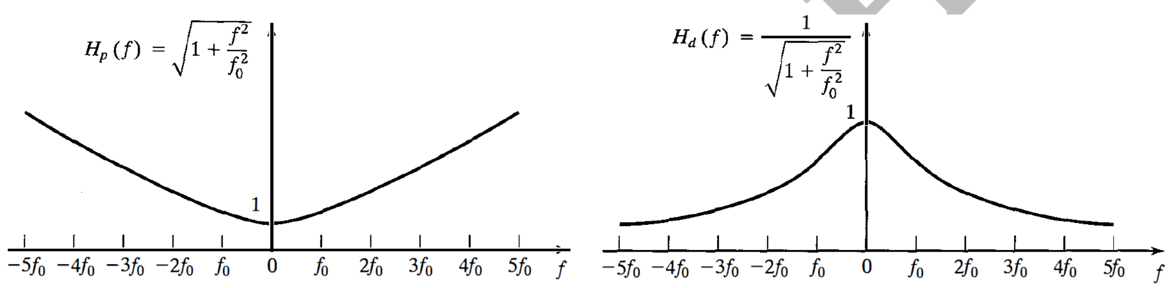

Figure 3: Preemphasis and deemphasis filter characteristics.

(a) Let $H_p(f)$ and $H_d(f)$ in Fig. 3 characterize the preemphasis and deemphasis filters. Find the SNR after the deemphasis filter provided that A_c is the carrier amplitude, k_f is the frequency deviation constant, $N_0/2$ is the power spectral density of the AWGN noise, and W is the bandwidth of the message signal $m(t)$.

Clearly, $H_p(f)H_d(f) = 1$ or equivalently, $h_p(t) * h_d(t) = \delta(t)$. Here, the demodulated signal is

$$y(t) = k_f m(t) * h_p(t) * h_d(t) + [\frac{1}{2\pi}\frac{d}{dt}Y_n(t)] * h_d(t) = k_f m(t) + [\frac{1}{2\pi}\frac{d}{dt}Y_n(t)] * h_d(t)$$

We know that the power spectral density of $\frac{1}{2\pi}\frac{d}{dt}Y_n(t)$ after the output W-Hz LPF is $\frac{N_0 f^2}{A_c^2}\sqcap(\frac{f}{2W})$. So, the power spectral density of the noise after the deemphasis filter is

$$S_{n_o}(f) = |H_d(f)|^2 \frac{N_0 f^2}{A_c^2}\sqcap(\frac{f}{2W}) = \frac{f_0^2}{f_0^2+f^2}\frac{N_0 f^2}{A_c^2}\sqcap(\frac{f}{2W})$$

The noise power is obtained as

$$P_{n_o} = \int_{-W}^{+W} S_{n_o}(f)df = \frac{2f_0^2 N_0}{A_c^2}(W - f_0\tan^{-1}(\frac{W}{f_0}))$$

Question 2 continued on next page...

The signal power is $P_{s_0} = k_f^2 P_m$ and the SNR equals

$$(\frac{S}{N})_o = \frac{k_f^2 P_m A_c^2}{2f_0^3 N_0} \frac{1}{\frac{W}{f_0} - \tan^{-1}(\frac{W}{f_0})} = \frac{W^3}{3f_0^3[\frac{W}{f_0} - \tan^{-1}(\frac{W}{f_0})]} \frac{3k_f^2 A_c^2 P_m}{2N_0 W^3}$$

(b) How is SNR improved compared to when no emphasis filter is applied?

Clearly, the SNR is scaled by the coefficient $\frac{W^3}{3f_0^3[\frac{W}{f_0} - \tan^{-1}(\frac{W}{f_0})]}$ compared to the conventional FM without emphasis filters having the SNR $3k_f^2 A_c^2 P_m/(2N_0 W^3)$. We claim that the coefficient is greater than 1 indicating an SNR improvement. Let $x = \frac{W}{f_0} > 0$. The coefficient can be rewritten in terms of $x > 0$ as

$$\frac{x^3}{3x - 3\tan^{-1}(x)}$$

Now, note that

$$f(x) = x^3 - 3x + 3\tan^{-1}(x)$$

is a strictly ascending function since

$$f'(x) = 3x^2 - 3 + \frac{3}{1+x^2} \geq 0 \Leftrightarrow 3(x^2+1)(x^2-1) + 3 \geq 0 \Leftrightarrow x^4 \geq 0$$

with a unique extremum point at $x = 0$. So, for $x > 0$, $f(x) > f(0) = 0$ and therefore,

$$x^3 > 3x - 3\tan^{-1}(x) \Rightarrow \frac{x^3}{3x - 3\tan^{-1}(x)} > 1$$

by the fact that $x \geq \tan^{-1}(x), \quad \forall x \geq 0$.

Question 3

In an analog communication system, demodulation gain is defined as the ratio of the SNR at the output of the demodulator to the SNR at the output of the noise-limiting bandpass filter at the reciver front end. Find expressions for the demodulation gain in each of the following cases.

(a) DSB.

The output of the receiver noise-limiting filter is

$$r(t) = A_c m(t)\cos(2\pi f_c t) + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

The power of the received signal is $P_{s,l} = \frac{A_c^2}{2} P_m$, whereas the power of the noise

$$P_{n,l} = \frac{1}{2}P_{n_c} + \frac{1}{2}P_{n_s} = P_n$$

Question 3 continued on next page. . .

Hence, the SNR at the output of the noise-limiting filter is

$$\left(\frac{S}{N}\right)_{o,\mathrm{lim}} = \frac{A_c^2 P_m}{2P_n}$$

Assuming coherent demodulation, the output of the demodulator is

$$y(t) = \frac{1}{2}\left[A_c m(t) + n_c(t)\right]$$

The output signal power is $P_{s,o} = \frac{1}{4}A_c^2 P_m$ whereas the output noise power

$$P_{n,o} = \frac{1}{4}P_{n_c} = \frac{1}{4}P_n$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\mathrm{dem}} = \frac{A_c^2 P_m}{P_n}$$

and the demodulation gain is given by

$$\frac{\left(\frac{S}{N}\right)_{o,\mathrm{dem}}}{\left(\frac{s}{N}\right)_{o,\mathrm{lim}}} = 2$$

(b) SSB.

In the case of SSB, the output of the receiver noise-limiting filter is

$$r(t) = A_c m(t)\cos(2\pi f_c t) \pm A_c \hat{m}(t)\sin(2\pi f_c t) + n(t)$$

The received signal power is $P_{s,l} = A_c^2 P_m$, whereas the received noise power is $P_{n,l} = P_n$. At the output of the demodulator

$$y(t) = \frac{A_c}{2}m(t) + \frac{1}{2}n_c(t)$$

with $P_{s,o} = \frac{1}{4}A_c^2 P_m$ and $P_{n,o} = \frac{1}{4}P_{n_c} = \frac{1}{4}P_n$. Therefore,

$$\frac{\left(\frac{S}{N}\right)_{o,\mathrm{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{\frac{A_c^2 P_m}{P_n}}{\frac{A_c^2 P_m}{P_n}} = 1$$

(c) Conventional AM with a modulation index a.

In the case of conventional AM modulation, the output of the receiver noise-limiting filter is

$$r(t) = \left[A_c\left(1 + a m_n(t)\right) + n_c(t)\right]\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

The total pre-detection power in the signal is

$$P_{s,l} = \frac{A_c^2}{2}\left(1 + a^2 P_{m_n}\right)$$

whereas the noise power is $P_{n,l} = P_0$. So,

$$\left(\frac{S}{N}\right)_{o,\mathrm{lim}} = \frac{A_c^2(1 + a^2 P_{m_n})}{2P_n}$$

. As discussed in the class, the SNR at the output of the coherent demodulator is

$$\left(\frac{S}{N}\right)_{o,\mathrm{dem}} = \frac{A_c^2 a^2 P_{m_n}}{P_n}$$

In this case, the demodulation gain is given by

$$\frac{\left(\frac{S}{N}\right)_{o,\text{ dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{2a^2 P_{m_n}}{1 + a^2 P_{m_n}}$$

The highest gain is achieved for $a = 1$, that is 100% modulation.

(d) *FM with a modulation index β_f.*

For an FM system, the output of the receiver front-end is

$$r(t) = A_c \cos\left(2\pi f_c t + \phi(t)\right) + n(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau\right) + n(t)$$

The total signal input power is $P_{s,l} = \frac{A_c^2}{2}$, whereas the pre-detection noise power is

$$P_{n,l} = \frac{N_0}{2} 2B_c = N_0 B_c = N_0 2\left(\beta_f + 1\right) W$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\mathrm{lim}} = \frac{A_c^2}{2N_0 2\left(\beta_f + 1\right) W}$$

The output post-detection signal to noise ratio is

$$\left(\frac{S}{N}\right)_{o,\mathrm{dem}} = \frac{3k_f^2 A_c^2 P_m}{2N_0 W^3}$$

Thus, the demodulation gain is

$$\frac{\left(\frac{S}{N}\right)_{o,\text{ dem}}}{\left(\frac{S}{N}\right)_{o,\text{ lim}}} = \frac{3\beta_f^2 P_m 2\left(\beta_f + 1\right)}{(\max[|m(t)|])^2} = 6\beta_f^2\left(\beta_f + 1\right) P_{m_n}$$

Question 3 continued on next page...

(e) PM with a modulation index β_p.

Similarly, for the PM case, we find that

$$\left(\frac{S}{N}\right)_{o,\text{ lim}} = \frac{A_c^2}{2N_0 2\left(\beta_p+1\right)W}$$

and

$$\left(\frac{S}{N}\right)_{o,\text{dem}} = \frac{k_p^2 A_c^2 P_m}{2N_0 W}$$

Thus, the demodulation gain for a PM system is

$$\frac{\left(\frac{S}{N}\right)_{o,\text{ dem}}}{\left(\frac{S}{N}\right)_{o,\text{ lim}}} = \frac{\beta_p^2 P_m 2\left(\beta_p+1\right)}{(\max[|m(t)|])^2} = 2\beta_p^2\left(\beta_p+1\right)P_{m_n}$$

Question 4

Show that if an FM system and a PM system are employed for transmitting a message signal and these systems have the same output SNR and the same carrier amplitude, then

$$\frac{B_{c_{PM}}}{B_{c_{FM}}} = \frac{\sqrt{3}\beta_f+1}{\beta_f+1}$$

, where β_f is the FM index.

We have

$$\left(\frac{S}{N}\right)_{\text{FM}} = \left(\frac{S}{N}\right)_{\text{PM}} \Rightarrow \frac{3k_f^2 A_c^2 P_m}{2N_0 W^3} = \frac{k_p^2 A_c^2 P_m}{2N_0 W} \Rightarrow \frac{\sqrt{3}k_f}{W} = k_p \Rightarrow \sqrt{3}\beta_f = \beta_p$$

So,

$$\frac{B_{c_{\text{PM}}}}{B_{c_{\text{FM}}}} = \frac{2(\beta_p+1)W}{2(\beta_f+1)W} = \frac{\sqrt{3}\beta_f+1}{\beta_f+1}$$

SOFTWARE QUESTIONS

Question 5

Develop a MATLAB mfile that takes a message, passes it through an FM modulator, and plots the spectrum of the modulated signal. Plot the spectrum of the modulated signal for several input messages and use the results to validate Carson's bandwidth rule. You may use modulation index, carrier amplitude, and so on as the input arguments to your developed mfile.

Question 5 continued on next page...

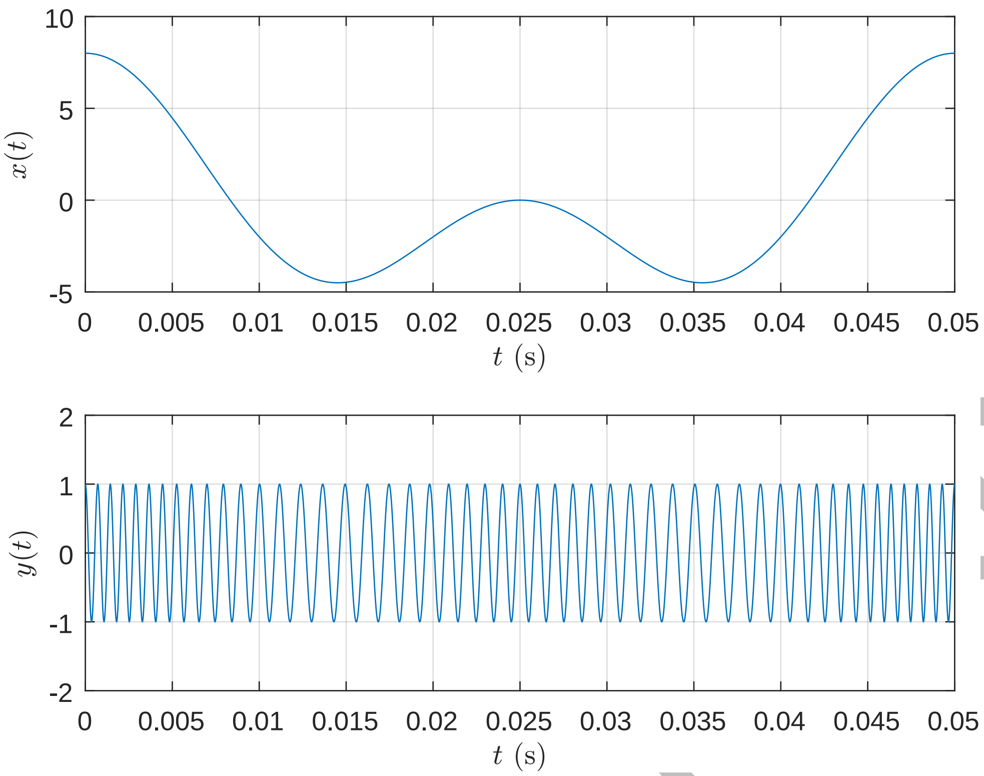

Figure 4: Time-domain representation of a signal and its corresponding FM signal.

Here is an implementation of the required function.

```
function [t,y]=fm_plot(t,x,Ac,fc,kf)

close all

% fm modulation
y = Ac*cos(2*pi*fc*t+2*pi*kf*cumsum(x)*abs(t(2)-t(1)));

% take the DFT and approximate the spectrum using the DFT
Fs = 1/abs(t(2)-t(1));
L = length(t);
f = Fs*((1:L)-L/2)/L;
Y = fftshift(fft(y))/L;
X = fftshift(fft(x))/L;

% plot time-domian curve
subplot(2,1,1);
plot(t,x)
subplot(2,1,2);
plot(t,y)

% plot frequency-domian curve
figure
subplot(2,1,1);
hold on
plot(f,real(X));
plot(f,imag(X),'r');
subplot(2,1,2);
hold on
plot(f,real(Y));
plot(f,imag(Y),'r');
```

Time-domain and frequency-domain representation of the output of the demodulator for

Question 5 continued on next page. . .

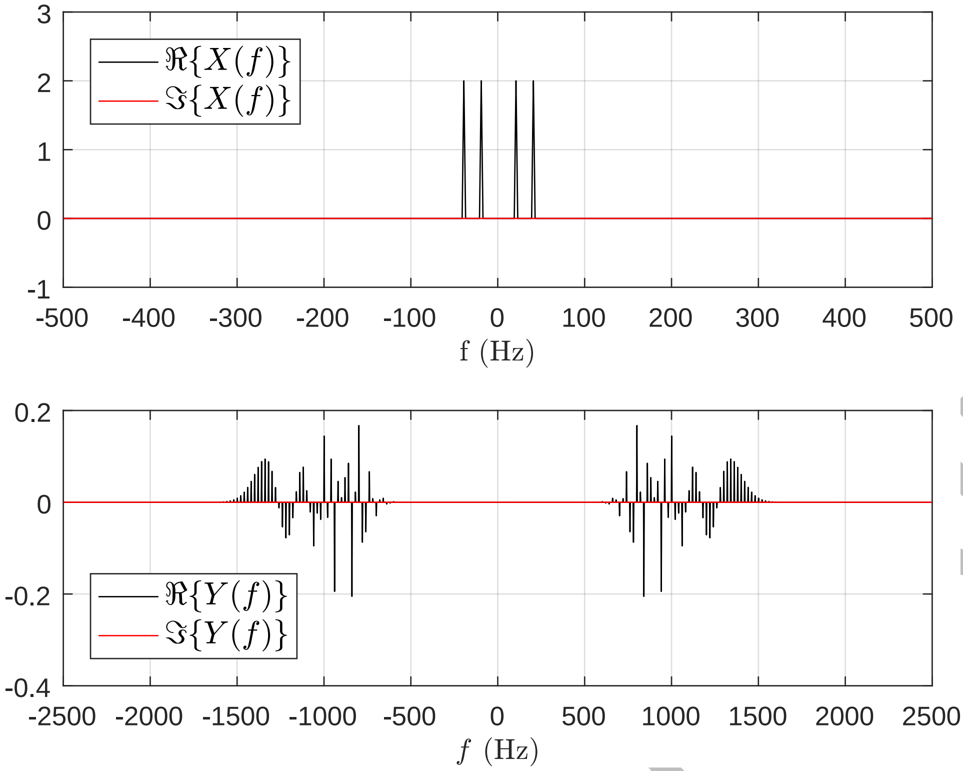

Figure 5: Frequency-domain representation of a signal and its corresponding FM signal.

$x(t) = 4\cos(40\pi t) + 4\cos(80\pi t)$, $A_c = 1$, $f_c = 1000$, and $k_f = 50$ are shown in Figs. 4 and 5. As can be seen in Fig. 5, the input signal bandwidth is $W = 40$ Hz. So, the modulation index equals $\beta_f = k_f \max\{|m(t)|\}/W = 50 \times 8/40 = 10$ and consequently, $B_c = 2(\beta_f + 1)W = 880$ Hz by Carson's rule, which coincides with the spectrum of the modulated signal.

BONUS QUESTIONS

Question 6

Chernoff bound is a useful tool in communication analysis.

(a) Prove the Chernoff inequality

$$P\{X \geq a\} \leq e^{-ta} E\{e^{tX}\}$$

Question 6 continued on next page...

for any random variable X, $t \geq 0$, and $a \in \mathbb{R}$.

According to Markov's inequality, for $t \geq 0$,

$$P\{X \geq a\} = P\{tX \geq ta\} = P\{e^{tX} \geq e^{ta}\} \leq \frac{E\{e^{tX}\}}{e^{ta}} = e^{-ta}E\{e^{tX}\}$$

(b) How does the Chernoff bound relate to the characteristic function?

$$P\{X \geq a\} \leq e^{-ta}E\{e^{tX}\} = e^{-ta}E\{e^{j\frac{t}{j}X}\} = e^{-ta}\phi_X(\frac{t}{j})$$

(c) Find the Chernoff bound for the Gaussian random variable $\mathcal{N}(\mu, \sigma^2)$.

We know that for the Gaussian random variable $\mathcal{N}(\mu, \sigma^2)$,

$$\phi_X(t) = e^{-\frac{1}{2}\sigma^2 t^2 + j\mu t}$$

. So, the corresponding Chernoff bound is

$$e^{-ta}\phi_X(\frac{t}{j}) = e^{-ta}e^{\frac{1}{2}\sigma^2 t^2 + \mu t} = e^{\frac{1}{2}\sigma^2 t^2 + (\mu - a)t}$$

(d) Tighten the derived Chernoff bound for the Gaussian random variable $\mathcal{N}(\mu, \sigma^2)$ by selecting a suitable value for t.

We know that,

$$\frac{d}{dt}e^{\frac{1}{2}\sigma^2 t^2 + (\mu - a)t} = [\sigma^2 t + (\mu - a)]e^{\frac{1}{2}\sigma^2 t^2 + (\mu - a)t}$$

. There is an extremum point at $t_{opt} = \frac{a-\mu}{\sigma^2}$. If $\mu > a$, then $t_{opt} < 0$ and the bound is ascending over $t \geq 0$. So, the minimum bound is achieved for $t = 0$ and equals to the trivial value of 1. If $\mu \leq a$, then the extremum falls somewhere over $[0, \infty)$. So, the minimum bound is

$$e^{\frac{1}{2}\sigma^2 t_{opt}^2 + (\mu - a)t_{opt}} = e^{-\frac{(\mu - a)^2}{2\sigma^2}}$$

Overall,

$$P\{X \geq a\} \leq \begin{cases} 1, & \mu > a \\ e^{-\frac{(\mu-a)^2}{2\sigma^2}}, & \mu \leq a \end{cases}$$

Question 7

Return your answers by filling the LaTeXtemplate of the assignment.

EXTRA QUESTIONS

Question 8

Feel free to solve the following questions from the book *Fundamentals of Communication Systems* by J. Proakis and M. Salehi.

1. **Chapter 6, question 2.**
2. **Chapter 6, question 4.**
3. **Chapter 6, question 11.**