Question 1

The ABCD parameters of the two-port $\mathcal{N}$ and the turn ratio of the transformer in Fig. 1 are given.

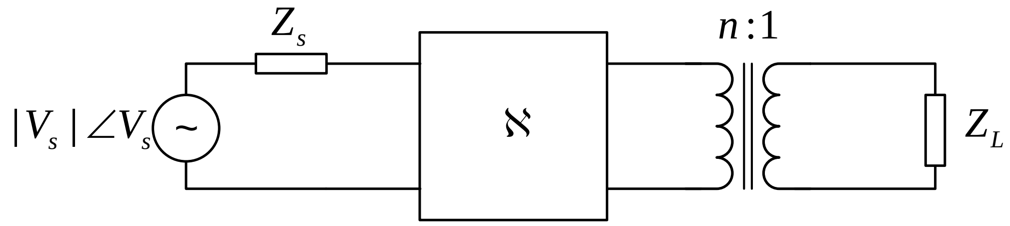

Figure 1: An LTI circuit for which the maximum power delivery condition is required.

(a) Find $Z_L(j\omega)$ that adsorbs the maximum power from the source.

The cascade connection of the two-port and transformer has the transmittance matrix

$$\boldsymbol{T} = \begin{bmatrix} A(j\omega) & B(j\omega) \\ C(j\omega) & D(j\omega) \end{bmatrix} \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} nA(j\omega) & \frac{B(j\omega)}{n} \\ nC(j\omega) & \frac{D(j\omega)}{n} \end{bmatrix}$$

The impedance seen from the load port is

$$Z_{eq}(j\omega) = \frac{\frac{B(j\omega)}{n} + \frac{D(j\omega)}{n} Z_s(j\omega)}{nA(j\omega) + nC(j\omega)Z_s(j\omega)} = \frac{1}{n^2} \frac{B(j\omega) + D(j\omega)Z_s(j\omega)}{A(j\omega) + C(j\omega)Z_s(j\omega)}$$

The maximum power is delivered if

$$Z_L(j\omega) = Z_{eq}^*(j\omega) = \frac{1}{n^2} \frac{B^*(j\omega) + D^*(j\omega)Z_s^*(j\omega)}{A^*(j\omega) + C^*(j\omega)Z_s^*(j\omega)}$$

(b) Evaluate the solution for $A(j\omega) = 1 + j, B(j\omega) = -1 + j4, C(j\omega) = \frac{1}{3}, D(j\omega) = 1 + \frac{j}{3}, n = 1, Z_s(j\omega) = 1$.

We have

$$Z_L(j\omega) = \frac{1}{n^2} \frac{B^*(j\omega) + D^*(j\omega)Z_s^*(j\omega)}{A^*(j\omega) + C^*(j\omega)Z_s^*(j\omega)} = \frac{-1 - j4 + 1 - \frac{j}{3}}{1 - j + \frac{1}{3}} = \frac{39}{25} - j\frac{52}{25}$$

Question 2

In Fig. 2, $\mathcal{N}_1$ and $\mathcal{N}_2$ are two identical LTI networks with different initial conditions. Here, the network $\mathcal{N}_1$ has nonzero initial conditions while network $\mathcal{N}_2$ is in rest and has zero initial conditions. In the first experimental setup, $\hat{v}(t) = u(t) + \delta(t)$ and $\hat{i}(t) = \delta(t) + \delta'(t)$. Find $v_L(t), t \geq 0$ in the second experiment if $v_C(0^-) = 8$ and $i_L(0^-) = -0.5$.

Question 2 continued on next page...

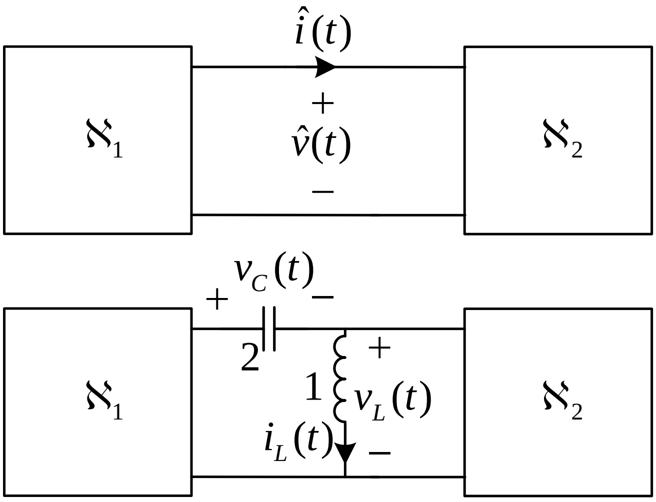

Figure 2: Two LTI networks in two experimental scenarios.

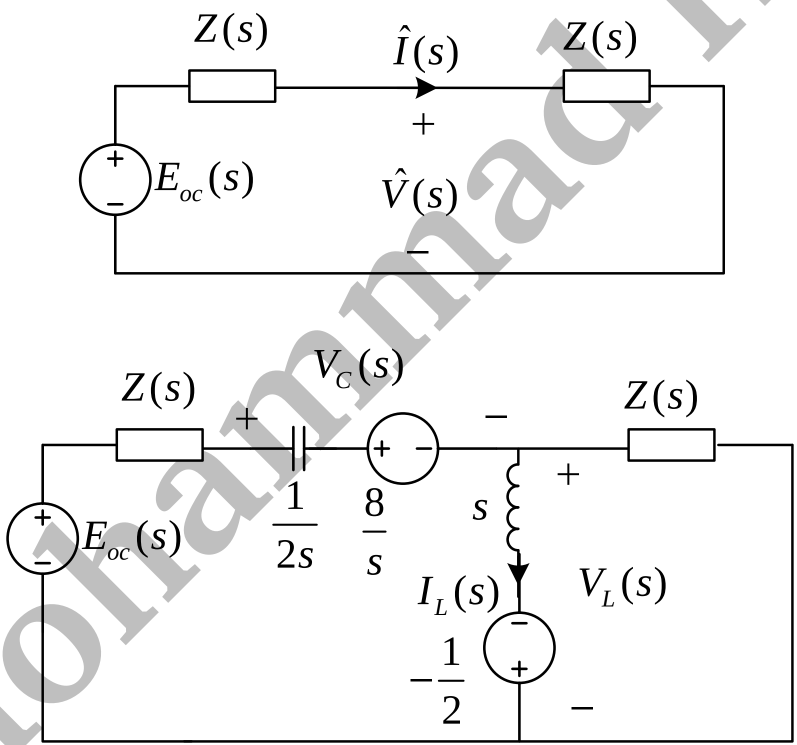

Figure 3: Equivalent circuits for the two experimental scenarios in Fig. 2.

The LTI networks can be replaced with their Thevenin equivalent circuits in the two experimental scenarios as shown in Fig. 3. Note that since $\mathcal{N}_2$ is in rest, no open circuit voltage exists in its Thevenin equivalent circuit. For the first experimental scenario,

$$\hat{V}(s) = \frac{1}{s} + 1, \quad \hat{I}(s) = 1 + s$$

So,

$$Z(s) = \frac{\hat{V}(s)}{\hat{I}(s)} = \frac{1}{s}, \quad E_{oc}(s) = 2Z(s)\hat{I}(s) = 2 + \frac{2}{s}$$

No, consider the second experimental setup whose circuit schematic is drawn in the Laplace domain in Fig. 3. We have,

$$\frac{V_L + \frac{8}{s} - 2 - \frac{2}{s}}{\frac{1}{2s} + \frac{1}{s}} + \frac{V_L - \frac{1}{2}}{s} + \frac{V_L}{\frac{1}{s}} = 0 \Rightarrow V_L = \frac{8s^2 - 24s + 3}{2\left(3 + 5s^2\right)} = -\frac{24s}{(10s^2+6)} - \frac{9}{5\left(10s^2+6\right)} + \frac{4}{5}$$

Finally,

$$v_L(t) = \frac{4}{5}\delta\left(t\right) - \frac{12}{5}\cos\left(\sqrt{\frac{3}{5}}t\right)u(t) - \frac{3\sqrt{3}}{10\sqrt{5}}\sin\left(\sqrt{\frac{3}{5}}t\right)u(t)$$

Question 3

For the circuit of Fig. 4,

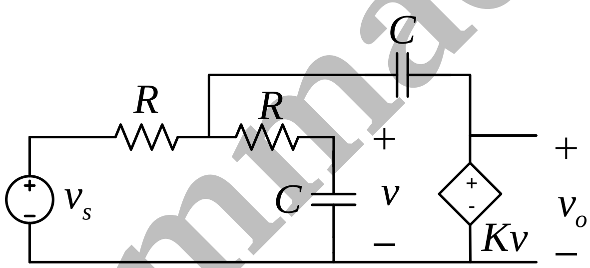

Figure 4: An LTI circuit.

(a) Find the transfer function $H(s) = \frac{V_o(s)}{V_s(s)}$.

Question 3 continued on next page. . .

Using the shortcut mesh analysis in Laplace domain,

$$\begin{bmatrix} 2R+\frac{1}{Cs} & -R-\frac{1}{Cs} \\ -R-\frac{1}{Cs} & R+\frac{2}{Cs} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_s \\ -K\frac{1}{Cs}(I_1-I_2) \end{bmatrix}$$

$$\begin{bmatrix} 2R+\frac{1}{Cs} & -R-\frac{1}{Cs} \\ -R-\frac{1}{Cs}+\frac{K}{Cs} & R+\frac{2}{Cs}-\frac{K}{Cs} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix}$$

$$H(s)=\frac{V_o}{V_s}=\frac{K(I_1-I_2)}{CsV_s}=\frac{K}{CsV_s}\frac{(RCs+2-K)CsV_s-(RCs+1-K)CsV_s}{(2RCs+1)(RCs+2-K)-(RCs+1)(RCs+1-K)}$$

So,

$$H(s)=\frac{K}{(RC)^2s^2+(3-K)RCs+1}$$

(b) *Find the values of K for which the zero-state response is stable.*

For stability, the poles should be in the left-side s-plane. We have

$$p_{1,2}=\frac{K-3\pm\sqrt{(K-5)(K-1)}}{2RC}$$

Now,

- If $K>5$, we have two real poles. The product of the two poles is the positive value $\frac{1}{R^2C^2}$. Since the sum of the two poles is $\frac{K-3}{R^2C^2}>0$, the two real poles are positive and the circuit is unstable.
- If $K<1$, we have two real poles. The product of the two poles is the positive value $\frac{1}{R^2C^2}$. Since the sum of the two poles is $\frac{K-3}{R^2C^2}<0$, the two real poles are negative and the circuit is stable.
- If $K=1$, we have two repeated real negative poles $\frac{-2}{RC}$ and the circuit is stable.
- If $K=5$, we have two repeated real positive poles $\frac{2}{RC}$ and the circuit is unstable.
- If $1<K<3$, we have two complex conjugate poles with negative real part and the circuit is stale.
- If $3<K<5$, we have two complex conjugate poles with positive real part and the circuit is unstable.
- If $K=3$, we have two imaginary conjugate poles with zero real part and the circuit is marginally stale.

Overall, the circuit is stable for $K<3$, is marginally stable for $K=3$, and is unstable for $K>3$.

(c) *Find the value of K for which the circuit acts like an oscillator.*

In this case the circuit should have its poles on the $j\omega$ axis. So, $K=3$.

Question 3 continued on next page...

(d) Find the impulse response if $K = 1$ and $RC = 0.5$.

We have,

$$H(s) = \frac{K}{(RC)^2s^2 + (3-K)RCs + 1} = \frac{4}{s^2+4s+4} = \frac{4}{(s+2)^2}$$

So,

$$h(t) = 4te^{-2t}u(t)$$

Question 4

For the circuit of Fig. 5,

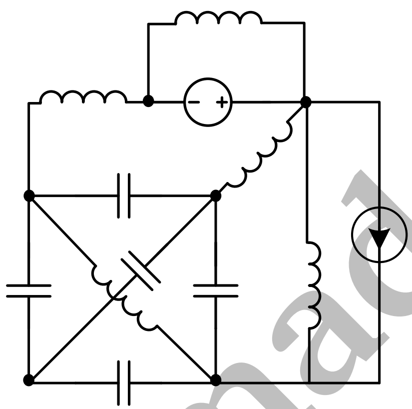

Figure 5: An LTI LC circuit.

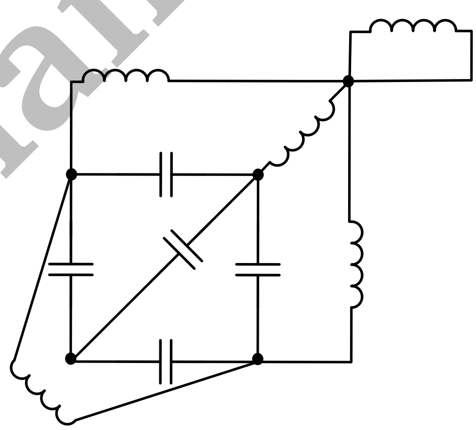

Figure 6: The unforced version of the circuit shown in Fig. 5.

(a) *Find the number of natural frequencies of the network.*

The circuit has 10 energy storage elements. Killing the independent sources as shown in Fig. 6, the unforced circuit has two independent capacitive loops and one independent inductive cut set. So, it has $10 - 2 - 1 = 7$ natural frequencies.

(b) *Find the number of zero natural frequencies of the network.*

Killing the independent sources, the unforced circuit has two independent inductive loops and one independent capacitive cut set. So, it has $2 + 1 = 3$ zero natural frequencies.

(c) *Find the number of state variables.*

The number of state variables equals the number of network natural frequencies, which is 7.

(d) *Find the circuit order.*

The circuit order equals the number of network natural frequencies, which is 7.

Question 5

A delta-connected positive-sequence balanced three-phase source drives the loads in Fig. 7. The source line voltage is $V_{ab} = 10\sqrt{3}\underline{/30^\circ}$ Vrms and $Z_1 = 18 + j12$, $Z_2 = 6 + j4$, and $Z_3 = 1 + j$.

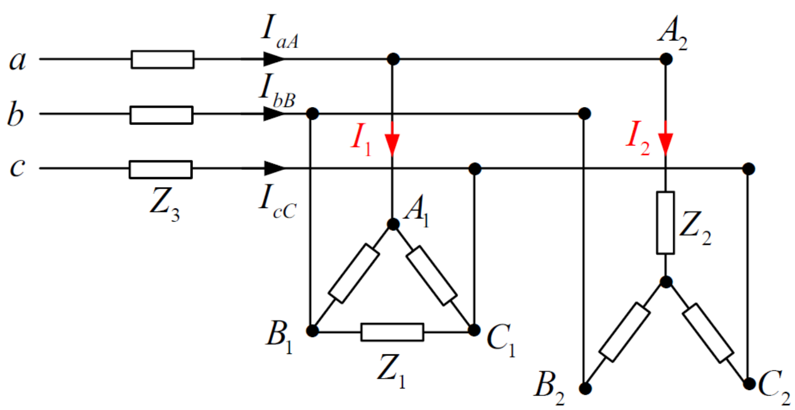

Figure 7: Two balanced interconnected three-phase loads.

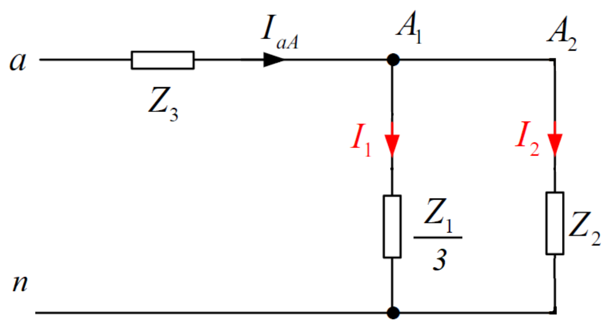

Figure 8: The equivalent single-phase circuit for the three-phase circuit shown in Fig. 7.

(a) Calculate I_1 and I_2.

The equivalent single-phase circuit is shown in Fig. 8, where $V_{an} = \frac{1}{\sqrt{3}}\angle{-30^\circ} V_{ab} = 10$. So,

$$I_{aA} = \frac{V_{an}}{\frac{Z_1}{3}||Z_2 + Z_3} = \frac{10}{(6+j4)||(6+j4)+(1+j)} = \frac{10}{3+j2+1+j} = \frac{10}{4+j3} = \frac{10}{5\angle{36.87^\circ}}$$

So,

$$I_{aA} = 2\angle{-36.87^\circ}$$
$$I_1 = \frac{Z_2}{\frac{Z_1}{3}+Z_2} I_{aA} = 1\angle{-36.87^\circ}$$
$$I_2 = \frac{\frac{Z_1}{3}}{\frac{Z_1}{3}+Z_2} I_{aA} = 1\angle{-36.87^\circ}$$

(b) Calculate I_{cC}.

$$I_{cC} = I_{aA} 1\angle{120^\circ} = 2\angle{83.13^\circ}$$

(c) Calculate the complex, reactive, and active powers delivered by the source.

$$S = 3V_{an}I_{aA}^* = 3 \times 10 \times 2\angle{36.87^\circ} = 60\angle{36.87^\circ}$$
$$P = \Re\{S\} = 60\cos(36.87^\circ) = 60 * 0.8 = 48$$
$$Q = \Im\{S\} = 60\sin(36.87^\circ) = 60 * 0.6 = 36$$

(d) Compute the load power factor seen by the source.

PF $= \cos(\phi) = 0.8$ lag

Question 6

For the circuit of Fig. 9, the initial conditions are $i_1(0^-) = I_{01}$, $i_2(0^-) = I_{02}$, $v_1(0^-) = V_{01}$, and $v_2(0^-) = V_{02}$.

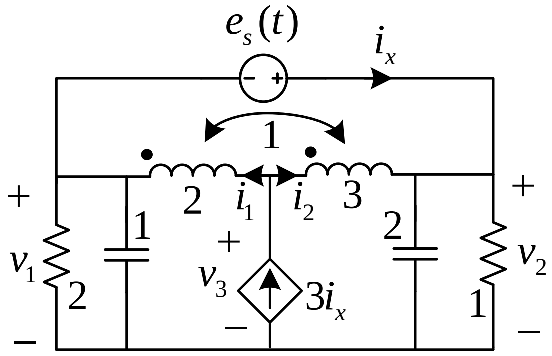

Figure 9: An LTI circuit for which different circuit equations are required.

(a) Find the Laplace-domain modified node equations.

Adding I_1, I_2, and I_x to the conventional unknown list V_1, V_2, V_3, we have

$$\begin{cases} sV_1 - V_{01} - I_1 + \frac{V_1}{2} + I_x = 0 \\ I_1 + I_2 - 3I_x = 0 \\ 2sV_2 - 2V_{02} - I_2 + V_2 - I_x = 0 \end{cases}$$

$$\begin{cases} V_2 - V_1 = E_s \\ V_1 - V_3 = 2s(-I_1) - 2(-I_{01}) + sI_2 - I_{02} \\ V_3 - V_2 = 3sI_2 - 3I_{02} + s(-I_1) - (-I_{01}) \end{cases}$$

Now,

$$\begin{bmatrix} s+\frac{1}{2} & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -3 \\ 0 & 2s+1 & 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2s & -s & 0 \\ 0 & -1 & 1 & s & -3s & 0 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ I_1(s) \\ I_2(s) \\ I_x(s) \end{bmatrix} = \begin{bmatrix} V_{01} \\ 0 \\ 2V_{02} \\ E_s(s) \\ 2I_{01} - I_{02} \\ I_{01} - 3I_{02} \end{bmatrix}$$

(b) Find the time-domain modified node equations.

$$\begin{bmatrix} D+\frac{1}{2} & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -3 \\ 0 & 2D+1 & 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2D & -D & 0 \\ 0 & -1 & 1 & D & -3D & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ i_1(t) \\ i_2(t) \\ i_x(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e_s(t) \\ 0 \\ 0 \end{bmatrix}$$

Question 6 continued on next page...

(c) *Find the phasor-domain modified node equations.*

$$\begin{bmatrix} j\omega+\frac{1}{2} & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -3 \\ 0 & 2j\omega+1 & 0 & 0 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2j\omega & -j\omega & 0 \\ 0 & -1 & 1 & j\omega & -3j\omega & 0 \end{bmatrix} \begin{bmatrix} V_1(j\omega) \\ V_2(j\omega) \\ V_3(j\omega) \\ I_1(j\omega) \\ I_2(j\omega) \\ I_x(j\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ E_s(j\omega) \\ 0 \\ 0 \end{bmatrix}$$

(d) *Find the state equations.*

We have,

$$\begin{cases} v_2 - v_1 = e_s \Rightarrow v_2 = v_1 + e_s \\ i_1 + i_2 - 3i_x = 0 \Rightarrow i_x = \frac{i_1+i_2}{3} \\ \frac{v_1}{2} + \frac{dv_1}{dt} - i_1 + i_x = 0 \Rightarrow \frac{dv_1}{dt} = -\frac{1}{2}v_1 + \frac{2}{3}i_1 - \frac{1}{3}i_2 \\ v_2 + 2\frac{dv_2}{dt} - i_2 - i_x = 0 \Rightarrow i_2 = \frac{1}{2}i_1 + \frac{1}{2}e_s + \frac{de_s}{dt} \\ e_s - 2\frac{di_1}{dt} + \frac{di_2}{dt} + 3\frac{di_2}{dt} - \frac{di_1}{dt} = 0 \Rightarrow \frac{di_1}{dt} = \frac{4}{3}\frac{di_2}{dt} + \frac{1}{3}e_s \end{cases}$$

Clearly, i_2 and v_2 depend on i_1 and v_1 and are not among the state variables. So, the state equations are

$$\begin{cases} \frac{dv_1}{dt} = -\frac{1}{2}v_1 + \frac{1}{2}i_1 - \frac{1}{6}e_s - \frac{1}{3}\frac{de_s}{dt} \\ \frac{di_1}{dt} = \frac{2}{3}e_s + \frac{de_s}{dt} + 2\frac{d^2e_s}{dt^2} \end{cases}$$

(e) *Find the network natural frequencies.*

The zero-input state equations are

$$\begin{cases} \frac{dv_1}{dt} = -\frac{1}{2}v_1 + \frac{1}{2}i_1 \\ \frac{di_1}{dt} = 0 \end{cases}$$

Or,

$$\frac{d}{dt}\begin{bmatrix} v_1(t) \\ i_1(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ i_1(t) \end{bmatrix}$$

The network natural frequencies are

$$\Delta(s) = \det[s\boldsymbol{I} - \boldsymbol{A}] = \begin{vmatrix} s+\frac{1}{2} & -\frac{1}{2} \\ 0 & s \end{vmatrix} = s(s+\frac{1}{2}) = 0 \Rightarrow s = 0, -\frac{1}{2}$$

(f) *Find the natural frequencies of* i_1.

The zero-input response of i_1 is

$$\frac{di_1}{dt} = 0 \Rightarrow i_1(t) = K$$

and it has a single zero natural frequency.