Coupled Circuits

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Fall 2021

Overview

Coupled Inductors

Coupled Inductors

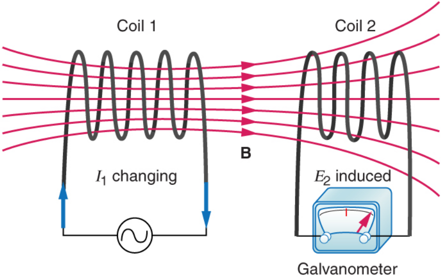


Figure: Coupled inductors as a method of wireless power transfer.

- Ampere's law: $\phi \propto f(i)$
- Faraday's law $v \propto \phi'$
- Ferromagnetic materials
- Mutual induction
- Self induction

Coupled Inductors

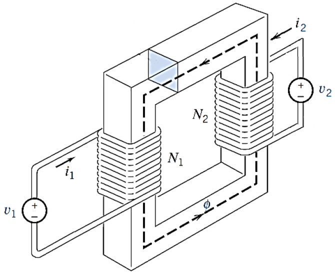


Figure: Coupled inductors.

- NTV coupled inductors: $\begin{cases} f_1(\phi_1, \phi_2, i_1, i_2, t) = 0 \\ f_2(\phi_1, \phi_2, i_1, i_2, t) = 0 \end{cases}$
- LTI coupled inductors: $\begin{cases} \phi_1 = L_1 i_1 \pm M_{12} i_2 \\ \phi_2 = \pm M_{21} i_1 + L_2 i_2 \end{cases} \Rightarrow \begin{cases} v_1(t) = L_1 i_1'(t) \pm M_{12} i_2'(t) \\ v_2(t) = \pm M_{21} i_1'(t) + L_2 i_2'(t) \end{cases}$

Dot Convention

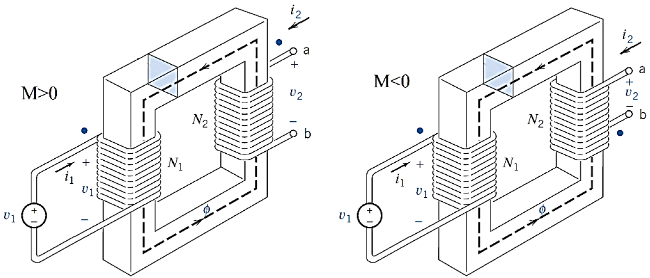


Figure: Additive and subtractive flux in coupled inductors.

- Additive flux:

$$\begin{cases} v_1(t) = L_1 i_1'(t) + M_{12} i_2'(t) \\ v_2(t) = +M_{21} i_1'(t) + L_2 i_2'(t) \end{cases}$$

- Subtractive flux:

$$\begin{cases} v_1(t) = L_1 i_1'(t) - M_{12} i_2'(t) \\ v_2(t) = -M_{21} i_1'(t) + L_2 i_2'(t) \end{cases}$$

Dot Convention

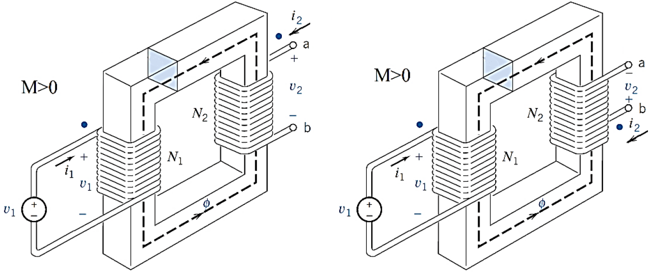


Figure: Dot convention, where the port are marked according to passive sign convention and the currents flow to the terminals that provide additive flux.

- Additive flux:

$$\begin{cases} v_1(t) = L_1 i_1'(t) + M_{12} i_2'(t) \\ v_2(t) = +M_{21} i_1'(t) + L_2 i_2'(t) \end{cases}$$

- Additive flux:

$$\begin{cases} v_1(t) = L_1 i_1'(t) + M_{12} i_2'(t) \\ v_2(t) = +M_{21} i_1'(t) + L_2 i_2'(t) \end{cases}$$

Absorbed Energy

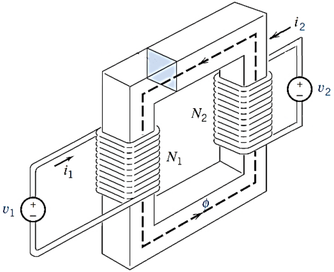


Figure: $M_{12} = M_{21} = M$ in coupled inductors.

- $(i_1(t_0), i_2(t_0)) = (0, 0)$: $\mathcal{E} = 0 + 0$
- $(i_1(t_1), i_2(t_1)) = (I_1, 0)$: $\mathcal{E} = 0.5L_1 I_1^2 + 0$
- $(i_1(t_2), i_2(t_2)) = (I_1, I_2)$: $\mathcal{E} = 0.5L_1 I_1^2 + M_{12} I_2 I_1 + 0.5L_2 I_2^2$
- $(i_1(t_0), i_2(t_0)) = (0, 0)$: $\mathcal{E} = 0 + 0$
- $(i_1(t_1), i_2(t_1)) = (0, I_2)$: $\mathcal{E} = 0 + 0.5L_2 I_2^2 + 0$
- $(i_1(t_2), i_2(t_2)) = (I_1, I_2)$: $\mathcal{E} = 0.5L_1 I_1^2 + M_{21} I_2 I_1 + 0.5L_2 I_2^2$
- $M_{12} = M_{21} = M$

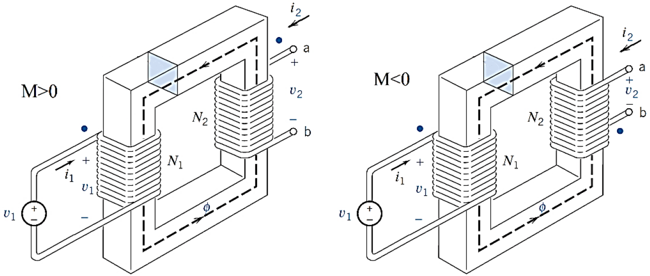


Figure: Absorbed energy in passive coupled inductors. The initial energy is zero.

- Additive flux:

$$\mathcal{E} = 0.5L_1 i_1^2(t) + M i_1(t) i_2(t) + 0.5 L_2 i_2^2(t) \geq 0$$

- Subtractive flux:

$$\mathcal{E} = 0.5L_1 i_1^2(t) - M i_1(t) i_2(t) + 0.5 L_2 i_2^2(t) \geq 0$$

- Absorbed energy: $\mathcal{E} = 0.5(\sqrt{L_1} i_1(t) - \sqrt{L_2} i_2(t))^2 + i_1(t) i_2(t)[\sqrt{L_1 L_2} \pm M] \geq 0$
- Coupling coefficient: $|M| \leq \sqrt{L_1 L_2} \Rightarrow k = \frac{|M|}{\sqrt{L_1 L_2}} \leq 1$

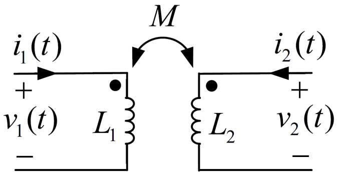


Figure: Circuit model of coupled inductors.

- Time-domain model: $\begin{cases} v_1(t) = L_1 i_1'(t) + M i_2'(t) \\ v_2(t) = M i_1'(t) + L_2 i_2'(t) \end{cases}$
- Time-domain model: $\begin{cases} i_1(t) = i_1(0) + \Gamma_{11} \int_0^t v_1(t')dt' + \Gamma_{12} \int_0^t v_2(t')dt' \\ i_2(t) = i_2(0) + \Gamma_{21} \int_0^t v_1(t')dt' + \Gamma_{22} \int_0^t v_2(t')dt' \end{cases}$
- Phasor-domain model: $\begin{cases} V_1 = j\omega L_1 I_1 + j\omega M I_2 \\ V_2 = j\omega M I_1 + j\omega L_2 I_2 \end{cases}$
- Phasor-domain model: $\begin{cases} I_1 = \frac{\Gamma_{11}}{j\omega} V_1 + \frac{\Gamma_{12}}{j\omega} V_2 \\ I_2 = \frac{\Gamma_{21}}{j\omega} V_1 + \frac{\Gamma_{22}}{j\omega} V_2 \end{cases}$
- Reciprocal inductance matrix: $\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} = L^{-1} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}^{-1}$

Circuit Model

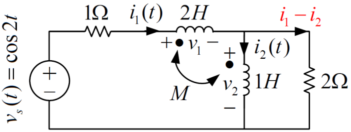
Example (Mesh phasor analysis)

Mesh analysis for the circuit below with $M = 0$ H yields $i_1(t) = 0.19\cos(2t - \angle 68.2^\circ)$ and $i_2(t) = 0.13\cos(2t - \angle 113.2^\circ)$.

$$\begin{cases} -1\angle 0 + I_1 + 4jI_1 + 2jI_2 = 0 \\ -2jI_2 + 2(I_1 - I_2) = 0 \end{cases}$$

$$\begin{cases} I_1 = 0.19\angle -68.2^\circ \\ I_2 = 0.13\angle -113.2^\circ \end{cases}$$

$$\begin{cases} i_1(t) = 0.19\cos(2t - \angle 68.2^\circ) \\ i_2(t) = 0.13\cos(2t - \angle 113.2^\circ) \end{cases}$$



Circuit Model

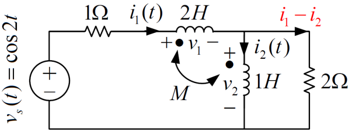
Example (Mesh phasor analysis)

Mesh analysis for the circuit below with $M = 1$ H yields $i_1(t) = 0.13\cos(2t - \underline{/50.2^\circ})$ and $i_2(t) = 0.13\cos(2t - \underline{/140.2^\circ})$.

$$\begin{cases} -1\underline{/0} + I_1 + 4jI_1 + 2jI_2 + 2jI_2 + 2jI_1 = 0 \\ -(2jI_2 + 2jI_1) + 2(I_1 - I_2) = 0 \end{cases}$$

$$\begin{cases} I_1 = 0.13\underline{/-50.2^\circ} \\ I_2 = 0.13\underline{/-140.2^\circ} \end{cases}$$

$$\begin{cases} i_1(t) = 0.13\cos(2t - \underline{/50.2^\circ}) \\ i_2(t) = 0.13\cos(2t - \underline{/140.2^\circ}) \end{cases}$$



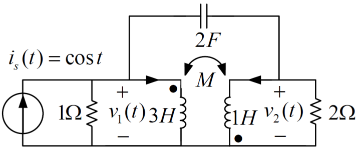
Example (Nodal phasor analysis)

Nodal analysis for the circuit below with $M = 0$ H yields $v_1(t) = 0.33\cos(t + \angle 30.53^\circ)$ and $v_2(t) = 0.13\cos(t + \angle 57.09^\circ)$.

$$\begin{cases} -1\angle 0 + \frac{V_1}{1} + \frac{V_1}{j3} + \frac{V_1 - V_2}{\frac{1}{j2}} = 0 \\ \frac{V_2}{2} + \frac{V_2}{j} + \frac{V_2 - V_1}{\frac{1}{j2}} = 0 \end{cases}$$

$$\begin{cases} V_1 = 0.33\angle 30.53^\circ \\ V_2 = 0.59\angle 57.09^\circ \end{cases}$$

$$\begin{cases} v_1(t) = 0.33\cos(t + \angle 30.53^\circ) \\ v_2(t) = 0.13\cos(t + \angle 57.09^\circ) \end{cases}$$



Circuit Model

Example (Nodal phasor analysis)

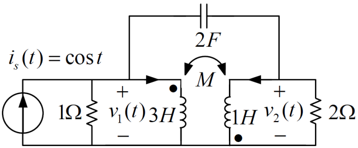
Nodal analysis for the circuit below with $M = 1$ H yields $v_1(t) = 0.12\cos(t + \underline{/33.23^\circ})$ and $v_2(t) = 0.41\cos(t + \underline{/78.23^\circ})$.

$$L = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \Gamma = L^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$$

$$\begin{cases} -1\underline{/0} + \frac{V_1}{1} + \frac{0.5V_1}{j} + \frac{0.5V_2}{j} + \frac{V_1 - V_2}{\frac{1}{j2}} = 0 \\ \frac{V_2}{2} + \frac{0.5V_1}{j} + \frac{1.5V_2}{j} + \frac{V_2 - V_1}{\frac{1}{j2}} = 0 \end{cases}$$

$$\begin{cases} V_1 = 0.12\underline{/33.23^\circ} \\ V_2 = 0.41\underline{/78.23^\circ} \end{cases}$$

$$\begin{cases} v_1(t) = 0.12\cos(t + \underline{/33.23^\circ}) \\ v_2(t) = 0.41\cos(t + \underline{/78.23^\circ}) \end{cases}$$



Multi-winding Inductors

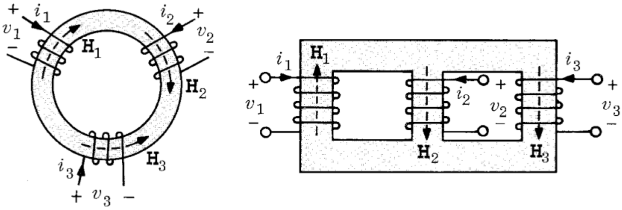


Figure: Passive three-winding coupled inductors.

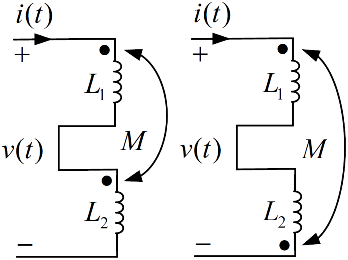
- Inductance matrix: $L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$
- Passivity condition: non-negative definite L

Example (Series connection of coupled inductors)

The inductance of the series connection of two coupled inductors is $L_{eq} = L_1 + L_2 + \pm 2|M|$.

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) \\ &= L_1 i'(t) + Mi'(t) + Mi'(t) + L_2 i'(t) \\ &= (L_1 + L_2 + 2M) i'(t) \end{aligned}$$

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) \\ &= L_1 i'(t) - Mi'(t) - Mi'(t) + L_2 i'(t) \\ &= (L_1 + L_2 - 2M) i'(t) \end{aligned}$$

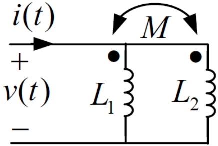


Series and Parallel Connection of Coupled Inductors

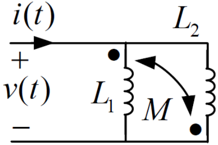
Example (Parallel connection of coupled inductors)

The reciprocal inductance of the parallel connection of two coupled inductors is $\Gamma_{eq} = \Gamma_{11} + \Gamma_2 22 + \pm 2|\Gamma_{12}|$.

$$
\begin{aligned}
i(t) &= i_1(t) + i_2(t) \\
&= \Gamma_{11} \int_0^t v(t')dt' + \Gamma_{12} \int_0^t v(t')dt' \\
&+ \Gamma_{12} \int_0^t v(t')dt' + \Gamma_{22} \int_0^t v(t')dt' \\
&= (\Gamma_{11} + \Gamma_{22} + 2\Gamma_{12}) \int_0^t v(t')dt'
\end{aligned}
$$



$$
\begin{aligned}
i(t) &= i_1(t) + i_2(t) \\
&= \Gamma_{11} \int_0^t v(t')dt' - \Gamma_{12} \int_0^t v(t')dt' \\
&- \Gamma_{12} \int_0^t v(t')dt' + \Gamma_{22} \int_0^t v(t')dt' \\
&= (\Gamma_{11} + \Gamma_{22} - 2\Gamma_{12}) \int_0^t v(t')dt'
\end{aligned}
$$

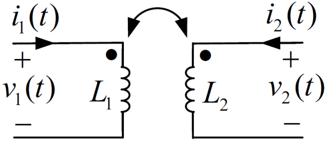


Equivalent Circuits for Coupled Inductors

Example (T equivalent circuit for coupled inductors)

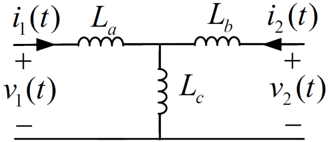
Two coupled inductors can be replaced with an inductive T network.

$$v_1(t) = L_1 i_1'(t) + M i_2'(t)$$
$$v_2(t) = M i_1'(t) + L_2 i_2'(t)$$



$$v_1(t) = L_a i_1'(t) + L_c(i_1'(t) + i_2'(t))$$
$$v_2(t) = L_b i_2'(t) + L_c(i_1'(t) + i_2'(t))$$

$$L_a = L_1 - M$$
$$L_b = L_2 - M$$
$$L_c = M$$



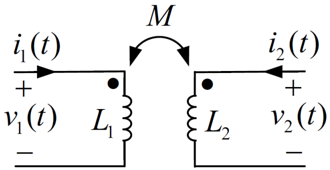
Equivalent Circuits for Coupled Inductors

Example ($\prod$ equivalent circuit for coupled inductors)

Two coupled inductors can be replaced with an inductive $\prod$ network.

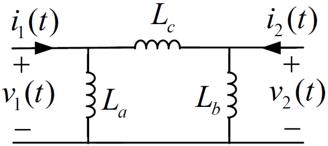
$$i_1(t) = \Gamma_{11} \int_0^t v_1(t')dt' + \Gamma_{12} \int_0^t v_2(t')dt'$$

$$i_2(t) = \Gamma_{21} \int_0^t v_1(t')dt' + \Gamma_{22} \int_0^t v_2(t')dt'$$



$$i_1(t) = \Gamma_a \int_0^t v_1(t')dt' + \Gamma_c \int_0^t [v_1(t') - v_2(t')]dt'$$

$$i_2(t) = \Gamma_b \int_0^t v_2(t')dt' + \Gamma_{22} \int_0^t [v_2(t') - v_1(t')]dt'$$



$$\Gamma_a = \Gamma_{11} + \Gamma_{12}$$

$$\Gamma_b = \Gamma_{22} + \Gamma_{12}$$

$$\Gamma_c = -\Gamma_{12}$$

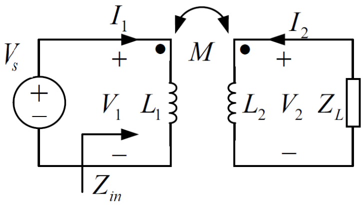
Example (Impedance changing by coupled inductors)

The input impedance seen from a coupled inductor is $Z_{in} = j\omega L_1 + M^2\omega^2/(j\omega L_2 + Z_L)$.

$$\begin{cases} V_s = j\omega L_1 I_1 + j\omega M I_2 \\ V_2 = j\omega L_2 I_2 + j\omega M I_1 \\ V_2 = -Z_L I_2 \end{cases}$$

$$\Rightarrow V_1 = [j\omega L_1 + M^2\omega^2/(j\omega L_2 + Z_L)] I_1$$

$$\Rightarrow Z_{in} = \frac{V_1}{I_1} = j\omega L_1 + \frac{M^2\omega^2}{j\omega L_2 + Z_L}$$



Example (Double-tuned Circuit)

Double-tuned circuit can have a wider bandwidth than the single-tined circuit.

$$I_+ = \frac{I_1 + I_2}{2}, \quad I_- = \frac{I_1 - I_2}{2}$$

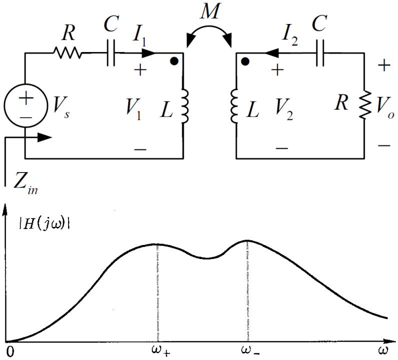
$$\omega_+^2 = \frac{1}{LC(1+k)}, \quad Q_+ = \omega_+ \frac{L+M}{R}$$

$$\omega_-^2 = \frac{1}{LC(1-k)}, \quad Q_- = \omega_- \frac{L-M}{R}$$

$$H_+(j\omega) = \frac{I_+}{V_s} = \frac{0.5/R}{1 + jQ_+(\frac{\omega}{\omega_+} - \frac{\omega_+}{\omega})}$$

$$H_-(j\omega) = \frac{I_-}{V_s} = \frac{0.5/R}{1 + jQ_-(\frac{\omega}{\omega_-} - \frac{\omega_-}{\omega})}$$

$$H(j\omega) = \frac{V_o}{V_s} = -R\frac{I_2}{V_s} = -R[H_+(j\omega) - H_-(j\omega)]$$



Ideal Transformers

Ideal Transformer

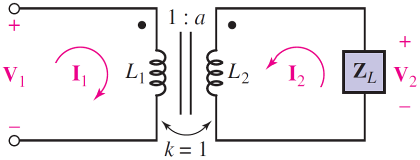


Figure: An ideal transformer is connected to a general load impedance.

- Ideal transformer assumptions: $L_1, L_2 \to \infty,\ L_2/L_1 = (n_2/n_1)^2 = a^2,\ k = 1$
- Circuit analysis: $\begin{cases} V_1 = j\omega L_1 I_1 + j\omega M I_2 \\ j\omega M I_1 + j\omega L_2 I_2 - Z_L I_2 = 0 \end{cases}$
- Current ratio: $\frac{I_2}{I_1} = \frac{j\omega M}{Z_L - j\omega L_2} \approx -\frac{M}{L_2} = -\sqrt{\frac{L_1}{L_2}} = -\frac{n_1}{n_2} = -\frac{1}{a}$
- Voltage ratio: $\frac{V_2}{V_1} = \frac{-I_2 Z_L}{I_1 Z_{in}} = \frac{1}{a}\frac{Z_L}{j\omega L_1 + \frac{\omega^2 M^2}{Z_L + j\omega L_2}} = \frac{1}{a}\frac{Z_L}{j\omega L_1 + \frac{\omega^2 a^2 L_1^2}{Z_L + j\omega a^2 L_1}} = \frac{1}{a}\frac{Z_L}{\frac{j\omega L_1 Z_L}{Z_L + j\omega a^2 L_1}} = a$

Multi-winding Ideal Transformer

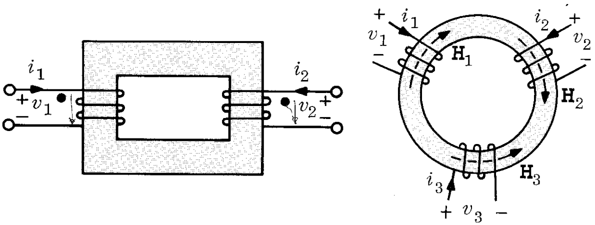


Figure: Two- and three-winding ideal transformers with the assumption of lossless operation without flux leakage.

- Two-winding transformer:
  - Voltage ratio: $\frac{v_2(t)}{v_1(t)} = \frac{\phi_2'(t)}{\phi_1'(t)} = \frac{n_2\phi'(t)}{n_1\phi'(t)} = \frac{n_2}{n_1}$
  - Current ratio: $v_1(t)i_1(t) + v_2(t)i_2(t) = 0 \Rightarrow \frac{i_2(t)}{i_1(t)} = -\frac{n_1}{n_2}$
- Three-winding transformer:
  - Voltage equation: $\frac{v_1(t)}{n_1} = \frac{v_2(t)}{n_2} = \frac{v_3(t)}{n_3}$
  - Current equation: $v_1(t)i_1(t) + v_2(t)i_2(t) + v_3(t)i_3(t) = 0$

Circuit Model

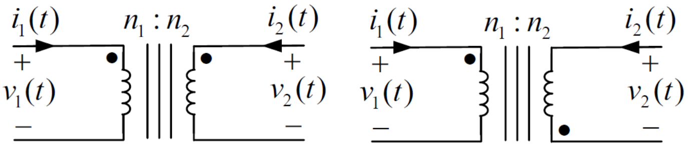


Figure: Circuit model of ideal transformers.

- Satisfied dot convention:

$$\begin{cases} \frac{v_2(t)}{v_1(t)} = \frac{n_2}{n_1} \\ \frac{i_2(t)}{i_1(t)} = -\frac{n_1}{n_2} \end{cases}$$

- Unsatisfied dot convention:

$$\begin{cases} \frac{v_2(t)}{v_1(t)} = -\frac{n_2}{n_1} \\ \frac{i_2(t)}{i_1(t)} = \frac{n_1}{n_2} \end{cases}$$

Impedance Changing by Ideal Transformer

Example (Impedance changing by ideal transformer)

The input impedance seen from an ideal transformer is $Z_{in} = (n_1/n_2)^2 Z_L$.

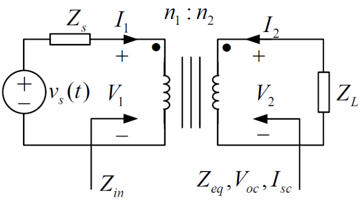
$$\begin{cases} \frac{V_1}{V_2} = \frac{n_1}{n_2} \\ \frac{I_1}{I_2} = -\frac{n_2}{n_1} \\ V_1 = V_s - Z_s I_1 \\ V_2 = -Z_L I_2 \end{cases}$$

$$Z_{in} = \frac{V_1}{I_1} = \frac{n_1 V_2/n_2}{-n_2 I_2/n_1} = (\frac{n_1}{n_2})^2 \frac{-V_2}{I_2} = (\frac{n_1}{n_2})^2 Z_L$$

$$Z_{eq} = \frac{V_2}{I_2}|_{V_s=0} = \frac{n_2 V_1/n_1}{-n_1 I_1/n_2} = (\frac{n_2}{n_1})^2 \frac{-V_1}{I_1} = (\frac{n_2}{n_1})^2 Z_s$$

$$V_{oc} = V_2|_{I_2=0} = \frac{n_2}{n_1} V_1 = \frac{n_2}{n_1} V_s$$

$$I_{sc} = -I_2|_{V_2=0} = \frac{n_1}{n_2} \frac{V_s}{Z_s}$$



Superposition in Ideal Transformers

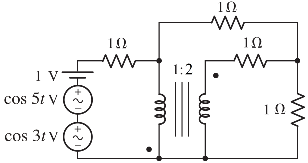
Example (Superposition in ideal transformers)

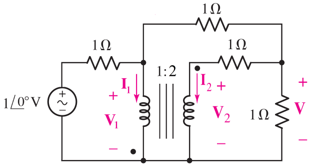
The average power consumed by the rightmost resistor is 0.0036 W.

$$\begin{cases} \frac{V_2}{-V_1} = 2 \\ \frac{I_2}{-I_1} = -\frac{1}{2} \\ V_1 + 1(I_1 + \frac{V_1 - V}{1}) = 1\angle 0^\circ \\ 1(I_2) + V_2 - V = 0 \\ -V_1 + 1(\frac{1-V_1}{1} - I_1) + 1(\frac{1-V_1}{1} - I_1 - I_2) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2V_1 + V_2 + 0I_1 + 0I_2 + 0V = 0 \\ 0V_1 + 0V_2 + I_1 - 2I_2 + 0V = 0 \\ 2V_1 + 0V_2 + I_1 + 0I_2 - V = 1 \\ 0V_1 + V_2 + 0I_1 + I_2 - V = 0 \\ -3V_1 + 0V_2 - 2I_1 - I_2 + 0V = -2 \end{cases}$$

$$\Rightarrow \begin{cases} V_1 = 0.18 \\ V_2 = -0.35 \\ I_1 = 0.59 \\ I_2 = 0.29 \\ V = -0.06 \end{cases}$$





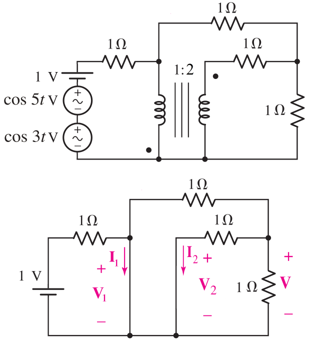
Example (Superposition in ideal transformers)

The average power consumed by the rightmost resistor is 0.0036 W.

$$v(t) = v_1(t) + v_2(t) + v_3(t)$$
$$= 0 - 0.06\cos(3t) - 0.06\cos(5t)$$

$$p(t) = [-0.06\cos(3t) - 0.06\cos(5t)]^2$$

$$P = \frac{0.06^2}{2} + \frac{0.06^2}{2} = 0.0036$$



The End