Natural Frequencies

Mohammad Hadi

mohammad.hadi@sharif.edu

@MohammadHadiDastgerdi

Fall 2021

Overview

Natural Frequencies of a Network Variable

Zero-input Response

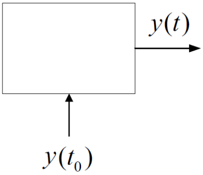


Figure: Zero-input response for an LTI circuit. Natural frequencies of a variable are the constant coefficients appeared in the exponents of the zero-input response giving that variable. The number of natural frequencies of a variable is equal to or less than the circuit order. s_i is a frequency response of order n_i.

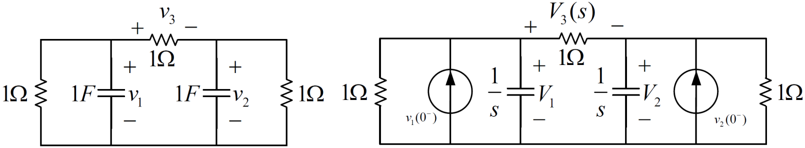
$$\sum_{k=0}^{n} a_k y^{(k)}(t) = 0, \quad y(0^-), y'(0^-), \cdots, y^{(n-1)}(0^-)$$

$$\sum_{k=0}^{n} \left[a_k s^k Y(s) - \sum_{k'=1}^{k} s^{k-k'} y^{k'-1}(0^-)\right] = 0 \Rightarrow Y(s) \sum_{k=0}^{n} a_k s^k - F_0(s) = 0 \Rightarrow Y(s) = \frac{F_0(s)}{\sum_{k=0}^{n} a_k s^k}$$

$$y(t) = \sum_{i=1}^{r} \sum_{j=1}^{n_i} K_{i,j} t^{j-1} e^{s_i t}$$

Example (Natural frequencies of a network variable)

The voltages V_1 and V_2 in the circuit below have two simple natural frequencies.



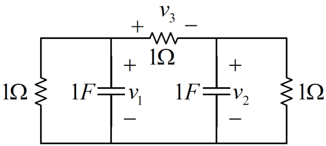
$$\begin{bmatrix} s+2 & -1 \\ -1 & s+2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} v_1(0^-) \\ v_2(0^-) \end{bmatrix} \Rightarrow \begin{cases} V_1 = \frac{(s+2)v_1(0^-)+v_2(0^-)}{(s+1)(s+3)} \\ V_2 = \frac{(s+2)v_2(0^-)+v_1(0^-)}{(s+1)(s+3)} \end{cases} \Rightarrow s = -1, -3$$

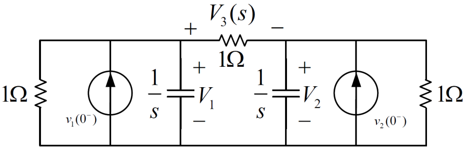
$$(v_1(0^-), v_2(0^-)) = (1,0) \Rightarrow \begin{cases} V_1 = \frac{s+2}{(s+1)(s+3)} \\ V_2 = \frac{1}{(s+1)(s+3)} \end{cases} , \quad (v_1(0^-), v_2(0^-)) = (0,0) \Rightarrow \begin{cases} V_1 = 0 \\ V_2 = 0 \end{cases}$$

$$(v_1(0^-), v_2(0^-)) = (1,-1) \Rightarrow \begin{cases} V_1 = \frac{1}{s+3} \\ V_2 = \frac{-1}{s+3} \end{cases} , \quad (v_1(0^-), v_2(0^-)) = (1,1) \Rightarrow \begin{cases} V_1 = \frac{1}{s+1} \\ V_2 = \frac{1}{s+1} \end{cases}$$

Example (Natural frequencies of a network variable)

The voltages V_1 and V_2 in the circuit below have two simple natural frequency.

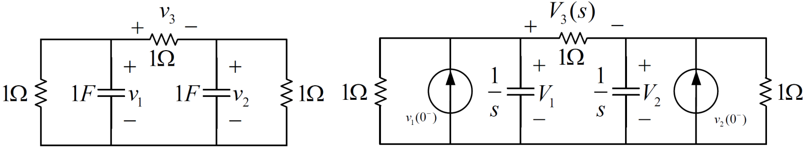




$$\begin{cases} v_1''(t) + 4v_1'(t) + 3v_1(t) = 0 \\ v_2''(t) + 4v_2'(t) + 3v_2(t) = 0 \end{cases} \Rightarrow s^2 + 4s + 3 = 0 \Rightarrow s = -1, -3$$

Example (Natural frequencies of a network variable)

The voltages V_3 in the circuit below has one simple natural frequency.



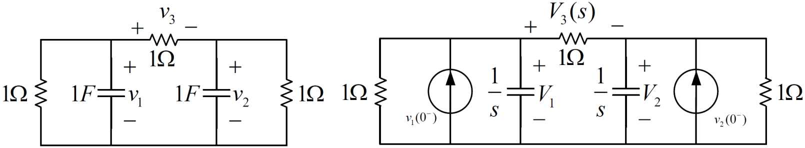
$$\begin{bmatrix} s+2 & -1 \\ -1 & s+2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} v_1(0^-) \\ v_2(0^-) \end{bmatrix} \Rightarrow \begin{cases} V_1 = \frac{(s+2)v_1(0^-)+v_2(0^-)}{(s+1)(s+3)} \\ V_2 = \frac{(s+2)v_2(0^-)+v_1(0^-)}{(s+1)(s+3)} \end{cases}$$

$$V_3 = V_1 - V_2 = \frac{(s+1)v_1(0^-) - (s+1)v_2(0^-)}{(s+1)(s+3)} = \frac{v_1(0^-) - v_2(0^-)}{s+3} \Rightarrow s = -3$$

$$(v_1(0^-), v_2(0^-)) = (0.5, -0.5) \Rightarrow V_3 = \frac{1}{s+3}, \quad (v_1(0^-), v_2(0^-)) = (1, 1) \Rightarrow V_3 = 0$$

Example (Natural frequencies of a network variable)

The voltages V_3 in the circuit below has one simple natural frequencies.



$$\begin{cases} v_1''(t) + 4v_1'(t) + 3v_1(t) = 0 \\ v_2''(t) + 4v_2'(t) + 3v_2(t) = 0 \end{cases}, v_3(t) = v_1(t) - v_2(t) \Rightarrow v_3''(t) + 4v_3'(t) + 3v_3(t) = 0 \Rightarrow s^2 + 4s + 3 = 0$$

$$v_3(t) = K_1 e^{-t} + K_2 e^{-3t}, \quad v_3(0^-) = v_1(0^-) - v_2(0^-), \quad v_3'(0^-) = -3[v_1(0^-) - v_2(0^-)]$$

$$\Rightarrow K_1 = 0, K_2 = v_1(0^-) - v_2(0^-) \Rightarrow v_3(t) = [v_1(0^-) - v_2(0^-)]e^{-3t}$$

Types of Natural Frequencies

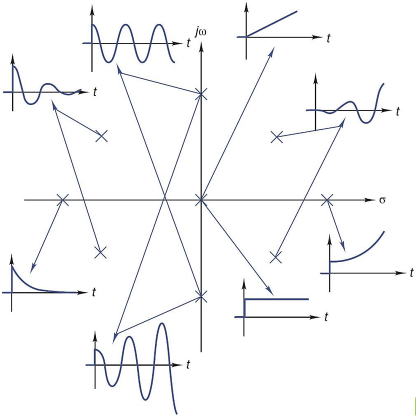


Figure: Natural frequencies can be simple or repeated, real, imaginary, or complex, and located in LHS, RHS, or $j\omega$ axis of the complex plane.

Types of Natural Frequencies

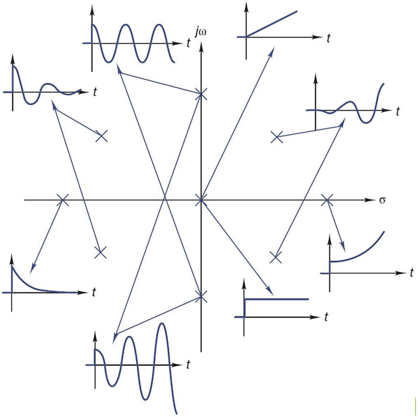


Figure: Types of natural frequencies.

- Transfer function denominator factors: $s, (s+\alpha), (s+\alpha)^m, (s^2+\omega^2)$ $, (s^2+\omega^2)^m, (s+\alpha)^2+\omega^2], [(s+\alpha)^2+\omega^2]^m$
- Time-domain behavior: $1, e^{-\alpha t}, t^m e^{-\alpha t}, \cos(\omega t+\theta), t^m \cos(\omega t+\theta)$ $, e^{-\alpha t}\cos(\omega t+\theta), t^m e^{-\alpha t}\cos(\omega t+\theta)$

Types of Natural Frequencies

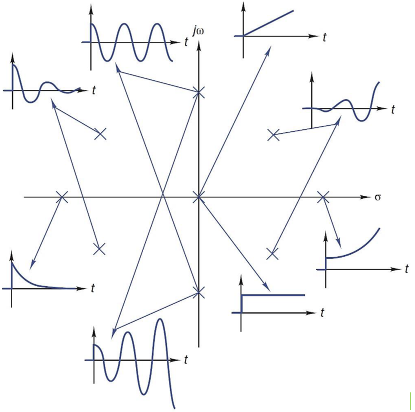


Figure: Impact of Natural frequencies on zero-input response stability.

- Stable circuit (strictly passive circuit): $\Re\{s_i\} < 0, \forall i$
- Marginally stable circuit (passive circuit): $\Re\{s_i\} \leq 0, \forall i$
- Unstable circuit (active circuit): $\Re\{s_i\} \geq 0, \exists i$

Zero Natural Frequencies

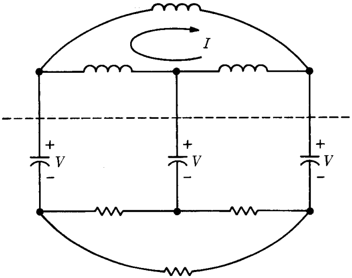


Figure: Zero natural frequency in capacitive cut sets and inductive loops of a passive circuit.

- Transfer function denominator factor: s
- Time-domain behavior: 1
- Zero natural frequency creation: Capacitive cut set, inductive loop, dependent sources (active circuit)
- Voltage and current of basic elements: May have different zero natural frequencies

Natural Frequencies of a Network

Natural Frequencies of a Network

Definition (Natural Frequencies of an LTI Network)

s_i is a natural frequency of an LTI network if s_i is a natural frequency of some voltage or a natural frequency of some current in the network.

Theorem (Calculation of the Natural Frequencies of a Network)

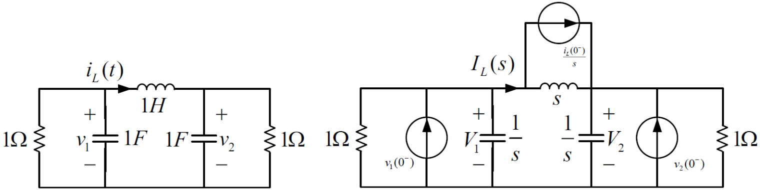
The nonzero natural frequencies of any linear time-invariant network are identical to the nonzero roots of the equation $\Delta(s) = det[\boldsymbol{P}(s)] = 0$, *where* $\boldsymbol{P}(s)$ *is the matrix of any system of differential equations which describe the network.*

- Admittance matrix: $\boldsymbol{Y}_n(s)\boldsymbol{E} = \boldsymbol{I}_0 + \boldsymbol{I}_s \Rightarrow \Delta_n(s) = \det[\boldsymbol{Y}_n(s)]$
- Impedance matrix: $\boldsymbol{Z}_m(s)\boldsymbol{I} = \boldsymbol{E}_0 + \boldsymbol{E}_s \Rightarrow \Delta_m(s) = \det[\boldsymbol{Z}_m(s)]$
- State matrix: $(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X} = \boldsymbol{X}_0 + \boldsymbol{B}\boldsymbol{W} \Rightarrow \Delta(s) = \det[s\boldsymbol{I} - \boldsymbol{A}]$
- Zero natural frequencies: Zero eigen values of $\boldsymbol{A}$ in state equations
- Zero natural frequencies: Capacitive cut sets and inductive loops in a circuit without dependent sources
- Number of natural frequencies of a network: Circuit order

Natural Frequencies of a Network

Example (Node analysis)

The circuit below has three simple natural frequencies.

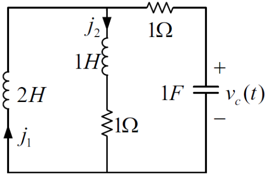


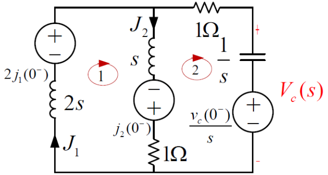
$$\begin{bmatrix} s+1+\frac{1}{s} & -\frac{1}{s} \\ -\frac{1}{s} & s+2+\frac{1}{s} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} v_1(0^-) - \frac{i_L(0^-)}{s} \\ v_2(0^-) + \frac{i_L(0^-)}{s} \end{bmatrix} \Rightarrow \begin{cases} V_1 = \frac{(s^2+s+1)v_1(0^-)+v_2(0^-)-(s+1)i_L(0^-)}{(s+1)(s^2+s+2)} \\ V_2 = \frac{(s^2+s+1)v_2(0^-)+v_1(0^-)+(s+1)i_L(0^-)}{(s+1)(s^2+s+2)} \\ I_L = \frac{V_1-V_2}{s} + \frac{i_L(0^-)}{s} = \frac{v_1(0^-)-v_2(0^-)+(s+1)i_L(0^-)}{s^2+s+2} \end{cases}$$

$$\Delta_n(s) = \det[\boldsymbol{Y}_n(s)] = \frac{(s+1)(s^2+s+2)}{s} = 0 \Rightarrow s = -1, \frac{-1 \pm j\sqrt{7}}{2}$$

Example (Mesh analysis)

The circuit below has one simple and two repeated natural frequencies.





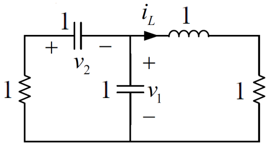
$$\begin{bmatrix} 3s+1 & -s-1 \\ -s-1 & s+2+\frac{1}{s} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 2j_1(0^-)+j_2(0^-) \\ -j_2(0^-)-\frac{v_c(0^-)}{s} \end{bmatrix} \Rightarrow \begin{cases} I_1 = \frac{2(s+1)j_1(0^-)+j_2(0^-)-v_c(0^-)}{(s+1)(2s+1)} \\ I_2 = \frac{2s(s+1)j_1(0^-)2s^2j_2(0^-)-(3s+1)v_c(0^-)}{(s+1)^2(2s+1)} \\ V_c = \frac{I_2}{s}+\frac{v_c(0^-)}{s} = \frac{2(s+1)j_1(0^-)-2sj_2(0^-)-(3s+5)v_c(0^-)}{(s+1)^2(2s+1)} \end{cases}$$

$$\Delta_m(s) = \det[\boldsymbol{Z}_m(s)] = \frac{(s+1)^2(2s+1)}{s} = 0 \Rightarrow s = -1, -1, -0.5$$

Natural Frequencies of a Network

Example (State analysis)

The circuit below has three repeated natural frequencies.



$$\boldsymbol{X}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ i_L(t) \end{bmatrix}, \boldsymbol{X}_0 = \boldsymbol{X}(0) = \begin{bmatrix} v_1(0^-) \\ v_2(0^-) \\ i_L(0^-) \end{bmatrix} \Rightarrow \frac{d\boldsymbol{X}}{dt} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \boldsymbol{X}$$

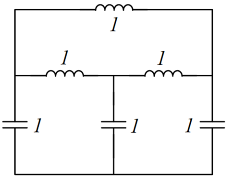
$$s\boldsymbol{I} - \boldsymbol{A} = \begin{bmatrix} s+1 & 1 & 1 \\ 1 & s+1 & 0 \\ -1 & 0 & s+1 \end{bmatrix} \Rightarrow \Delta(s) = \det[s\boldsymbol{I} - \boldsymbol{A}] = -[0 - (s+1)] + (s+1)[(s+1)^2 - 1]$$

$$\Delta(s) = (s+1)^3 \Rightarrow s = -1, -1, -1$$

Natural Frequencies of a Network

Example (Number of network natural frequencies)

The circuit below has order 6, so it has 6 natural frequencies including two zero natural frequencies and 4 repeated imaginary conjugate natural frequencies.



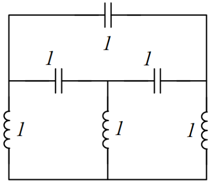
$$\boldsymbol{Y}_n(s) = \begin{bmatrix} s + \frac{1}{s} + \frac{1}{s} & -\frac{1}{s} & -\frac{1}{s} \\ -\frac{1}{s} & s + \frac{1}{s} + \frac{1}{s} & -\frac{1}{s} \\ -\frac{1}{s} & -\frac{1}{s} & s + \frac{1}{s} + \frac{1}{s} \end{bmatrix}$$

$$\Delta_n(s) = \det[\boldsymbol{Y}_n(s)] = \frac{(s^2+3)^2}{s} = 0 \Rightarrow s = -j\sqrt{3}, +j\sqrt{3}, -j\sqrt{3}, +j\sqrt{3}$$

Natural Frequencies of a Network

Example (Number of network natural frequencies)

The circuit below has order 4, so it has 4 natural frequencies including 4 repeated imaginary conjugate natural frequencies.



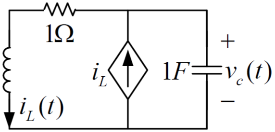
$$\boldsymbol{Y}_n(s) = \begin{bmatrix} s+s+\frac{1}{s} & -s & -s \\ -s & s+s+\frac{1}{s} & -s \\ -s & -s & s+s+\frac{1}{s} \end{bmatrix}$$

$$\Delta_n(s) = \det[\boldsymbol{Y}_n(s)] = \frac{(3s^2+1)^2}{s} = 0 \Rightarrow s = -j\frac{1}{\sqrt{3}}, +j\frac{1}{\sqrt{3}}, -j\frac{1}{\sqrt{3}}, +j\frac{1}{\sqrt{3}}$$

Natural Frequencies of a Network

Example (Number of network natural frequencies)

The circuit below has order 2, so it has 2 natural frequencies including 1 simple natural frequency and 1 simple nonzero real natural frequency.



$$\begin{cases} v_C' = i_L - i_L = 0 \\ i_L' = v_C - i_L \end{cases} \quad \Rightarrow \boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

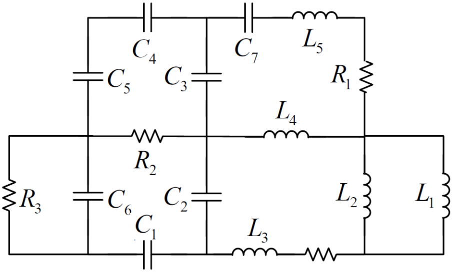
$$\Delta(s) = |s\boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} s & 0 \\ -1 & s+1 \end{vmatrix} = s(s+1) = 0 \Rightarrow s = 0, -1$$

Natural Frequencies of a Network

Example (Number of network natural frequencies)

The circuit below has

1. 7 capacitors and 5 inductors
2. 1 independent capacitive loop and 2 independent inductive cut set
3. 2 independent capacitive cut set and 1 independent inductive loop
4. $7 + 5 - 1 - 2 = 9$ natural frequencies and order 9
5. $1 + 2 = 3$ zero natural frequencies and $9 - 3 = 6$ nonzero natural frequencies



Minimal Differential Equation

Definition (Minimal Differential Equation)

Minimal differential equation of a variable is the smallest-order differential equation whose corresponding characteristic equation contains all the natural frequencies of the variable.

- D Operator: $Df(t) = \frac{df(t)}{dt}, \quad D^{-1}f(t) = \int_{0^-}^{t} f(\tau)d\tau$
- Elimination method: systematic method to obtain minimal differential equation
  - Variable reordering
  - Integral replacement
  - Elementary row operations
  - Upper triangle (diagonal) differential matrix equation
- Elementary row operations:
  - Row swap: $\mathcal{E}_k \leftrightarrow \mathcal{E}_j$
  - Row multiplication: $\mathcal{E}_k \leftarrow m\mathcal{E}_k$
  - Row sum: $\mathcal{E}_k \leftarrow \mathcal{E}_k + \mathcal{E}_j$
  - Row mixed operation: $\mathcal{E}_k \leftarrow m\mathcal{E}_k + p(D)\mathcal{E}_j$

Minimal Differential Equation

Example (Minimal differential equation)

The minimal differential equation of the voltage in a zero-input RC circuit is first-order.

$$C\frac{dv(t)}{dt}+\frac{v(t)}{R}=0, v(0)=V_0 \Rightarrow Cs+R=0 \Rightarrow s=-\frac{1}{RC} \Rightarrow v(t)=V_0e^{-\frac{t}{RC}}$$

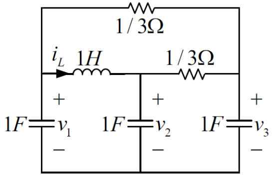
$$C\frac{d^2v(t)}{dt^2}+\frac{1}{R}\frac{dv(t)}{dt}=0, v(0)=V_0, v'(0)=-\frac{V_0}{RC} \Rightarrow Cs^2+Rs=0$$

$$\Rightarrow s=0,-\frac{1}{RC} \Rightarrow v(t)=K_1+K_2e^{-\frac{t}{RC}} \Rightarrow \begin{cases} K_1+K_2=V_0 \\ K_2\frac{-1}{RC}=-\frac{V_0}{RC} \end{cases} \Rightarrow K_1=0, K_2=V_0 \Rightarrow v(t)=V_0e^{-\frac{t}{RC}}$$

Minimal Differential Equation

Example (Minimal differential equation)

The minimal differential equation of v_3 in the circuit below can be obtained using the elimination method.



$$\begin{cases} Dv_1 + D^{-1}(v_1 - v_2) + i_L(0^-) + 3(v_1 - v_3) = 0 \\ Dv_2 + D^{-1}(v_2 - v_1) - i_L(0^-) + 3(v_2 - v_3) = 0 \\ Dv_3 + 3(v_3 - v_1) + 3(v_3 - v_2) = 0 \end{cases}$$

$$\begin{bmatrix} D+3+D^{-1} & -D^{-1} & -3 \\ -D^{-1} & D+3+D^{-1} & -3 \\ -3 & -3 & D+6 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = \begin{bmatrix} -i_L(0^-) \\ i_L(0^-) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} D^2+3D+1 & -1 & -3 \\ -1 & D^2+3D+1 & -3 \\ -3D & -3D & D+6 \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \\ v_3(t) \end{bmatrix} = \begin{bmatrix} -i_L(0^-) \\ i_L(0^-) \\ 0 \end{bmatrix}$$

Minimal Differential Equation

Example (Minimal differential equation (cont.))

The minimal differential equation of v_3 in the circuit below can be obtained using the elimination method.

$$\begin{bmatrix} D+3+D^{-1} & -D^{-1} & -3 & -i_L(0^-) \\ -D^{-1} & D+3+D^{-1} & -3 & i_L(0^-) \\ -3 & -3 & D+6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} D^2+3D+1 & -1 & -3 & -i_L(0^-) \\ -1 & D^2+3D+1 & -3 & i_L(0^-) \\ -3D & -3D & D+6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & D^2+3D+1 & -3 & i_L(0^-) \\ D^2+3D+1 & -1 & -3 & -i_L(0^-) \\ -3D & -3D & D+6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & D^2+3D+1 & -3 & i_L(0^-) \\ 0 & -1+(D^2+3D+1)(D^2+3D+1) & -3-3(D^2+3D+1) & -i_L(0^-)+(D^2+3D+1)i_L(0^-) \\ 0 & -3D-3D(D^2+3D+1) & D+6-3D(-3) & 0-3Di_L(0^-) \end{bmatrix}$$

$$\begin{bmatrix} -1 & D^2+3D+1 & -3 & i_L(0^-) \\ 0 & D^4+6D^3+11D^2+6D & -3D^2-9D-6 & 0 \\ 0 & -3D^3-9D^2-6D & 10D+6 & 0 \end{bmatrix}$$

Minimal Differential Equation

Example (Minimal differential equation (cont.))

The minimal differential equation of v_3 in the circuit below can be obtained using the elimination method.

$$\begin{bmatrix} -1 & D^2+3D+1 & -3 & i_L(0^-) \\ 0 & -3D^3-9D^2-6D & 10D+6 & 0 \\ 0 & D^4+6D^3+11D^2+6D & -3D^2-9D-6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & D^2+3D+1 & -3 & i_L(0^-) \\ 0 & -3D^3-9D^2-6D & 10D+6 & 0 \\ 0 & 0 & -3D^2-9D-6+(\frac{1}{3}D+1)(10D+6) & 0+(\frac{1}{3}D+1)0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & D^2+3D+1 & -3 & i_L(0^-) \\ 0 & -3D^3-9D^2-6D & 10D+6 & 0 \\ 0 & 0 & \frac{1}{3}D^2+3D & 0 \end{bmatrix}$$

$$(\frac{1}{3}D^2+3D)v_3(t)=0 \Rightarrow v_3''+9v_3'=0 \Rightarrow s^2+9s=0 \Rightarrow s=0,-9$$

The End