# MATHEMATICAL QUESTIONS

## Question 1

**In transmission of telephone signals over line-of-sight microwave links, a combination of FDM-SSB and FM is often employed. A block diagram of such a system is shown in Fig. 1. Each of the signals $m_i(t)$ is bandlimited to $W$ Hz, and these signals are USSB modulated on carriers $c_i(t) = A_i \cos(2\pi f_i t)$, where $f_i = (i-1)W, 1 \le i \le K$, and $m(t)$ is the sum of all USSB-modulated signals. This signal FM modulates a carrier with frequency $f_c$ with a modulation index of $\beta$ and an amplitude of $A_c$.**

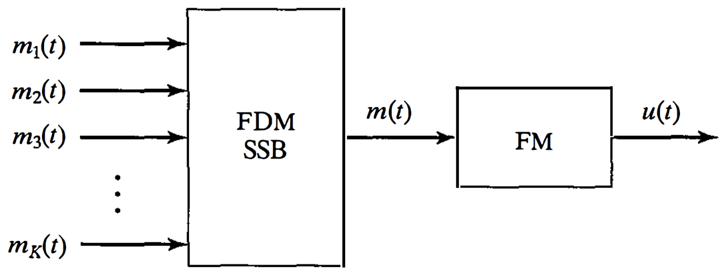


Figure 1: Combined FDM-SSB and FM.

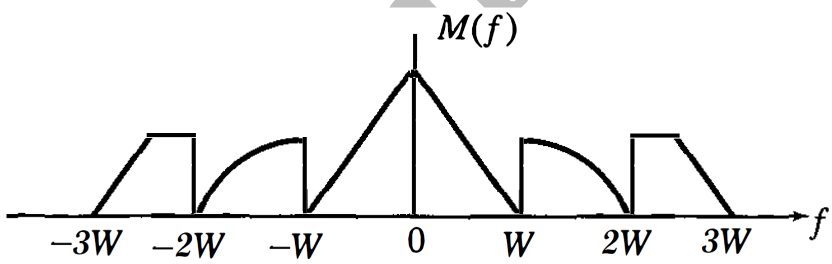


Figure 2: A typical spectrum of $m(t)$.

*(a) Plot a typical spectrum of the USSB-modulated signal $m(t)$.*

A typical spectrum of $m(t)$ for $K = 3$ is shown in Fig. 2.

*(b) Determine the bandwidth of $m(t)$.*

The bandwidth of the signal $m(t)$ is $W_m = KW$.

*(c) At the receiver side, the received signal $r(t) = u(t) + n_W(t)$ is first FM demodulated and then passed through a bank of USSB demodulators, where $n_W(t)$ is an AWGN with a power spectral density of $\frac{N_0}{2}$. Show that the noise power entering these demodulators depend on $i$.*

After FM demodulation, We get

$$\begin{aligned} & k_f \sum_{i=1}^{K} [A_i m_i(t)\cos(2\pi f_i t) - A_i \hat{m}_i(t)\sin(2\pi f_i t)] + n(t) \\ =& k_f A_1 m_1(t) + k_f \sum_{i=2}^{K} [A_i m_i(t)\cos(2\pi f_i t) - A_i \hat{m}_i(t)\sin(2\pi f_i t)] + n(t) \end{aligned}$$

, where $n(t)$ is the noise at the output of the FM demodulator with the power spectral density of $S_{n,o}(f) = \frac{N_0}{A_c^2} f^2$. There is a BPF over $[(i-1)W, iW)] \cup [-iW, -(i-1)W]$ at the input o the $i$th USSB demodulator. The input noise power, after the noise-limiting BPF of the $i$th demodulator, is

$$P_{in,o_i} = \int_{(i-1)W}^{iW} \frac{N_0}{A_c^2} f^2 df + \int_{-iW}^{-(i-1)W} \frac{N_0}{A_c^2} f^2 df = \frac{2N_0}{A_c^2} \int_{(i-1)W}^{iW} f^2 df = \frac{2N_0 W^3}{3A_c^2}(3i^2 - 3i + 1)$$

, which clearly depends on $i$.

*(d) Determine an expression for the ratio of the noise power entering the demodulator, whose carrier frequency is $f_i$ to the noise power entering the demodulator with the carrier frequency $f_j, 1 \leq i, j \leq K$.*

Clearly,

$$\frac{P_{in,o_i}}{P_{in,o_j}} = \frac{3i^2 - 3i + 1}{3j^2 - 3j + 1}$$

*(e) How should the carrier amplitudes $A_i$ be chosen to guarantee that, after USSB demodulation, the SNR for all channels is the same?*

For the first channel with $i = 1$ and $f_1 = 0$, the USSB modulated signal is

$$A_1 m_1(t)\cos(2\pi f_1 t) - A_1 \hat{m}_1(t)\sin(2\pi f_1 t) = A_1 m_1(t)$$

. At the output of the FM demodulator, the input to the first USSB demodulator is

$$A_1 k_f m_1(t) + n(t)$$

, where $n(t)$ is the noise at the output of the FM demodulator with the power spectral density of $S_{n,o}(f) = \frac{N_0}{A_c^2} f^2$. Here, the message is itself in the baseband and no mixing is required in the USSB demodulator. Applying the output LPF over $[-W, W]$, the noise power is

$$P_{n,o_1} = P_{n_1} = 2\int_0^W \frac{N_0}{A_c^2} f^2 df = \frac{2N_0 W^3}{3A_c^2}$$

*Question 1 continued on next page. . .* 

. The SNR of the first received USSB signal is

$$(\frac{S}{N})_{o_1} = \frac{P_{s,o_1}}{P_{n,o_1}} = \frac{3k_f^2 A_c^2 A_1^2 P_{m_1}}{2N_0 W^3}$$

For $i \geq 2$, $f_i > 0$, a mixing is required at the USSB demodulator. The $i$th USSB signal occupies the frequency band $[-iW, -(i-1)W] \cup [(i-1)W, iW]$ and equals

$$A_i m_i(t)\cos(2\pi f_i t) - A_i \hat{m}_i(t)\sin(2\pi f_i t)$$

At the output of the FM demodulator, the input to the $i$th USSB demodulator is

$$A_i k_f m_i(t)\cos(2\pi f_i t) - A_i k_f \hat{m}_i(t)\sin(2\pi f_i t) + n(t)$$

, where $n(t)$ is the noise at the output of the FM demodulator with the power spectral density of $S_{n,o}(f) = \frac{N_0}{A_c^2} f^2$. After the mixing and lowpass filtering at the USSB demodulator,

$$\frac{1}{2} A_i k_f m_i(t) + \frac{1}{2} n_c(t)$$

Applying the output LPF over $[-W, W]$, the noise power is

$$P_{n,o_i} = \frac{1}{4} P_{n_i} = \frac{1}{4} 2 \int_{(i-1)W}^{iW} \frac{N_0}{A_c^2} f^2 df = \frac{N_0 W^3}{6A_c^2}(3i^2 - 3i + 1)$$

The SNR of the $i$th received USSB signal is

$$(\frac{S}{N})_{o_i} = \frac{P_{s,o_i}}{P_{n,o_i}} = \frac{3k_f^2 A_c^2 A_i^2 P_{m_i}}{2N_0 W^3 (3i^2 - 3i + 1)}$$

To have the same SNR for all the USSB channels,

$$(\frac{S}{N})_{o_i} = (\frac{S}{N})_{o_1}, \quad i = 2, 3, \cdots, K$$

So,

$$\frac{3k_f^2 A_c^2 A_i^2 P_{m_i}}{2N_0 W^3 (3i^2 - 3i + 1)} = \frac{3k_f^2 A_c^2 A_1^2 P_{m_1}}{2N_0 W^3}, \quad i = 2, 3, \cdots, K$$

Finally,

$$A_i^2 = (3i^2 - 3i + 1)\frac{P_{m_1} A_1^2}{P_{m_i}} \Rightarrow A_i = A_1 \sqrt{(3i^2 - 3i + 1)\frac{P_{m_1}}{P_{m_i}}}, \quad i = 2, 3, \cdots, K$$

. When the messages have the same power,

$$A_i = A_1 \sqrt{3i^2 - 3i + 1}, \quad i = 2, \cdots, K$$

## Question 2

**A superheterodyne FM receiver operates in the frequency range of 88-108 MHz. The IF and local oscillator frequencies are chosen such that $f_{IF} < f_{LO}$. We require that the image fre-**

**quency $f_c'$ fall outside of the 88-108 MHz region. Determine the minimum required $f_{IF}$ and the range of variation in $f_{LO}$.**

Since $88$ MHz $< f_c < 108$ MHz and $f_{IF} < f_{LO}$,

$$|f_c - f_c'| = 2f_{IF}$$

. We conclude that in order for the image frequency $f_c'$ to fall outside the interval $[88, 108]$ MHz, the minimum frequency $f_{IF}$ is such that

$$2f_{IF} = 108 - 88 \Rightarrow f_{IF} = 10 \text{ MHz}$$

If $f_{IF} = 10$ MHz, then the range of $f_{LO}$ is $[88 + 10, 108 + 10] = [98, 118]$ MHz.

## Question 3

**Show that the overall noise figure of a cascade of $n$ amplifiers with gains $\mathcal{G}_i$ and noise figures $\mathcal{F}_i$, as shown in Fig. 3, is**

$$F = F_1 + \frac{F_2 - 1}{\mathcal{G}_1} + \frac{F_3 - 1}{\mathcal{G}_1\mathcal{G}_2} + \cdots + \frac{F_n - 1}{\mathcal{G}_1\mathcal{G}_2\cdots\mathcal{G}_{n-1}}$$

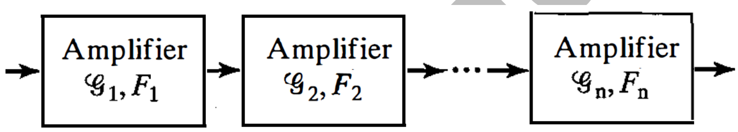


Figure 3: Cascade of several amplifiers.

Let the amplifier have the same physical temperature $T$ and the same noise equivalent bandwidth $B_{neq}$. The total output noise is

$$P_{n_o} = \mathcal{G}_1\cdots\mathcal{G}_nP_{n_s} + \mathcal{G}_2\cdots\mathcal{G}_nP_{n_{a_1}} + \mathcal{G}_3\cdots\mathcal{G}_nP_{n_{a_2}} + \cdots + P_{n_{a_n}}$$

, where $P_{n_s} = KTB_{neq}$ is the input noise to the first amplifier and $P_{n_{ai}}$ is the internal noise of amplifier $i$. Dividing both sides of the noise expression by $\mathcal{G}_1...\mathcal{G}_nP_{n_s}$, we get

$$\frac{P_{n_o}}{\mathcal{G}_1...\mathcal{G}_nP_{n_s}} = 1 + \frac{P_{n_{a_1}}}{\mathcal{G}_1P_{n_s}} + \frac{P_{n_{a_2}}}{\mathcal{G}_1\mathcal{G}_2P_{n_s}} + \cdots + \frac{P_{n_{a_n}}}{\mathcal{G}_1...\mathcal{G}_nP_{n_s}}$$

We know that the output noise of an amplifier is

$$P_{n_{o_i}} = P_{n_{ai}} + \mathcal{G}_iKTB_{neq} = P_{n_{ai}} + \mathcal{G}_iP_{n_s} = \mathcal{G}_iKB_{neq}(T + T_{e_i}) = \mathcal{G}_iP_{n_s}(1 + \frac{T_{e_i}}{T}) = \mathcal{G}_iP_{n_s}F_i$$

So,

$$F_i = 1 + \frac{P_{n_{ai}}}{\mathcal{G}_iP_{n_s}}, \quad \frac{P_{n_{o_i}}}{\mathcal{G}_iP_{n_s}} = F_i$$

*Question 3 continued on next page…*

If the cascade structure is modeled as an amplifier with the gain $\mathcal{G}_1 \cdots \mathcal{G}_n$, the physical temperature $T$, and the noise equivalent bandwidth $B_{neq}$, its corresponding noise figure

$$F = \frac{P_{n_{o_i}}}{\mathcal{G}_1 \cdots \mathcal{G}_n P_{n_s}} = [1 + \frac{P_{n_{a_1}}}{\mathcal{G}_1 P_{n_s}}] + \frac{1}{\mathcal{G}_1}[\frac{P_{n_{a_2}}}{\mathcal{G}_2 P_{n_s}}] + \cdots + \frac{1}{\mathcal{G}_1 ... \mathcal{G}_{n-1}} \frac{P_{n_{an}}}{\mathcal{G}_n P_{n_s}}$$

Therefore,

$$F = F_1 + \frac{F_2 - 1}{\mathcal{G}_1} + \frac{F_3 - 1}{\mathcal{G}_1 \mathcal{G}_2} + \cdots + \frac{F_n - 1}{\mathcal{G}_1 \mathcal{G}_2 \cdots \mathcal{G}_{n-1}}$$

## Question 4

**The lowpass signal $x(t)$ with a bandwidth of $W$ is sampled with a sampling interval of $T_s$, and the signal**

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT_s) p(t - nT_s)$$

**is reconstructed from the samples, where $p(t)$ is an arbitrary-shaped pulse (not necessarily time limited to the interval $[0, T_s]$).**

*(a) Find the Fourier transform of $x_p(t)$.*

We have

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT_s) p(t - nT_s) = p(t) * \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s) = p(t) * x_\delta(t)$$

Applying the Fourier Transform to both sides and using the convolution property, we get

$$X_p(f) = P(f) X_\delta(f) = \frac{P(f)}{T_s} \sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s})$$

*(b) Find the conditions for perfect reconstruction of $x(t)$ from $x_p(t)$.*

In order to avoid aliasing $\frac{1}{T_s} > 2W$. Furthermore, the spectrum $P(f)$ should be invertible for $|f| < W$.

*(c) Determine the required reconstruction filter.*

Considering that the conditions of perfect reconstruction are satisfied, we have

$$X(f) = X_p(f) H(f) = X_p(f) [\frac{T_s}{P(f)} \sqcap (\frac{f}{2W'})]$$

, where $W \leq W' \leq \frac{1}{T_s} - W$.

# Question 5

**Show that power spectral density of the pulse amplitude modulation signal $x(t) = \sum_{k=-\infty}^{\infty} a_k p(t-kD)$ is**

$$S_x(f) = \frac{1}{D}|P(f)|^2 \sum_{n=-\infty}^{\infty} R_a[n]e^{-j2\pi nfD}$$

**, where $P(f)$ is the Fourier transform of $p(t)$ and $R_a[n] = E\{a_{n+k}a_k\}$ is the autocorrelation of the stationary discrete random process $a_k$.**
**Hint: Use the definition of the power spectral density of a random process on page 68 of the slides on "Probability and Random Processes".**

According to the definition,

$$S_x(f) = \lim_{T\to\infty} \frac{1}{T}E\{|X_T(f)|^2\}$$

in which $X_T(f)$ is the Fourier transform of a truncated sample function $x_T(t) = x(t)\sqcap(\frac{t}{T})$. Let $T = (2K+1)D$. So, the limit $T \to \infty$ corresponds to $K \to \infty$. Then, for $K \gg 1$,

$$x_T(t) = \sum_{k=-K}^{K} a_k p(t-kD) \Rightarrow X_T(f) = \sum_{k=-K}^{K} a_k P(f)e^{-j2\pi fkD}$$

and

$$|X_T(f)|^2 = X_T(f)X_T^*(f) = |P(f)|^2 \sum_{k=-K}^{K} a_k e^{-j2\pi fkD} \sum_{i=-K}^{K} a_i e^{-j2\pi fiD}$$

where, without loss of generality, we have assumed that the symbols $a_i$ are real. After interchanging the order of expectation and summation, we have

$$E\{|X_T(f)|^2\} = |P(f)|^2\rho_K(f)$$

, where

$$\rho_K(f) = \sum_{k=-K}^{K}\sum_{i=-K}^{K} E\{a_k a_i\}e^{-j2\pi f(k-i)D} = \sum_{k=-K}^{K}\sum_{i=-K}^{K} R_a[k-i])e^{-j2\pi f(k-i)D}$$

since $E\{a_k a_i\} = R_a[k-i]$. The double summation for $\rho_K(f)$ can be manipulated into the single sum

$$\rho_K(f) = (2K+1)\sum_{n=-2K}^{2K}\big(1 - \frac{|n|}{2K+1}\big)R_a[n]e^{-j2\pi fnD}$$

It usually happens that $\lim_{n\to\infty} R_a[n] = 0$ meaning the distant symbols are orthogonal (uncorrelated if they are zero-mean). Substituting these expressions in the definition of $S_x(f)$ finally gives

$$S_x(f) = \lim_{K\to\infty} \frac{1}{(2K+1)D}|P(f)|^2\rho_K(f) = \frac{1}{D}|P(f)|^2 \sum_{n=-\infty}^{\infty} R_a[n]e^{-j2\pi nfD}$$

# SOFTWARE QUESTIONS

## Question 6

**A quantizer with $2^\nu$ quantized levels working over the input range $[-1, 1]$ is fed with a zero-mean Gaussian random process having the power $\sigma^2$. Develop a MATLAB function to calculate the signal to quntization noise ratio when the quntization intervals are uniformly distributed and when the quantization intervals are nonuniformly distributed according to $\mu$-law companding method with the parameter $\mu$. Discuss the results for different values of $\nu$, $\sigma^2$, and $\mu$. Feel free to plot any suitable curve to better describe the observations.**

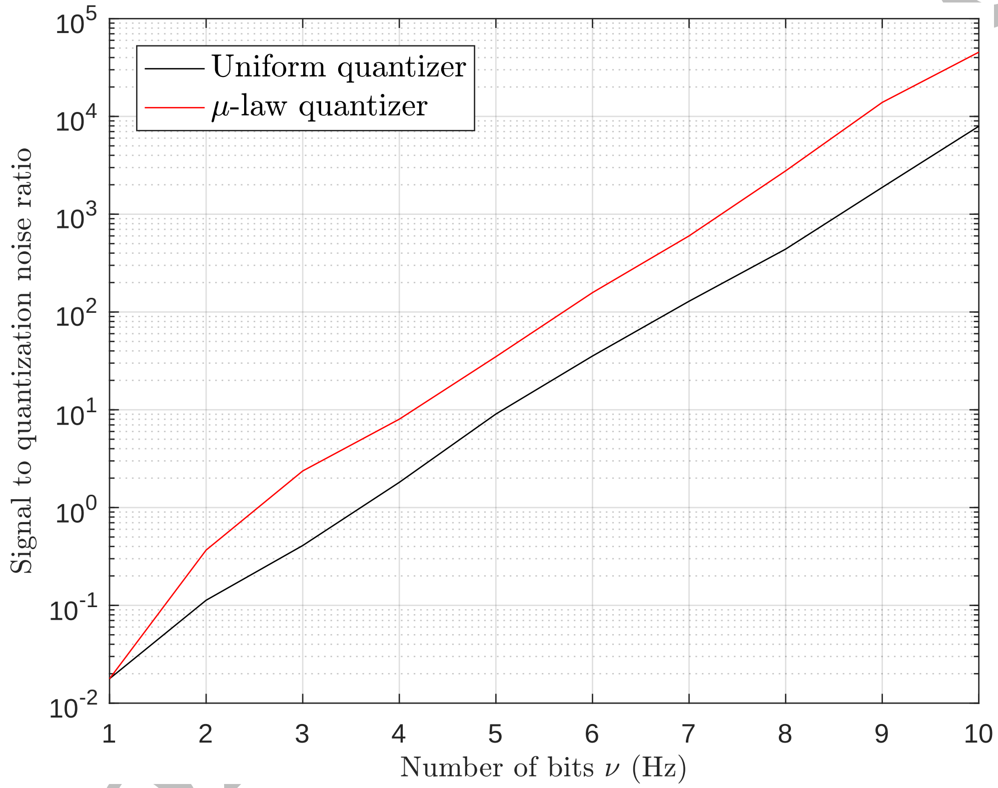


Figure 4: SQNR versus number of bits $\nu$ for $\sigma^2 = 0.01$ and $\mu = 22$.

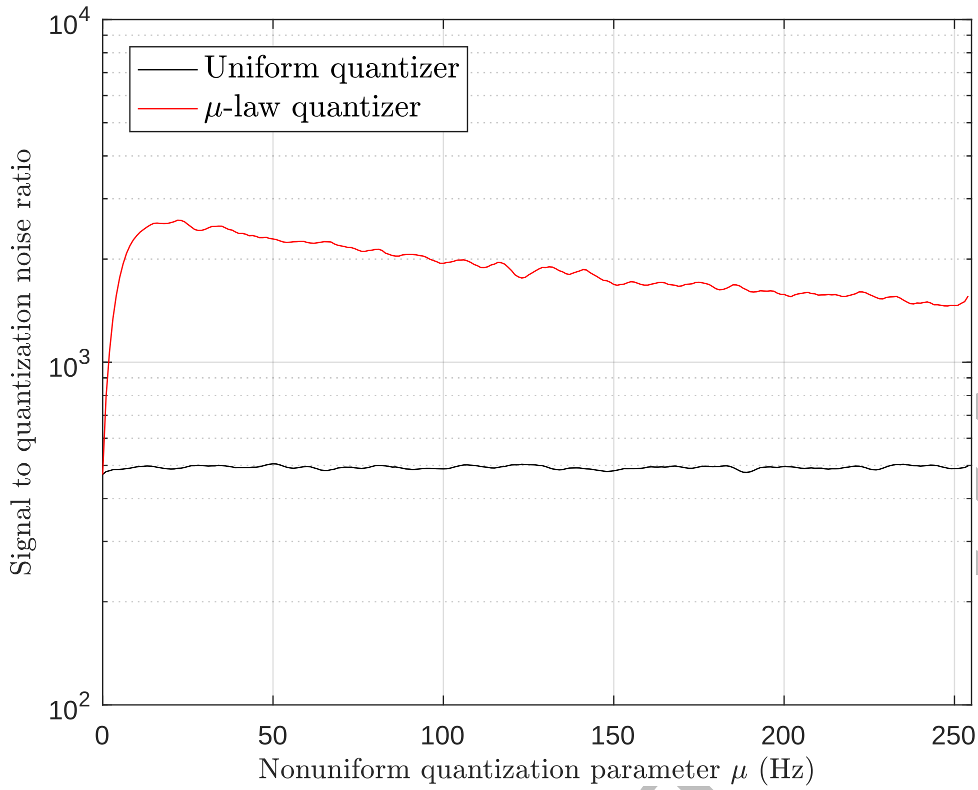


Figure 5: SQNR versus $\mu$-aw parameter $\mu$ for $\sigma^2 = 0.01$ and $\nu = 8$.

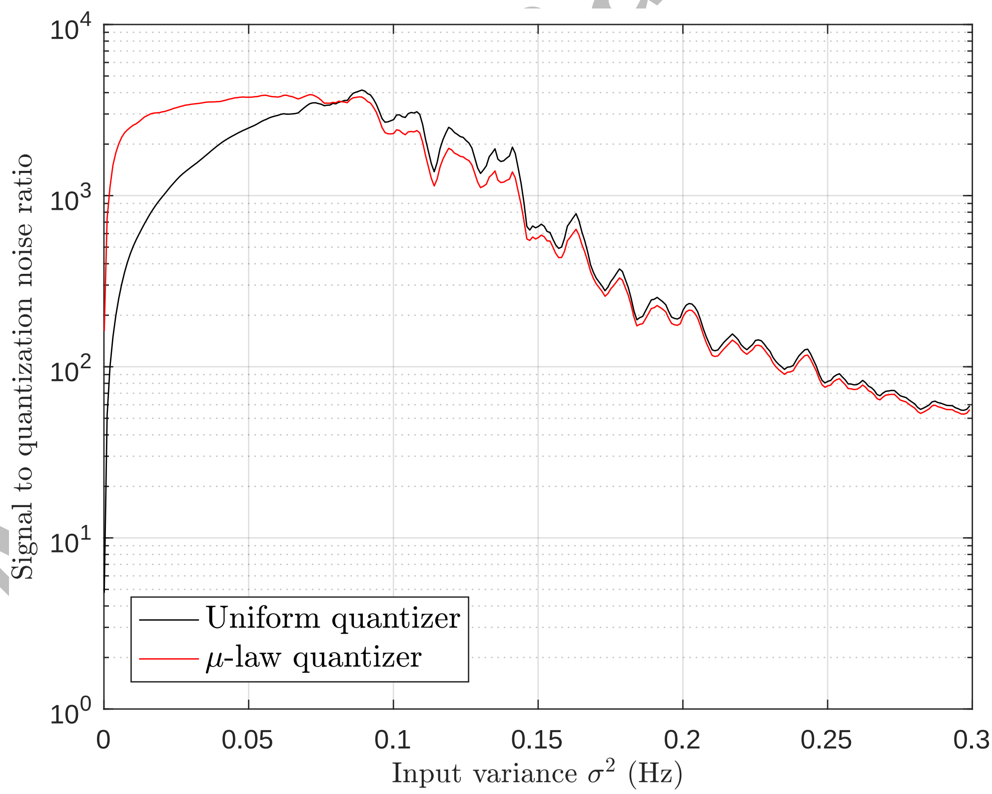


Figure 6: SQNR versus input variance $\sigma^2$ for $\mu = 22$ and $\nu = 8$.

*Question 6 continued on next page...*

The uniform quantizer can be implemented as.

```
function [value, level, bit] = uniform_quantizer(x, nu)
% quantization intervals
a=linspace(-1,1,2^nu+1);
a(1)=-Inf;
a(end)=[];
% quantization level
level=sum(x>=a)-1;
% quantization value
a(1)=-1;
value = a(level+1);
% quantization bit
bit=de2bi(bin2gray(level, 'pam', 2^nu), nu, 'left-msb');
```

A possible implementation of the $\mu$-law quantizer is

```
function [value, level, bit] = ulaw_quantizer(x, nu, mu)
% quantization intervals
a=linspace(-1,1,2^nu+1);
a(1)=-Inf;
a(end)=[];
b = ((exp(abs(a)*log(1+mu))-1)/mu).*sign(a);
% quantization level
level=sum(x>=b)-1;
% quantization value
b(1)=-1;
value = b(level+1);
% quantization bit
bit=de2bi(bin2gray(level, 'pam', 2^nu), nu, 'left-msb');
```

The following mfile calls the quantizers and computes their signal to quantization noise ratio

```
% settings
sigma = sqrt(0.01);
mu = 22;
nu = 8;
N = 100;

% average noise calculation
uniform_err = 0;
ulaw_err = 0;
for n=1:N
    x = sigma*randn;
    uniform_err = ((n-1)*uniform_err + (x-uniform_quantizer(x,nu))^2)/n;
    ulaw_err = ((n-1)*ulaw_err + (x-ulaw_quantizer(x,nu,mu))^2)/n;
end

% snr calculation
uniform_snr = sigma^2/uniform_err;
ulaw_snr = sigma^2/ulaw_err;

% snr gain
ulaw_snr/uniform_snr
```

As you can see in Fig. 4, increasing the number of bits $\nu$ improves SQNR. The SQNR of the $\mu$-law quantizer can be $5$ times the SQNR of the uniform quantizer. According to Fig. 5, the SQNR of the $\mu$-law quantizer is maximized for a suitable value of $\mu$. Finally, if the input variance is high, meaning that the input of the quantizer is not concentrated around zero, the $\mu$-law quantizer may work worse than the uniform quantizer, as can be seen in Fig. 6.

# BONUS QUESTIONS

## Question 7

**Mean-ergodicity is a useful feature, which allows to replace statistical means with time averages in communication analysis.**

*(a) Consider the real stationary process $X(t)$ with the statistical average $E\{X(t)\} = \eta$. Define the random variable $\eta_T$ as*

$$\eta_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt$$

*, which is called time average random variable. Show that $E\{\eta_T\} = \eta$.*

> Consider the random process
>
> $$W(t) = X(t) * [\frac{1}{2T} \sqcap (\frac{t}{2T})] = \frac{1}{2T}\int_{-T}^{T} X(t-\tau)d\tau = \frac{1}{2T}\int_{t-T}^{t+T} X(\alpha)d\alpha$$
>
> Clearly, $W(0) = \eta_T$. Further, $W(t)$ is a stationary process with the mean $H(0)\eta = \eta$, so $E\{\eta_T\} = \eta$, where $H(f) = \mathcal{F}\{\frac{1}{2T} \sqcap (\frac{t}{2T})\} = \text{sinc}(2Tf)$.

*(b) The process $X(t)$ is called mean-ergodic if and only if*

$$\lim_{T\to\infty} \textit{Var}\{\eta_T\} = 0$$

*. Show that this condition is equivalent to*

$$\lim_{T\to\infty} \frac{1}{T}\int_{0}^{2T} C(\alpha)(1 - \frac{\alpha}{2T})d\alpha = 0$$

*, where the autocovariance $C(\tau) = R(\tau) - \eta^2$ and $R(\tau)$ is the autocorrelation function of $X(t)$.*

> Clearly,
>
> $$S_W(f) = |H(f)|^2 S_X(f) = \text{sinc}^2(2Tf)S_X(f)$$
>
> So,
>
> $$R_W(\tau) = \mathcal{F}^{-1}\{S_W(f)\} = R_X(\tau) * [\frac{1}{2T}\Lambda(\frac{\tau}{2T})] = \frac{1}{2T}\int_{-2T}^{2T} R_X(\tau - \alpha)(1 - \frac{|\alpha|}{2T})d\alpha$$

Now,

$$\begin{aligned}
\text{Var}\{\eta_T\} &= C_W(0) = R_W(0) - \eta_W^2 \\
&= \frac{1}{2T}\int_{-2T}^{2T} R_X(-\alpha)(1-\frac{|\alpha|}{2T})d\alpha - \eta_X^2 \\
&= \frac{1}{2T}\int_{-2T}^{2T} R_X(\alpha)(1-\frac{|\alpha|}{2T})d\alpha - \eta_X^2\frac{1}{2T}\int_{-2T}^{2T}(1-\frac{|\alpha|}{2T})d\alpha \\
&= \frac{1}{2T}\int_{-2T}^{2T} (R_X(\alpha)-\eta_X^2)(1-\frac{|\alpha|}{2T})d\alpha \\
&= \frac{1}{2T}\int_{-2T}^{2T} C_X(\alpha)(1-\frac{|\alpha|}{2T})d\alpha \\
&= \frac{1}{2T}\int_{-2T}^{2T} C(\alpha)(1-\frac{|\alpha|}{2T})d\alpha \\
&= \frac{1}{T}\int_{0}^{2T} C(\alpha)(1-\frac{\alpha}{2T})d\alpha
\end{aligned}$$

*(c) Show that a zero-mean white process with a power spectral density of $\frac{N_0}{2}$ is mean-ergodic.*

We know that

$$R_X(\tau) = \frac{N_0}{2}\delta(\tau) \Rightarrow C_X(\tau) = \frac{N_0}{2}\delta(\tau) - 0^2 = \frac{N_0}{2}\delta(\tau)$$

. Therefore,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T} C(\alpha)(1-\frac{|\alpha|}{2T})d\alpha = \lim_{T\to\infty}\frac{N_0}{4T}\int_{-2T}^{2T}\delta(\alpha)d\alpha = \lim_{T\to\infty}\frac{N_0}{4T} = 0$$

*(d) How can we find the statistical mean of a zero-mean white process from the time average of its any given sample function?*

Since the process is mean-ergodic, the statistical average equals to the time average of any given sample function. So,

$$E\{X(t)\} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t)dt = \eta_\infty$$

## Question 8

**Return your answers by filling the LaTeXtemplate of the assignment.**

# EXTRA QUESTIONS

## Question 9

**Feel free to solve the following questions from the book *Fundamentals of Communication Systems* by J. Proakis and M. Salehi.**

1. **Chapter 4, question 19.**
2. **Chapter 6, question 14.**
3. **Chapter 6, question 15.**
4. **Chapter 7, question 6.**