

Network Functions

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Overview

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- 2 Frequency Response
- 3 Poles
- 4 Zeros

Transfer Functions

Complete Response

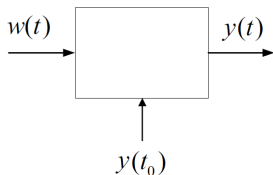


Figure: Laplace analysis confirms that for linear systems, the complete response is the sum of zero-input and zero-state responses. Further, it confirms that the zero-state response of an LTI system is totally described by a transfer function.

$$\sum_{k=0}^n a_k y^{(k)}(t) = \sum_{l=0}^m b_l w^{(l)}(t), \quad y(0^-), y'(0^-), \dots, y^{(n-1)}(0^-)$$

$$\sum_{k=0}^n [a_k s^k Y(s) - \sum_{k'=1}^k s^{k-k'} y^{(k'-1)}(0^-)] = \sum_{l=0}^m b_l s^l W(s)$$

$$Y(s) \sum_{k=0}^n a_k s^k - F_0(s) = W(s) \sum_{l=0}^m b_l s^l$$

$$Y(s) = \frac{\sum_{l=0}^m b_l s^l}{\sum_{k=0}^n a_k s^k} W(s) + \frac{F_0(s)}{\sum_{k=0}^n a_k s^k}$$

$$Y(s) = H(s)W(s) + \frac{F_0(s)}{\sum_{k=0}^n a_k s^k}$$

Zero-state Response



Figure: **Transfer function** for zero-state response of LTI systems.

- **Laplace-domain zero-state response:** $Y(s) = H(s)W(s)$
- **Transfer function:** $H(s) = \frac{\sum_{l=0}^m b_l s^l}{\sum_{k=0}^n a_k s^k}$
- **Time-domain zero-state response:** $y(t) = h(t) * w(t)$, $h(t) = \mathcal{L}^{-1}[H(s)]$
- **Frequency response:** $H(j\omega) = H(s)|_{s=j\omega}$
- **Multi-input zero-state response:**
 $Y(s) = \sum_i H_i(s)W_i(s)$, $H_i(s) = \frac{Y(s)}{W_i(s)}|_{W_k(s)=0, k \neq i}$

Transfer Function

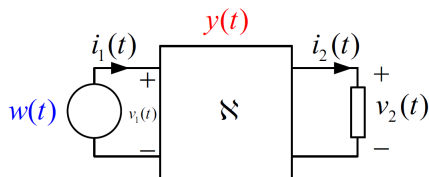


Figure: **Transfer function** for a network without internal independent sources.

- **Transfer function:** $H(s) = \frac{Y_{zs}(s)}{W(s)}$
- **Real rational function:** $H(s) = \frac{\sum_{l=0}^m b_l s^l}{\sum_{k=0}^n a_k s^k}$
- **Transfer function zeros:** $\{s_i \in \mathcal{C} \mid H(s_i) = 0\}$
- **Transfer function poles:** $\{s_i \in \mathcal{C} \mid H(s_i) = \infty\}$
- **Zero-pole decomposition:** $H(s) = \frac{b_0}{a_0} \frac{\prod_{l=1}^m (s - z_l)}{\prod_{k=1}^n (s - p_k)}$

Transfer Function

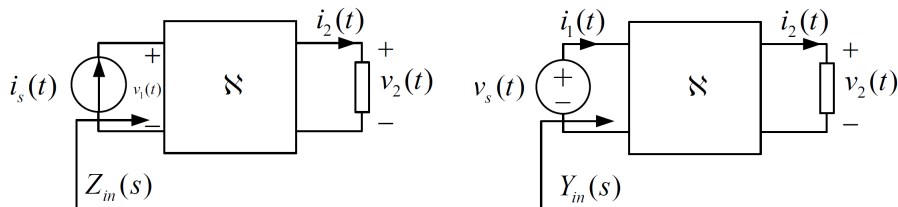


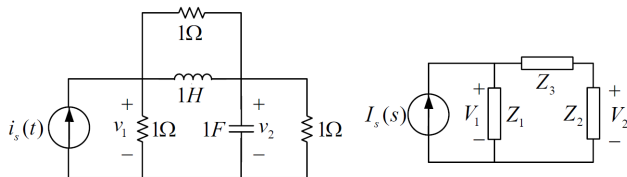
Figure: Types of **Transfer functions**.

- **Input impedance:** $H(s) = \frac{V_1(s)}{I_s(s)}$
- **Transfer impedance:** $H(s) = \frac{V_2(s)}{I_s(s)}$
- **Input admittance:** $H(s) = \frac{I_1(s)}{V_s(s)}$
- **Transfer admittance:** $H(s) = \frac{I_2(s)}{V_s(s)}$
- **Voltage gain:** $H(s) = \frac{V_2(s)}{V_s(s)}$
- **Current gain:** $H(s) = \frac{I_2(s)}{I_s(s)}$

Transfer Function

Example (Node analysis)

Node analysis can be used to find the transfer functions of the node voltages in the circuit below.



$$\begin{bmatrix} 1 + 1 + \frac{1}{s} & -1 - \frac{1}{s} \\ -1 - \frac{1}{s} & 1 + 1 + s + \frac{1}{s} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_s \\ 0 \end{bmatrix} \Rightarrow \begin{cases} V_1 = \frac{1}{2} I_s \\ V_2 = \frac{1}{2(s+1)} I_s \end{cases} \Rightarrow \begin{cases} H_1 = \frac{V_1}{I_s} = \frac{1}{2} \\ H_2 = \frac{V_2}{I_s} = \frac{1}{2(s+1)} \end{cases}$$

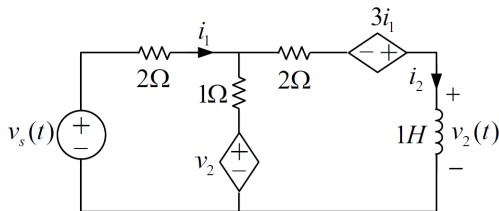
$$\begin{cases} H_1 = \frac{V_1}{I_s} = Z_{in} = Z_1 || (Z_2 + Z_3) = \frac{1}{2} \\ H_2 = \frac{V_2}{I_s} = \frac{V_1}{I_s} \frac{V_2}{V_1} = Z_{in} \frac{Z_2}{Z_2 + Z_3} = \frac{1}{2(s+1)} \end{cases}$$

$$S_1(s) = H_1(s) \frac{1}{s} = \frac{1}{2s} \Rightarrow s_1(t) = 0.5u(t), \quad H_2(s) = \frac{1}{2(s+1)} \Rightarrow h_2(t) = 0.5e^{-t}u(t)$$

Transfer Function

Example (Mesh analysis)

Node analysis can be used to find the transfer functions of the mesh currents in the circuit below.



$$\begin{bmatrix} 2 + 1 & -1 \\ -1 & 1 + 2 + s \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_s - s i_2 \\ s i_2 + 3 i_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -1 + s \\ -1 - 3 & s + 3 - s \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} i_1 = \frac{3}{4s+5} V_s \\ i_2 = \frac{4}{4s+5} V_s \end{cases} \Rightarrow \begin{cases} H_1 = \frac{i_2}{V_s} = \frac{3}{4s+5} \\ H_2 = \frac{i_1}{V_s} = \frac{4}{4s+5} \end{cases}$$

$$H_3 = \frac{V_2}{V_s} = s \frac{i_2}{V_s} = \frac{4s}{4s+5} \Rightarrow S_3(s) = \frac{s}{s+1.25} \frac{1}{s} \Rightarrow s_3(t) = e^{-1.25t} u(t)$$

Frequency Response

Frequency Response

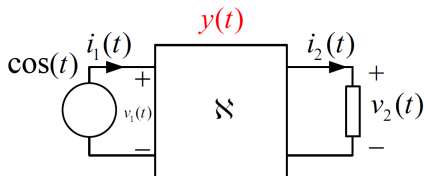


Figure: Frequency response for a network without internal independent sources.

- Measurable frequency response: $H(j\omega) = H(s)|_{s=j\omega}$
- Hermitian symmetry: $H(-j\omega) = H^*(j\omega)$
- Even magnitude response: $|H(j\omega)| = |H(-j\omega)|$
- Odd phase response: $\angle H(j\omega) = -\angle H(-j\omega)$
- Even real-part response: $\Re\{H(j\omega)\} = \Re\{H(-j\omega)\}$
- Odd imaginary-part response: $\Im\{H(j\omega)\} = -\Im\{H(-j\omega)\}$

Bode Diagram

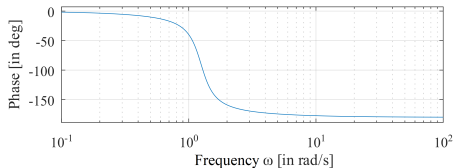
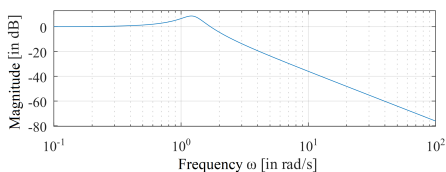


Figure: Frequency response represented in Bode diagram.

- **Bode diagram:** Curves of magnitude and phase responses for $\omega \geq 0$
- **Axis scale:** Linear or logarithmic
- **Output frequency response:** $Y(j\omega) = H(j\omega)W(j\omega)$
- **Output magnitude response:** $|Y(j\omega)| = |H(j\omega)||W(j\omega)|$
- **Output phase response:** $\angle Y(j\omega) = \angle H(j\omega) + \angle W(j\omega)$
- **Sinusoidal steady state response:** $y(t) = A|H(j\omega)| \cos(\omega t + \theta + \angle H(j\omega))$

Zero-Pole Diagram

- **Frequency response:**

$$H(j\omega) = \frac{b_0}{a_0} \frac{\prod_{l=1}^m (j\omega - z_l)}{\prod_{k=1}^n (j\omega - p_k)}$$

- **Magnitude response:**

$$|H(j\omega)| = \left| \frac{b_0}{a_0} \right| \frac{\prod_{l=1}^m |j\omega - z_l|}{\prod_{k=1}^n |j\omega - p_k|} = \left| \frac{b_0}{a_0} \right| \frac{\prod_{l=1}^m d_l}{\prod_{k=1}^n d_k}$$

- **Phase response:**

$$\angle H(j\omega) = \angle \frac{b_0}{a_0} + \sum_{l=1}^m \angle (j\omega - z_l) - \sum_{k=1}^n \angle (j\omega - p_k) = \angle \frac{b_0}{a_0} + \sum_{l=1}^m \phi_l - \sum_{k=1}^n \theta_k$$

- **Pole close to the $j\omega$ axis:** High magnitude and rapid phase change
- **Zero close to the $j\omega$ axis:** Low amplitude and rapid phase change

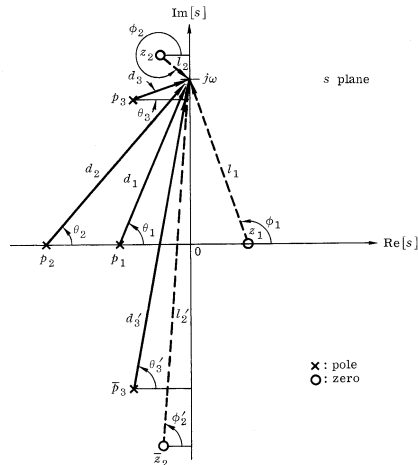


Figure: Interpretation of frequency response in zero-pole diagram.

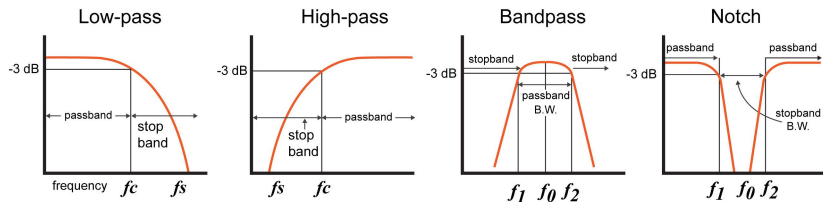


Figure: Frequency response of different filters.

- Filter types

- Lowpass filter (LPF)
- Highpass filter (HPF)
- Bandpass filter (BPF)
- Bandstop filter (Notch)

- 3-dB cut off frequency: $|H(j\omega_c)| = \frac{\max\{|H(j\omega)|\}}{\sqrt{2}}$

Example (Parallel RC LPF)

A parallel RC circuit can act like a lowpass filter.

$$H(s) = \frac{V}{I} = Z_{in} = \frac{1}{Y_{in}} = \frac{1}{G + Cs} = \frac{1}{C} \frac{1}{s + \frac{1}{RC}}$$

$$h(t) = \frac{1}{C} e^{-\frac{t}{RC}}$$

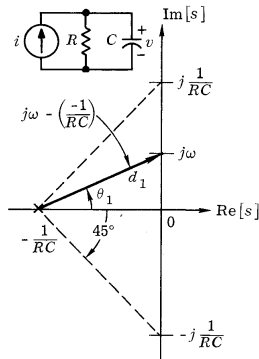
$$H(j\omega) = \frac{1}{C} \frac{1}{j\omega - (-\frac{1}{RC})}$$

$$|H(j\omega)| = \frac{R}{\sqrt{1 + (RC\omega)^2}} = \frac{1}{C} \frac{1}{\sqrt{\omega^2 + (\frac{1}{RC})^2}} = \frac{1}{C} \frac{1}{d_1}$$

$$\angle H(j\omega) = -\tan^{-1}(RC\omega) = -\theta_1$$

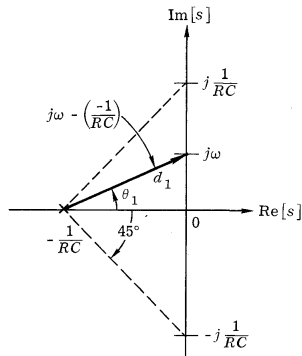
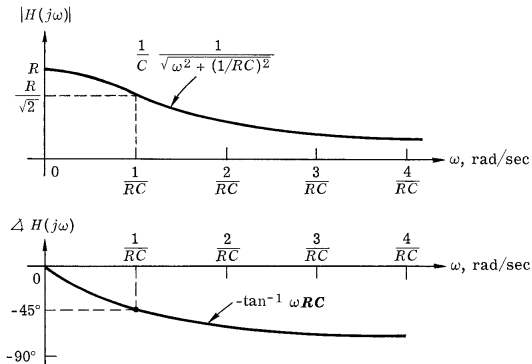
$$\begin{cases} |H(j0)| = R \\ |H(j\infty)| = 0 \\ |H(j\omega_c)| = |H(j\frac{1}{RC})| = \frac{R}{\sqrt{2}} \end{cases}$$

$$\begin{cases} \angle H(j0) = 0 \\ \angle H(j\infty) = -\frac{\pi}{2} \\ \angle H(j\omega_c) = \angle H(j\frac{1}{RC}) = -\frac{\pi}{4} \end{cases}$$



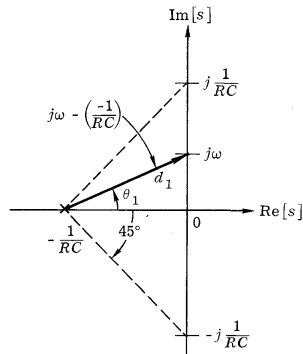
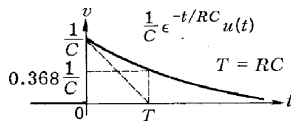
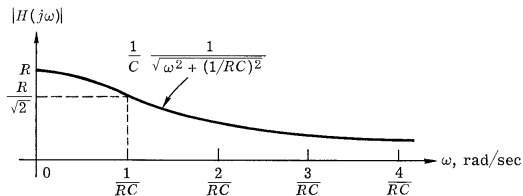
Example (Parallel RC LPF (cont.))

A parallel RC circuit can act like a lowpass filter.



Example (Parallel RC LPF (cont.))

A parallel RC circuit can act like a lowpass filter.



Example (Parallel RLC BPF)

A parallel RLC circuit can act like a bandpass filter.

$$H(s) = \frac{V}{I} = Z_{in} = \frac{1}{Y_{in}} = \frac{1}{\frac{1}{R} + \frac{1}{Ls} + Cs}$$

$$\alpha = \frac{1}{2RC} < \omega_0 = \frac{1}{\sqrt{LC}}, Q = \frac{\omega_0}{2\alpha}, \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

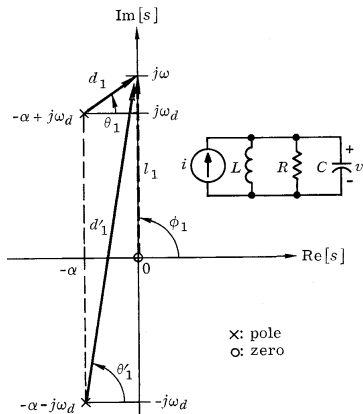
$$H(s) = \frac{s/C}{s^2 + 2\alpha s + \omega_0^2} = \frac{1}{C} \frac{s}{(s + \alpha + j\omega_d)(s + \alpha - j\omega_d)}$$

$$h(t) = \frac{1}{C} e^{-\alpha t} [\cos(\omega_d t) - \sin(\omega_d t)] u(t)$$

$$H(j\omega) = \frac{1}{C} \frac{j\omega - 0}{[j\omega - (-\alpha + j\omega_d)][j\omega - (-\alpha - j\omega_d)]}$$

$$|H(j\omega)| = \frac{R}{\sqrt{1 + Q^2 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}} = \frac{1}{C} \frac{h_1}{d_1 d'_1}$$

$$\angle H(j\omega) = -\tan^{-1} \left[Q \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \right] = \phi_1 - \theta_1 - \theta'_1$$

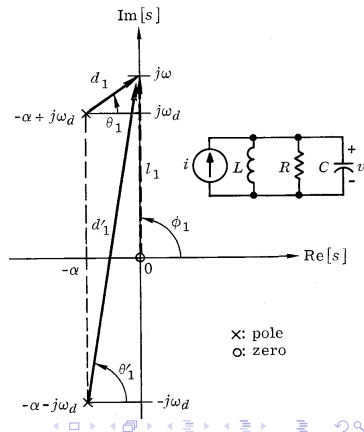


Example (Parallel RLC BPF)

A parallel RLC circuit can act like a bandpass filter.

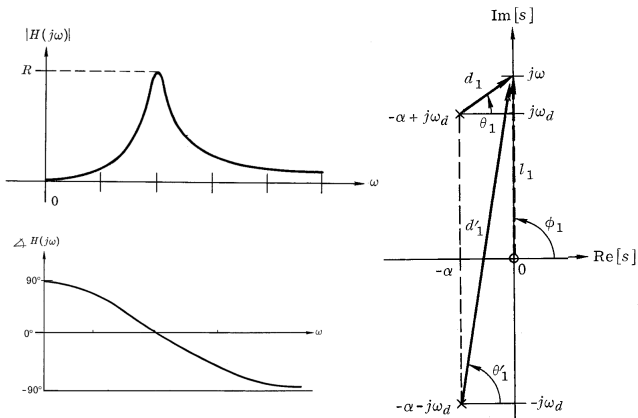
$$\begin{cases} |H(j0)| = 0 \\ |H(j\infty)| = 0 \\ |H(j\omega_d)| \approx \frac{1}{C} \frac{\omega_d}{2\omega_d\alpha} = R, \quad \omega_d \gg \alpha \end{cases}$$

$$\begin{cases} \angle H(j0) = \frac{\pi}{2} \\ \angle H(j\infty) = -\frac{\pi}{2} \\ \angle H(j\omega_d) \approx 0, \quad \omega_d \gg \alpha \end{cases}$$



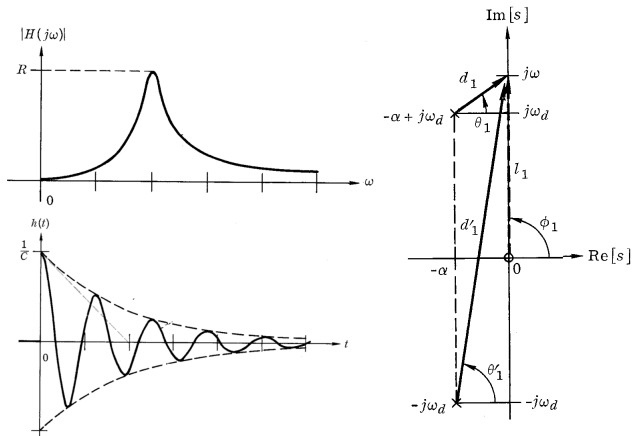
Example (Parallel RLC BPF (cont.))

A parallel RLC circuit can act like a bandpass filter.



Example (Parallel RLC BPF (cont.))

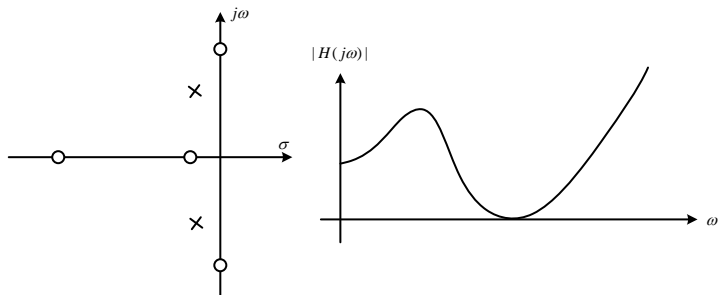
A parallel RLC circuit can act like a bandpass filter.



Frequency Response

Example (Approximated Frequency Response)

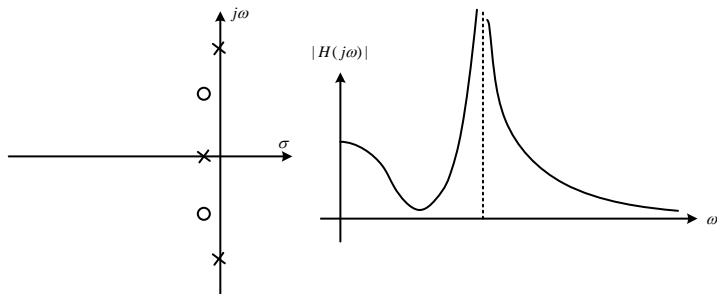
Frequency response can be approximately plotted using the corresponding zero-pole diagram.



Frequency Response

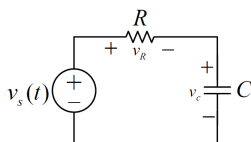
Example (Approximated Zero-Pole Diagram)

Zero-pole diagram can be approximately plotted using the corresponding frequency response.



Example (Series RC LPF)

A series RC circuit can act like a lowpass filter.

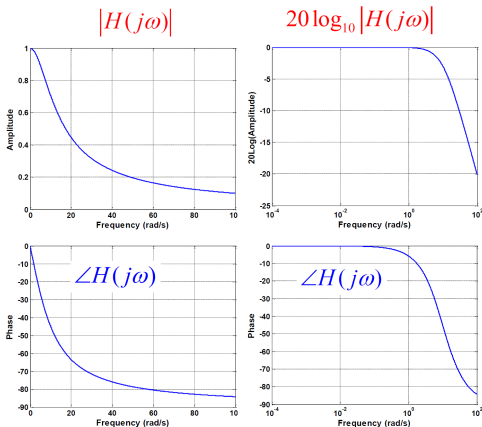


$$H(j\omega) = \frac{V_c}{V_s} = \frac{1}{1 + jRC\omega} = \frac{1}{1 + j\frac{\omega}{\omega_c}} = \begin{cases} 1 & , \quad |\omega| \ll \omega_c \\ j\frac{1}{\omega_c} & , \quad |\omega| \gg \omega_c \end{cases}, \quad \omega_c = \frac{1}{RC}$$

$$20 \log |H(j\omega)| = \begin{cases} 0 & , \quad |\omega| \ll \omega_c \\ 20 \log(\omega_c) - 20 \log(\omega) & , \quad |\omega| \gg \omega_c \end{cases}$$

Example (Series RC LPF)

A series RC circuit can act like a lowpass filter.



Poles

Statement (Poles of Network Function)

Any pole of a network function is a natural frequency of the corresponding network variable.

Statement (Natural Frequencies of a Network)

For an unforced network, appropriate independent sources can be applied to introduce various network functions. Natural frequencies of the unforced network is the union of the poles of the appropriate network functions.

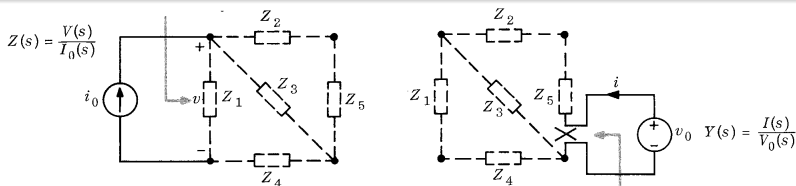


Figure: Illustration of adding independent sources to an **unforced network**. The application of the independent sources **does not alter** the **topology** nor the **element values** of the unforced network, and hence it does not alter its natural behavior.

Types of Poles

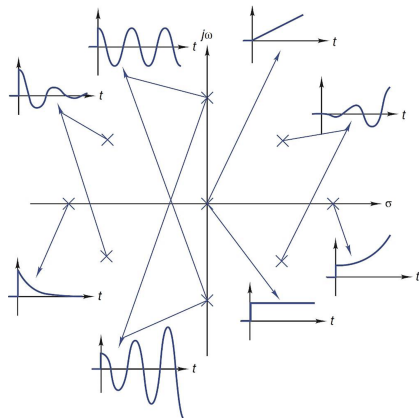


Figure: Poles can be simple or repeated, real, imaginary, or complex, and located in LHS, RHS, or $j\omega$ axis of the complex plane.

Types of Poles

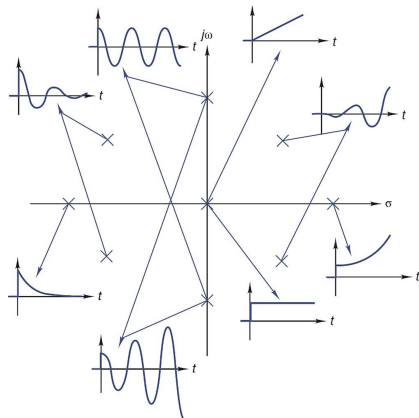


Figure: Types of poles.

- **Transfer function denominator factors:** $s, (s + \alpha), (s + \alpha)^m, (s^2 + \omega^2), (s^2 + \omega^2)^m, (s + \alpha)^2 + \omega^2, [(s + \alpha)^2 + \omega^2]^m$
- **Time-domain behavior of impulse response:** $1, e^{-\alpha t}, t^m e^{-\alpha t}, \cos(\omega t + \theta), t^m \cos(\omega t + \theta), e^{-\alpha t} \cos(\omega t + \theta), t^m e^{-\alpha t} \cos(\omega t + \theta)$

Types of Poles

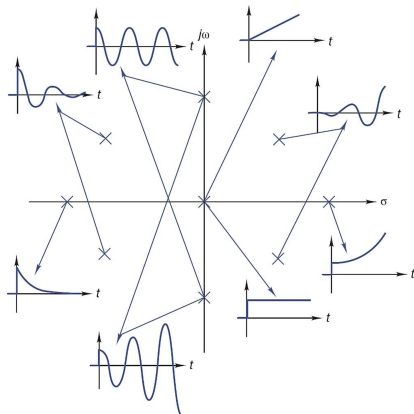
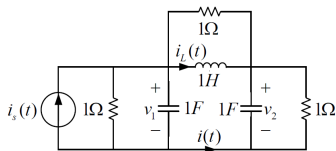


Figure: Impact of poles on zero-state response stability.

- Stable circuit (strictly passive circuit): $\Re\{s_i\} < 0, \forall i$
- Marginally stable circuit (passive circuit): $\Re\{s_i\} \leq 0, \forall i$
- Unstable circuit (active circuit): $\Re\{s_i\} \geq 0, \exists i$

Example (Poles and natural frequencies)

The number of poles is less than the number of natural frequencies of a circuit variable which in turn is less than the number of natural frequencies of the circuit network.



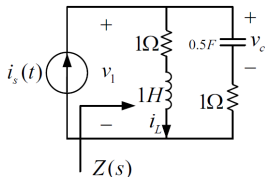
$$\begin{bmatrix} s + 2 + \frac{1}{s} & -1 - \frac{1}{s} \\ -1 - \frac{1}{s} & s + 2 + \frac{1}{s} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_s + v_1(0^-) - \frac{i_L(0^-)}{s} \\ v_2(0^-) + \frac{i_L(0^-)}{s} \end{bmatrix}$$

$$\Delta_n(s) = \frac{(s+2)(s+1)^2}{s} = 0 \Rightarrow s = -1, -1, -2$$

$$\begin{cases} V_1 = \frac{1}{s+2} I_s + \frac{(s+1)v_1(0^-) + v_2(0^-) - i_L(0^-)}{(s+1)(s+2)} \\ V_2 = \frac{1}{(s+1)(s+2)} I_s + \frac{(s+1)v_2(0^-) + v_1(0^-) + i_L(0^-)}{(s+1)(s+2)} \end{cases} \Rightarrow \begin{cases} H_1 = \frac{V_1}{I_s} = \frac{1}{s+2} \\ H_2 = \frac{V_2}{I_s} = \frac{1}{(s+1)(s+2)} \end{cases}$$

Example (Driving-point impedance)

The number of poles is less than the number of natural frequencies of a circuit variable which in turn is less than the number of natural frequencies of the circuit network.

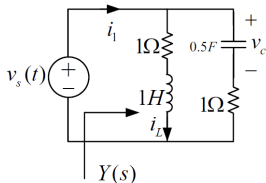


$$\Delta_m(s) = \frac{s^2 + 2s + 2}{s} = 0 \Rightarrow s = -1 \pm j$$

$$\begin{cases} V_c = \frac{2(s+1)}{s^2+2s+2} I_s + \frac{(s+1)v_c(0^-) - 2i_L(0^-)}{s^2+2s+2} \\ V_1 = \frac{(s+1)(s+2)}{s^2+2s+2} I_s + \frac{0.5v_c(0^-) - (s+2)i_L(0^-)}{s^2+2s+2} \end{cases} \Rightarrow Z_{in} = \frac{(s+1)(s+2)}{s^2+2s+2}$$

Example (Driving-point admittance)

The number of poles is less than the number of natural frequencies of a circuit variable which in turn is less than the number of natural frequencies of the circuit network.



$$\Delta_{m1}(s) = \frac{s+2}{s} = 0 \Rightarrow s = -2, \quad \Delta_{m2}(s) = s+1 = 0 \Rightarrow s = -1$$

$$\begin{cases} V_c = \frac{2}{s+2} V_s + \frac{v_c(0^-)}{s+2} \\ I_1 = \frac{s^2+2s+2}{(s+1)(s+2)} V_s + \frac{-(s+1)v_c(0^-)+(s+2)i_L(0^-)}{(s+1)(s+2)} \end{cases} \Rightarrow Y_{in} = \frac{s^2 + 2s + 2}{(s+1)(s+2)}$$

Zeros

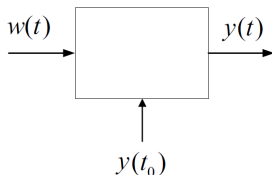


Figure: A **simple zero** can block an **exponential input** while a pair of **imaginary conjugate zeros** can block a **sinusoidal input**. A set of **initial conditions** and a **suitable input** can make a **circuit complete response zero**, especially when the corresponding transfer function has a zero.

$$Y(s) = H(s)W(s) + \frac{F_0(s)}{\sum_{k=0}^n a_k s^k} = \frac{B_1(s)(s - z_1)}{A_1(s)} W(s) + \frac{F_0(s)}{\sum_{k=0}^n a_k s^k}$$

$$w(t) = Ae^{z_1 t} u(t) \Rightarrow Y(s) = \frac{B_1(s)(s - z_1)}{A_1(s)} \frac{A}{s - z_1} + \frac{F_0(s)}{\sum_{k=0}^n a_k s^k} = A \frac{B_1(s)}{A_1(s)} + \frac{F_0(s)}{\sum_{k=0}^n a_k s^k}$$

$$Y(s) = H(s)W(s) + \frac{F_0(s)}{\sum_{k=0}^n a_k s^k} = \frac{B_1(s)(s^2 + \omega_0^2)}{A_1(s)} W(s) + \frac{F_0(s)}{\sum_{k=0}^n a_k s^k}$$

$$w(t) = A \cos(\omega_0 t + \phi) u(t) \Rightarrow Y(s) = \frac{B_1(s)(s^2 + \omega_0^2)}{A_1(s)} \frac{as + b}{s^2 + \omega_0^2} + \frac{F_0(s)}{\sum_{k=0}^n a_k s^k} = \frac{B_1(s)(as + b)}{A_1(s)} + \frac{F_0(s)}{\sum_{k=0}^n a_k s^k}$$

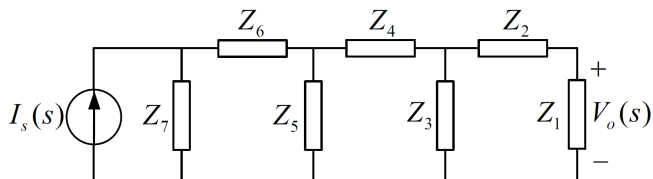


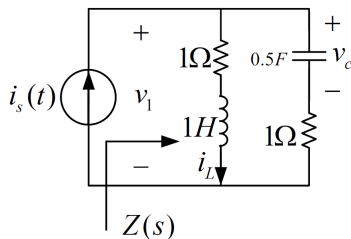
Figure: Different zeros can be realized using **shunt** and **series arms** in a **ladder network**. A zero may be simplified with a factor in the denominator.

$$H(s) = \frac{V_o(s)}{I_s(s)} = \frac{Z_1(s)Z_3(s)Z_5(s)Z_7(s)}{(\cdots)(\cdots)(\cdots)(\cdots)(\cdots)}$$

$$H(s) = \frac{V_o(s)}{I_s(s)} = \frac{Y_2(s)Y_4(s)Y_6(s)}{(\cdots)(\cdots)(\cdots)(\cdots)(\cdots)}$$

Example (A transfer function with zero)

A non-singular function and a set of initial conditions can make the capacitor voltage zero.



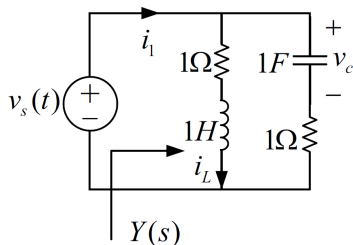
$$V_c(s) = \frac{2(s+1)}{s^2 + 2s + 2} I_s(s) + \frac{(s+1)v_c(0^-) - 2i_L(0^-)}{s^2 + 2s + 2}$$

$$i_s(t) = e^{-t}u(t) \Rightarrow V_c(s) = \frac{2}{s^2 + 2s + 2} + \frac{(s+1)v_c(0^-) - 2i_L(0^-)}{s^2 + 2s + 2}$$

$$i_s(t) = e^{-t}u(t), v_c(0^-) = 0, i_L(0^-) = 1 \Rightarrow V_c(s) = 0 \Rightarrow v_c(t) = 0, t > 0$$

Example (A transfer function without zero)

A singular function and a set of initial conditions can make the capacitor voltage zero.

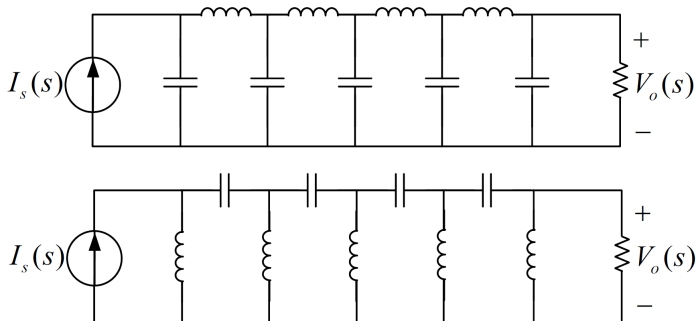


$$V_c(s) = \frac{1}{s+1} V_s(s) + \frac{v_c(0^-)}{s+1}$$

$$v_s(t) = -v_c(0^-)\delta(t) \Rightarrow V_c(s) = 0 \Rightarrow v_c(t) = 0, t > 0$$

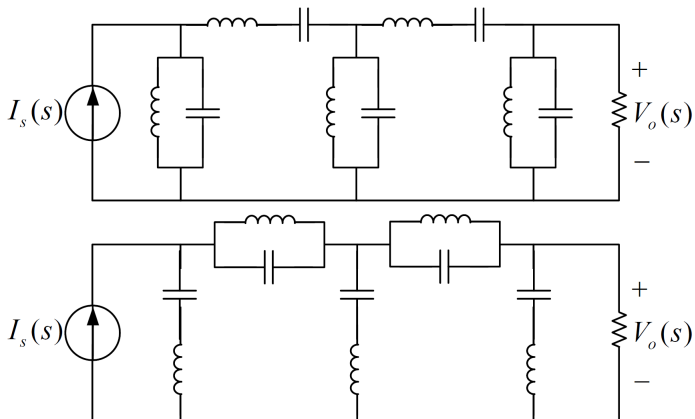
Example (LPF/HPF using ladder network)

A highpass or a lowpass filter can be implemented using a ladder network.



Example (BPF/BSF using ladder network)

A bandpass or a bandstop filter can be implemented using a ladder network.



The End