Network Graphs

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Overview

Graphs

Graphs

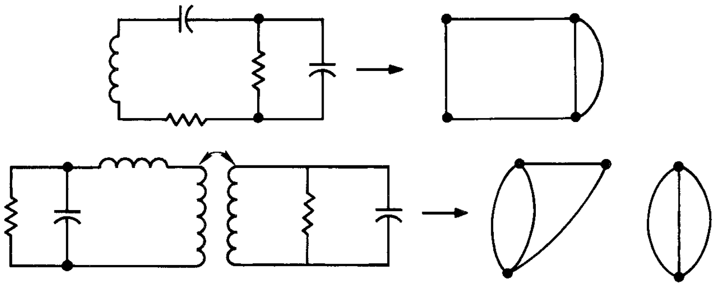


Figure: Each circuit can be represented by a network graph if each element is replaced with an edge having two ending nodes. The nature of elements is discarded in the network graph. A circuit may have a unconnected graph.

Definition (graph)

A graph is mathematically described by $G(\boldsymbol{N}, \boldsymbol{E})$, where $\boldsymbol{N}$ is the set of nodes and the set of edges $\boldsymbol{E} = \{(e_i, e_j) | e_i, e_j \in \boldsymbol{N}\}$.

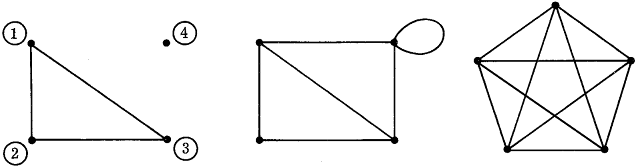


Figure: Graphs with isolated node and self-loop along with a complete graph.

Graphs

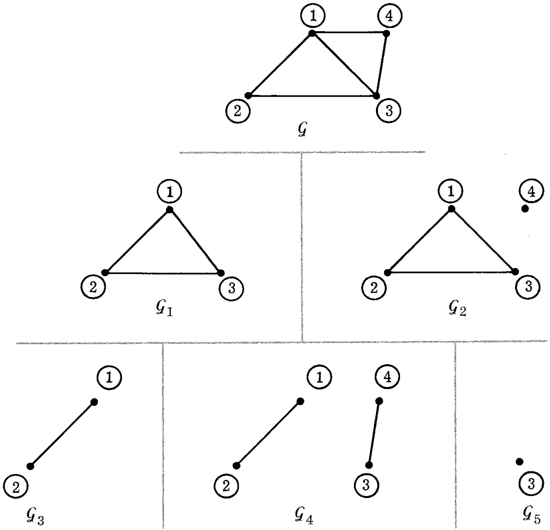


Figure: A graph and some of its subgraphs.

Graphs

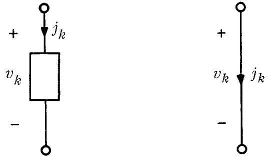


Figure: Associated reference directions for an element and for a branch.

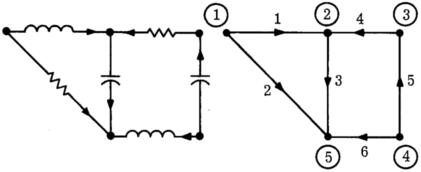


Figure: A network and its corresponding directed graph.

KCL

Cut Sets

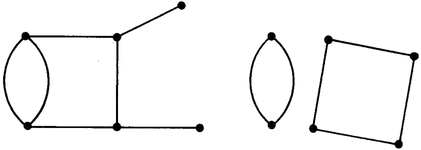


Figure: Connected and unconnected graphs. A unconnected graph have two or more separated parts.

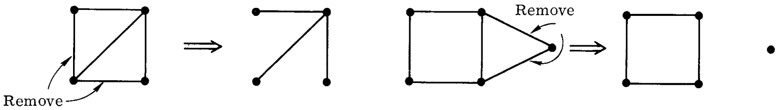


Figure: Branch removal operation.

Cut Set

Definition (Cut Set)

A cut set is the set of branches such that

- The removal of all the branches of the set adds a new separated part to the graph.
- The removal of all but any one of the branches of the set adds no new separated part to the graph.

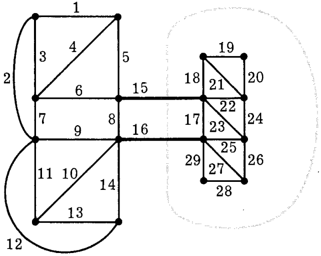


Figure: Example of cut sets.

Node and Gaussian surface

Statement (Node)

A node is a special cut set that only surrounds a node.

Statement (Gaussian surface)

A Gaussian surface is a generalized cut set that decomposes the graph into two or more separated parts.

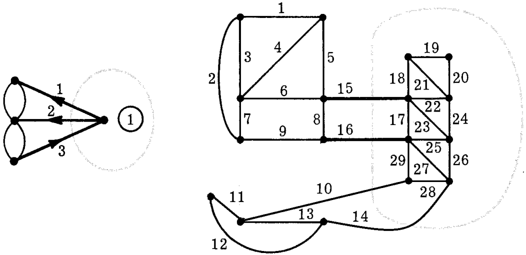


Figure: Examples of node and Gaussian surface.

Definition (KCL)

For any lumped network and at any time, the algebraic sum of all the branch currents entering (exiting) a cut set (node, Gaussian surface) branches is zero.

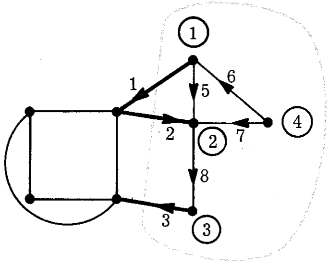


Figure: KCL for the shown cut set yields $j_1(t) - j_2(t) + j_3(t) = 0, \forall t$.

KCL

KCL equations

- originate from change conservation.
- are independent of the nature of the elements.
- are linear homogeneous equations with real coefficient $-1, 0, 1$.
- are dependent equations.

KVL

Loop

Definition (Loop)

A subgraph of a graph is a loop if

- The subgraph is connected.
- Two branches of the subgraph are incident with each node of the subgraph.

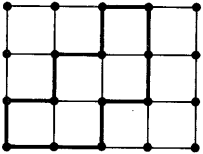


Figure: Example of loop.

Mesh and Super-mesh

Statement (Mesh)

A mesh is a loop of a planar graph without any inner branch.

Statement (Closed Chain)

A closed chain is a generalized loop of a planar graph that creates a closed path.

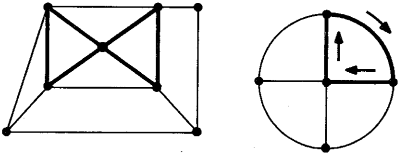


Figure: Examples of mesh and closed chain.

KVL

Definition (KVL)

For any lumped network and at any time, the algebraic sum of the aligned branch voltages around a loop (mesh, closed chain) is zero.

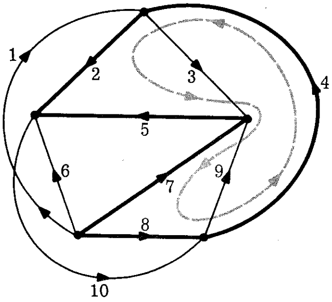


Figure: KVL for the shown loop yields $v_4(t) + v_2(t) - v_5(t) - v_7(t) + v_8(t) = 0, \forall t$.

KVL

KVL equations

- originate from conservativity of electric field.
- are independent of the nature of the elements.
- are linear homogeneous equations with real constant coefficient $-1, 0, 1$.
- are dependent equations.

Node-based Description

Number of Independent KCLs and Voltages

Theorem (Number of Independent KCLs)

In a connected graph, the $n_t - 1$ linear homogeneous algebraic equations obtained by applying KCL to each node except the reference node, constitute a set of linearly independent equations.

Theorem (Number of Independent Voltages)

In a connected graph, the $n_t - 1$ node voltages $\mathbf{e}$ measured with respect to the reference node constitute a set of linearly independent voltages.

Node-to-branch Incidence Matrix

Definition (Node-to-branch Incidence Matrix)

The node-to-branch incidence matrix $\boldsymbol{A_a}$ is a rectangular matrix whose (i,k)th element a_{ik} is defined by

$$a_{ik} = \begin{cases} 1, & \text{if branch } k \text{ leaves node } i \\ -1, & \text{if branch } k \text{ enters node } i \\ 0, & \text{if branch } k \text{ is not incident with node } i \end{cases}$$

The matrix $\boldsymbol{A_a}$ has dimension $n_t \times b$ and rank $n_t - 1$, where n_t and b are the number of nodes and branches, respectively.

Definition (Reduced Node-to-branch Incidence Matrix)

The reduced node-to-branch incidence matrix $\boldsymbol{A}$ is obtained from $\boldsymbol{A_a}$ by eliminating the row corresponding to the reference node. The matrix $\boldsymbol{A}$ has dimension $(n_t - 1) \times b$ and is of full rank $n_t - 1$.

KCL Matrix Equation for Nodes

Statement (KCL Matrix Equation for Nodes)

$\boldsymbol{A_a j} = 0$ *describes* n_t *KCL equations of the nodes, where* $\boldsymbol{j}$ *denotes branch currents vector.*

Statement (KCL Matrix Equation for Nodes)

$\boldsymbol{Aj} = 0$ *describes* $n_t - 1$ *linearly independent KCL equations of the nodes.*

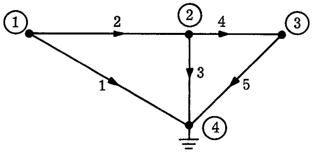
Statement (Branch Voltages)

The branch voltages $\boldsymbol{v}$ *are obtained from the linearly-independent node voltages* $\boldsymbol{e}$ *by the equation* $\boldsymbol{v} = \boldsymbol{A}^T\boldsymbol{e}$*, where* $\boldsymbol{A}^T$ *is the transpose of* $\boldsymbol{A}$*.*

KCL Matrix Equation for Nodes

Example (KCL Equation Matrix)

The circuit below has 3 independent KCL equations at its nodes.

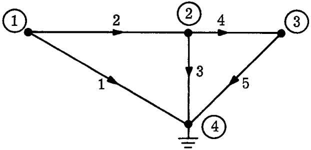


$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{j} = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix}, \quad \boldsymbol{Aj} = 0, \quad \begin{cases} j_1 + j_2 = 0 \\ -j_2 + j_3 + j_4 = 0 \\ -j_4 + j_5 = 0 \end{cases}$$

KCL Matrix Equation for Nodes

Example (Branch Voltages)

The circuit below has 5 branch voltages.



$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}, \quad v = A^T e, \quad \begin{cases} v_1 = e_1 \\ v_2 = e_1 - e_2 \\ v_3 = e_2 \\ v_4 = e_2 - e_3 \\ v_5 = e_3 \end{cases}$$

Mesh-based Description

Definition (Topological Graph)

Each different representation of a graph is called topological graph.

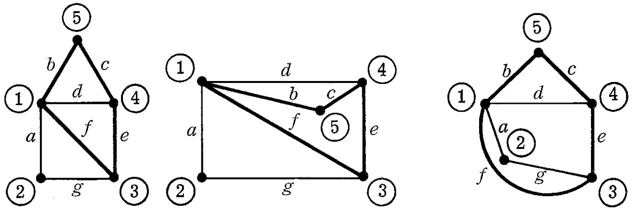


Figure: Three different topological graphs corresponding to a same graph. A loop remains unchanged for different topological graphs while a mesh may change.

Planar Graphs

Definition (Planar Graph)

A graph is planar if it can be drawn on the plane in such a way that no two branches intersect at a point which is not a node.

Definition (Mesh and Outer-Mesh)

Any loop of a planar graph for which there is no branch in its interior is called a mesh. The loop of a planar graph for which there is no branch in its exterior is called the outer-mesh.

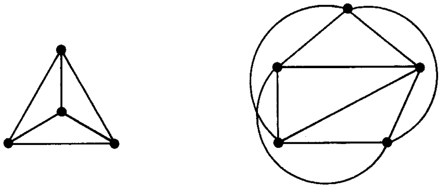


Figure: Examples of planar and non-planar graphs.

Hinged Graphs

Definition (Hinged Graph)

A graph is hinged if it can be partitioned into two non-isolated sub-graphs which are connected together by one node.

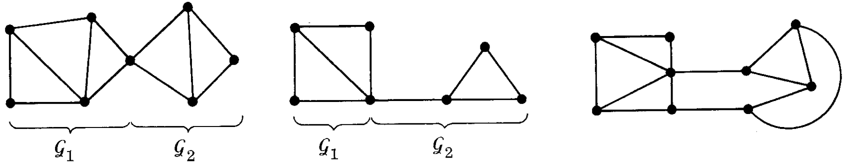


Figure: Examples of hinged and unhinged graphs. Circuit analysis of a hinged graph simplifies to separate analysis of its unhinged sub-graphs provided that there is no coupling between the unhinged sub-graphs.

Number of Meshes

Theorem (Number of Meshes)

For a connected unhinged planar graph, the number of meshes is equal $l = b - n_t + 1$, *where* b *is the number of branches and* n_t *is the number of nodes.*

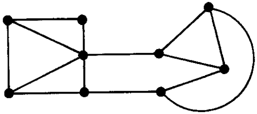


Figure: A planar unhinged graph with $n_t = 9$ nodes, $b = 14$ branches, and $l = 14 - 9 + 1 = 6$ meshes.

Number of Independent KVLs and Currents

Theorem (Number of Independent KVLs)

In a connected planar unhinged graph, the $b - n_t + 1$ *linear homogeneous algebraic equations obtained by applying KVL to each mesh except the outer mesh constitute a set of linearly independent equations.*

Theorem (Number of Independent Currents)

In a connected planar unhinged graph, the $b - n_t + 1$ *mesh currents* $\boldsymbol{i}$ *constitute a set of linearly independent currents.*

Mesh-to-branch Incidence Matrix

Definition (Mesh-to-branch Incidence Matrix)

The mesh-to-branch incidence matrix $\boldsymbol{M_a}$ is a rectangular matrix whose (i, k)th element m_{ik} is defined by

$$m_{ik} = \begin{cases} 1, & \text{if branch } k \text{ is in mesh or outer-mesh } i \text{ and their directions coincide} \\ -1, & \text{if branch } k \text{ is in mesh or outer-mesh } i \text{ and their directions don't coincide} \\ 0, & \text{if branch } k \text{ does not belong to mesh or outer-mesh } i \end{cases}$$

The matrix $\boldsymbol{M_a}$ has dimension $(l+1) \times b$ and rank l, where l and b are the number of meshes and branches, respectively.

Definition (Reduced Mesh-to-branch Incidence Matrix)

The reduced mesh-to-branch incidence matrix $\boldsymbol{M}$ is obtained from $\boldsymbol{M_a}$ by eliminating the row corresponding to the outer mesh. The matrix $\boldsymbol{M}$ has dimension $l \times b$ and is of full rank l.

KVL Matrix Equation for Meshes

Statement (KVL Matrix Equation for Meshes)

$\boldsymbol{M_a v} = 0$ *describes* $l + 1$ *KVL equations of the meshes, where* $\boldsymbol{v}$ *denotes branch voltages vector.*

Statement (KVL Matrix Equation for Meshes)

$\boldsymbol{M v} = 0$ *describes* l *linearly independent KVL equations of the meshes.*

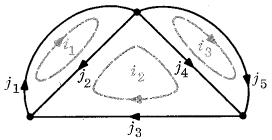
Statement (Branch Currents)

The branch currents $\boldsymbol{j}$ *are obtained from the linearly-independent mesh currents* $\boldsymbol{i}$ *by the equation* $\boldsymbol{j} = \boldsymbol{M}^T \boldsymbol{i}$, *where* $\boldsymbol{M}^T$ *is the transpose of* $\boldsymbol{M}$.

KVL Matrix Equation for Meshes

Example (KVL matrix equation)

The circuit below has 3 independent KVL equations at its meshes.

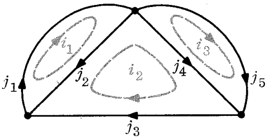


$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}, \quad \boldsymbol{M}\boldsymbol{v} = 0, \quad \begin{cases} v_1 + v_2 = 0 \\ -v_2 + v_3 + v_4 = 0 \\ -v_4 + v_5 = 0 \end{cases}$$

KVL Matrix Equation for Meshes

Example (Branch currents)

The circuit below has 5 branch currents.



$$\boldsymbol{M}^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}, \quad \boldsymbol{j} = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \end{bmatrix}, \quad \boldsymbol{j} = \boldsymbol{M}^T \boldsymbol{i}, \quad \begin{cases} j_1 = i_1 \\ j_2 = i_1 - i_2 \\ j_3 = i_2 \\ j_4 = i_2 - i_3 \\ j_5 = i_3 \end{cases}$$

Cut Set-based Description

Trees

Definition (Tree of a Connected Graph)

A graph is called the tree of a connected graph if

1. It is a connected sub-graph.
2. It contains all the nodes of the connected graph.
3. It contains no loops.

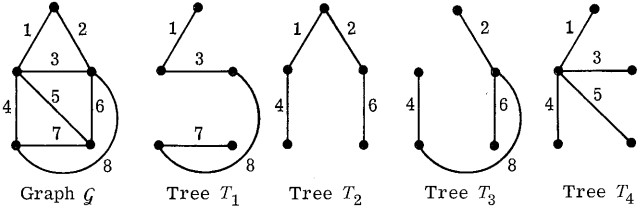


Figure: Examples of trees of a graph.

Trees

Definition (Tree Branch)

The branches of a tree of a connected graph are called tree branch.

Definition (link Branch)

The branches of a connected graph not in its associated tree are called link branch.

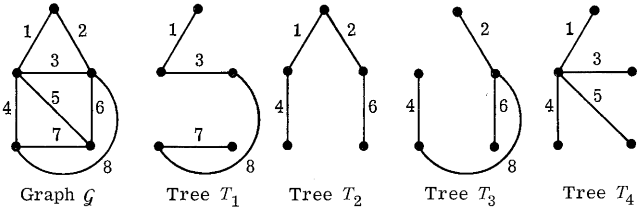


Figure: Examples of trees of a graph.

Trees

Theorem (Fundamental Theory of Graphs)

Given a connected graph G of n_t nodes and b branches, and a tree T of G,

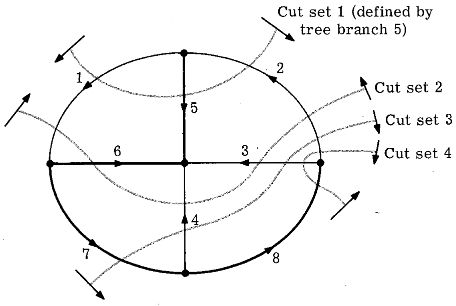
- *There is a unique path along the tree between any pair of nodes.*
- *There are $n_t - 1$ tree branches and $b - n_t + 1$ links.*
- *Every link of G and the unique tree path between its nodes constitute a unique loop (this is called the fundamental loop associated with the link).*
- *Every tree branch of T together with some links defines a unique cut set. This cut set is called a fundamental cut set associated with the tree branch.*

Corollary (Fundamental Theory of Graphs)

Suppose that G has n_t nodes, b branches, and s separate parts. Let $T_l, T_2, \cdots, T_s$ be trees of each separate part, respectively. The set $\{T_l, T_2, \cdots, T_s\}$ is called a forest of G. Then the forest has $n_t - s$ branches, G has $b - n_t + s$ links, and the remaining statements of the fundamental theorem are true.

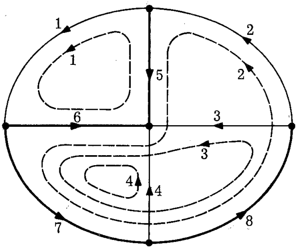
Example (Fundamental cut sets)

The circuit below has 4 fundamental cut sets. The direction of each cut set is inherited from the direction of its associated tree branch.



Example (Fundamental loops)

The circuit below has 4 fundamental loops. The direction of each loop is inherited from the direction of its associated link branch.



Links 1, 2, 3, 4

Tree branches 5, 6, 7, 8

Number of Independent KCLs and Voltages

Theorem (Number of Independent KCLs)

In a connected graph, the $n_t - 1$ linear homogeneous algebraic equations obtained by applying KCL to the fundamental cut sets of a tree of the graph, constitute a set of linearly independent equations.

Theorem (Number of Independent Voltages)

In a connected graph, the $n_t - 1$ tree branch voltages constitute a set of linearly independent voltages.

Fundamental Cut Set Matrix

Definition (Fundamental Cut Set Matrix)

The fundamental cut set matrix $\boldsymbol{Q}$ is a rectangular matrix whose (i,k)th element q_{ik} is defined by

$$q_{ik} = \begin{cases} 1, & \text{if branch } k \text{ belongs to cut set } i \text{ and has the same direction} \\ -1, & \text{if branch } k \text{ belongs to cut set } i \text{ and has the opposite direction} \\ 0, & \text{if branch } k \text{ does not belong to cut set } i \end{cases}$$

The matrix $\boldsymbol{Q}$ has dimension $(n_t - 1) \times b$ and is of full rank $n_t - 1$, where $n_t - 1$ and b are the number of tree branches and branches, respectively.

KCL Matrix Equation for Cut Sets

Statement (KCL Matrix Equation for Cut Sets)

$\boldsymbol{Qj} = 0$ *describes* $n_t - 1$ *linearly independent KCL equations of the cut sets, where* $\boldsymbol{j}$ *denotes branch currents vector.*

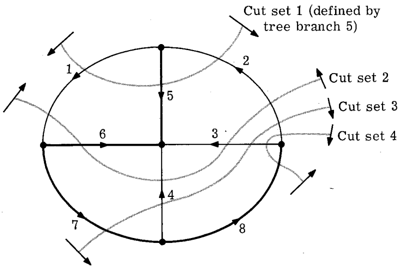
Statement (Branch Voltages)

The branch voltages $\boldsymbol{v}$ *are obtained from the linearly-independent tree branch voltages* $\boldsymbol{e}$ *by the equation* $\boldsymbol{v} = \boldsymbol{Q}^T\boldsymbol{e}$, *where* $\boldsymbol{Q}^T$ *is the transpose of* $\boldsymbol{Q}$.

KCL Matrix Equation for Cut Sets

Example (KCL matrix equation)

The circuit below has 4 independent KCL equations at its cut sets.

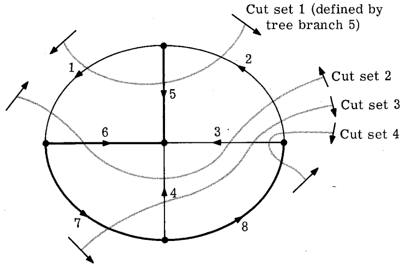


$$\boldsymbol{Q} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{j} = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix}, \quad \boldsymbol{Qj} = 0, \quad \begin{cases} j_1 - j_2 + j_5 = 0 \\ -j_1 + j_2 + j_3 + j_4 + j_6 = 0 \\ -j_2 - j_3 - j_4 + j_7 = 0 \\ -j_2 - j_3 + j_8 = 0 \end{cases}$$

KCL Matrix Equation for Cut Sets

Example (Branch voltages)

The circuit below has 8 branch voltages.



$$\boldsymbol{Q}^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix}, \quad \boldsymbol{v} = \boldsymbol{Q}^T\boldsymbol{e}, \quad \begin{cases} v_1 = e_1 - e_2 \\ v_2 = -e_1 + e_2 - e_3 - e_4 \\ v_3 = e_2 - e_3 - e_4 \\ v_4 = e_2 - e_3 \\ v_5 = e_1 \\ v_6 = e_2 \\ v_7 = e_3 \\ v_8 = e_4 \end{cases}$$

Loop-based Description

Number of Independent KVLs and Currents

Theorem (Number of Independent KVLs)

In a connected graph, the $l = b - n_t + 1$ *linear homogeneous algebraic equations obtained by applying KVL to the fundamental loops of a tree of the graph, constitute a set of linearly independent equations.*

Theorem (Number of Independent Currents)

In a connected graph, the $l = b - n_t + 1$ *link branch currents constitute a set of linearly independent voltages.*

Fundamental Loop Matrix

Definition (Fundamental Loop Matrix)

The fundamental loop matrix $\boldsymbol{B}$ is a rectangular matrix whose (i, k)th element b_{ik} is defined by

$$b_{ik} = \begin{cases} 1, & \text{if branch } k \text{ is in loop } i \text{ and their directions agree} \\ -1, & \text{if branch } k \text{ is in loop } i \text{ and their directions don't agree} \\ 0, & \text{if branch } k \text{ is not in loop } i \end{cases}$$

The matrix $\boldsymbol{B}$ has dimension $l \times b$ and is of full rank l, where l and b are the number of link branches and branches, respectively.

KVL Matrix Equation for Loops

Statement (KVL Matrix Equation for Loops)

$\boldsymbol{Bv} = 0$ *describes* $l = b - n_t + 1$ *linearly independent KVL equations of the loops, where* $\boldsymbol{v}$ *denotes branch voltages vector.*

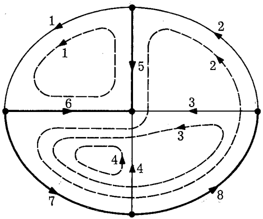
Statement (Branch Current)

The branch currents $\boldsymbol{j}$ *are obtained from the linearly-independent tree link currents* $\boldsymbol{i}$ *by the equation* $\boldsymbol{j} = \boldsymbol{B}^T\boldsymbol{i}$, *where* $\boldsymbol{B}^T$ *is the transpose of* $\boldsymbol{B}$.

KVL Matrix Equation for Loops

Example (KVL matrix equation)

The circuit below has 4 independent KVL equations at its loops.



Links 1, 2, 3, 4

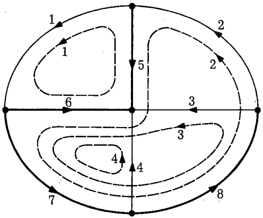
Tree branches 5, 6, 7, 8

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix}, \quad \boldsymbol{Bv} = 0, \quad \begin{cases} v_1 - v_5 + v_6 = 0 \\ -v_2 + v_5 - v_6 + v_7 + v_8 = 0 \\ v_3 - v_6 + v_7 + v_8 = 0 \\ v_4 - v_6 + v_7 = 0 \end{cases}$$

KVL Matrix Equation for Loops

Example (Branch currents)

The circuit below has 8 branch currents.



Links 1, 2, 3, 4

Tree branches 5, 6, 7, 8

$$\boldsymbol{B}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}, \quad \boldsymbol{j} = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix}, \quad \boldsymbol{j} = \boldsymbol{B}^T \boldsymbol{i}, \quad \begin{cases} j_1 = i_1 \\ j_2 = i_2 \\ j_3 = i_3 \\ j_4 = i_4 \\ j_5 = -i_1 + i_2 \\ j_6 = i_1 - i_2 - i_3 - i_4 \\ j_7 = i_2 + i_3 + i_4 \\ j_8 = i_2 + i_3 \end{cases}$$

Comparison of Different Descriptions

Different Equations

- Node-based: $\begin{cases} \text{KCL}: & \boldsymbol{Aj} = 0 \\ \text{KVL}: & \boldsymbol{v} = \boldsymbol{A}^T\boldsymbol{e} \end{cases}$
- Mesh-based: $\begin{cases} \text{KVL}: & \boldsymbol{Mv} = 0 \\ \text{KCL}: & \boldsymbol{j} = \boldsymbol{M}^T\boldsymbol{i} \end{cases}$
- Cut Set-based: $\begin{cases} \text{KCL}: & \boldsymbol{Qj} = 0 \\ \text{KVL}: & \boldsymbol{v} = \boldsymbol{Q}^T\boldsymbol{e} \end{cases}$
- Loop-based: $\begin{cases} \text{KVL}: & \boldsymbol{Bv} = 0 \\ \text{KCL}: & \boldsymbol{j} = \boldsymbol{B}^T\boldsymbol{i} \end{cases}$

Number of Items

- Number of linearly independent KCLs: $n = n_t - 1$
- Number of linearly independent voltages: $n = n_t - 1$
- Number of tree branches: $n = n_t - 1$

- Number of linearly independent KVLs: $l = b - n_t + 1$
- Number of linearly independent currents: $l = b - n_t + 1$
- Number of link branches: $l = b - n_t + 1$

- Number of trees: $|\boldsymbol{AA}^T| = |\boldsymbol{MM}^T| = |\boldsymbol{QQ}^T| = |\boldsymbol{BB}^T|$
- Number of trees in complete graph: $n_t^{n_t - 2}$

Relationship of Matrices

Statement (KVL Matrix Equation for Loops)

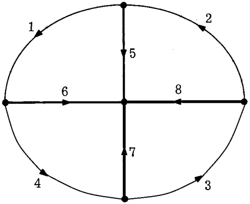
Call $\boldsymbol{B}$ *the fundamental loop matrix and* $\boldsymbol{Q}$ *the fundamental cut-set matrix of the same directed graph* G*, and let both matrices pertain to the same tree* T*. Then,* $\boldsymbol{B}\boldsymbol{Q}^T = 0$ *and* $\boldsymbol{Q}\boldsymbol{B}^T = 0$*. Furthermore, if we number the links from* 1 *to* l *and number the tree branches from* $l+1$ *to* b*, then* $\boldsymbol{B}_{l\times b} = [\boldsymbol{I}_{l\times l}|\boldsymbol{F}]$ *and* $\boldsymbol{Q}_{(n_t-1)\times b} = [-\boldsymbol{F}^T|\boldsymbol{I}_{(n_t-1)\times(n_t-1)}]$.

$$\boldsymbol{Q}\boldsymbol{j} = 0 \Rightarrow \quad \boldsymbol{Q}(\boldsymbol{B}^T\boldsymbol{i}) = 0 \Rightarrow (\boldsymbol{Q}\boldsymbol{B}^T)\boldsymbol{i} = 0 \Rightarrow \quad \boldsymbol{Q}\boldsymbol{B}^T = 0 \Rightarrow \quad \boldsymbol{B}\boldsymbol{Q}^T = 0$$

$$\boldsymbol{B}\boldsymbol{Q}^T = 0 \Rightarrow \left[\boldsymbol{I}_{l\times l} \quad | \quad \boldsymbol{F}_{l\times(n_t-1)}\right] \left[\frac{\boldsymbol{E}^T_{l\times(n_t-1)}}{\boldsymbol{I}_{(n_t-1)\times(n_t-1)}}\right] = \boldsymbol{E}^T_{l\times(n_t-1)} + \boldsymbol{F}_{l\times(n_t-1)} = 0$$

Example (Possible equality of $\boldsymbol{A}$ and $\boldsymbol{Q}$)

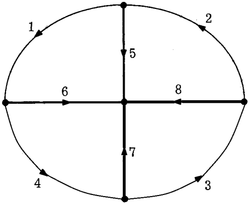
There may be a special tree for which the node-to-branch incident matrix $\boldsymbol{A}$ and fundamental cut set matrix $\boldsymbol{Q}$ are the same.



$$\boldsymbol{A} = \boldsymbol{Q} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example (Possible equality of $\boldsymbol{M}$ and $\boldsymbol{B}$)

There may be a special tree for which the mesh-to-branch incident matrix $\boldsymbol{M}$ and fundamental loop matrix $\boldsymbol{B}$ are the same.

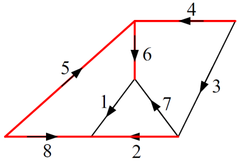


$$\boldsymbol{A} = \boldsymbol{Q} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{M} = \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{bmatrix}$$

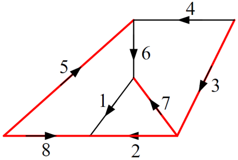
Example (Desired set of independent voltages)

The shown tree corresponds to a set of independent voltages that includes v_2 and v_6 and does not include v_1, v_3, and v_7.



Example (Desired set of independent currents)

The shown tree corresponds to a set of independent currents that includes j_4 and j_6 and does not include j_2 and j_7.



Duality

Dual Graphs

Statement (Dual Graphs)

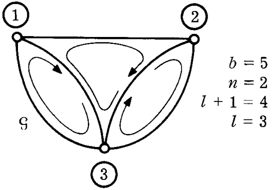
Two connected, unhinged, and planar topological graphs G and $\hat{G}$ are dual if,

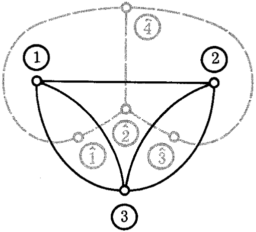
- *There is a one-to-one correspondence between the meshes of G (including the outer mesh) and the nodes of $\hat{G}$.*
- *There is a one-to-one correspondence between the meshes of $\hat{G}$ (including the outer mesh) and the nodes of G.*
- *There is a one-to-one correspondence between the branches of each graph in such a way that whenever two meshes of one graph have the corresponding branch in common, the corresponding nodes of the other graph have the corresponding branch connecting these nodes.*

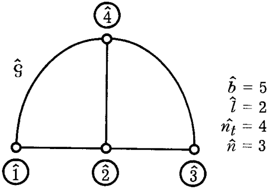
Dual Graphs

Example (Dual Graphs)

The two graphs below are dual of each other.



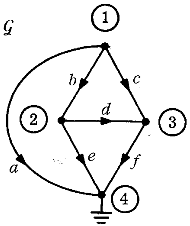


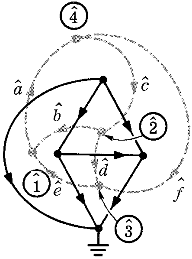


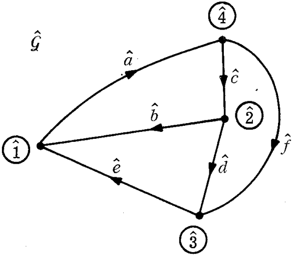
Dual Graphs

Example (Dual Directed Graphs)

The two directed graphs below are dual of each other.







Dual Circuits

Statement (Dual Circuits)

Two circuits are dual if,

- *Their associative graphs, G and $\hat{G}$, are dual.*
- *The governing circuit equations of $\hat{G}$ are obtained by the following replacement from governing circuit equations of G.*

$$j \to \hat{v}$$

$$v \to \hat{j}$$

$$q \to \hat{\phi}$$

$$\phi \to \hat{q}$$

Dual Circuits

G	$\hat{G}$
KVL	KCL
KCL	KVL
Node	Mesh
Mesh	Node
Refrence Node	Outer Mesh
Outer Mesh	Reference Node
Parallel Connection	Series Connection
Series Connection	Parallel Connection
Link Branch	Tree Branch
Tree Branch	Link Branch
Open Circuit	Short Circuit
Short Circuit	Open Circuit

Table: Dual items.

G	$\hat{G}$
v	$\hat{j}$
j	$\hat{v}$
e	$\hat{i}$
i	$\hat{e}$
$\boldsymbol{A}$	$\hat{\boldsymbol{M}}$
$\boldsymbol{M}$	$\hat{\boldsymbol{A}}$
$\boldsymbol{Q}$	$\hat{\boldsymbol{B}}$
$\boldsymbol{B}$	$\hat{\boldsymbol{Q}}$

Table: Dual items.

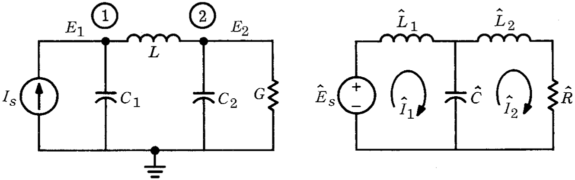
G	$\hat{G}$
q	$\hat{\phi}$
ϕ	$\hat{q}$
R	$\hat{G}$
G	$\hat{R}$
L	$\hat{C}$
C	$\hat{L}$
Z	$\hat{Y}$
Y	$\hat{Z}$

Table: Dual items.

Dual Circuits

Example (Dual Graphs)

Two circuits below are dual.



$$\frac{-1}{j\omega L}E_1 + (j\omega C_2 + G + \frac{1}{j\omega L})E_2 = 0, \quad \frac{-1}{j\omega \hat{C}}\hat{I}_1 + (j\omega \hat{L}_2 + \hat{R} + \frac{1}{j\omega \hat{C}})\hat{I}_2 = 0$$

Tellegen's Theorem

Tellegen's Theorem

Theorem (Tellegen's Theorem)

Consider an arbitrary lumped network whose graph G has b branches and n_t nodes. Suppose that to each branch of the graph we assign arbitrarily a branch voltage v_k and a branch current j_k for $k = 1, 2 \cdots, b$, and suppose that they are measured with respect to arbitrarily picked associated reference directions. If the branch voltages $v_1, v_2, \cdots, v_b$ satisfy all the constraints imposed by KVL and if the branch currents $j_1, j_2, \cdots, j_b$, satisfy all the constraints imposed by KCL, then

$$\sum_{k=1}^{b} v_k j_k = 0$$

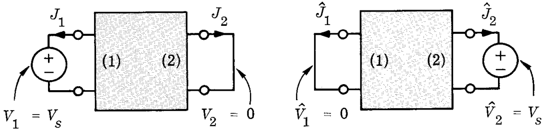
Tellegen's Theorem

- If the voltage sets $\{v_k | k = 1, \cdots, b\}$ and $\{\hat{v}_k | k = 1, \cdots, b\}$ and the current sets $\{j_k | k = 1, \cdots, b\}$ and $\{\hat{j}_k | k = 1, \cdots, b\}$ satisfy KVL and KCL requirements, then
$$\sum_{k=1}^{b} v_k j_k = 0, \quad \sum_{k=1}^{b} \hat{v}_k j_k = 0, \quad \sum_{k=1}^{b} v_k \hat{j}_k = 0, \quad \sum_{k=1}^{b} \hat{v}_k \hat{j}_k = 0$$
- Tellegen's theorem is independent of the nature of elements.
- Instantaneous and apparent power conservation are special cases of Tellegen's theorem.

Tellegen's Theorem

Example (Two-measurement experiment)

For the LTI RLC network below, Tellegen's theorem forces $\hat{J}_1 = J_2$ in the two illustrated measurement scenarios.



$$V_1\hat{J}_1 + V_2\hat{J}_2 + \sum_{k=3}^{b} V_k\hat{J}_k = V_1\hat{J}_1 + V_2\hat{J}_2 + \sum_{k=3}^{b} Z_K J_k\hat{J}_k = 0$$

$$\hat{V}_1 J_1 + \hat{V}_2 J_2 + \sum_{k=3}^{b} \hat{V}_k J_k = \hat{V}_1 J_1 + \hat{V}_2 J_2 + + \sum_{k=3}^{b} Z_K \hat{J}_k J_k = 0$$

$$V_1\hat{J}_1 + V_2\hat{J}_2 = \hat{V}_1 J_1 + \hat{V}_2 J_2$$

$$V_s\hat{J}_1 = V_s J_2$$

$$\hat{J}_1 = J_2$$

Driving-point Impedance

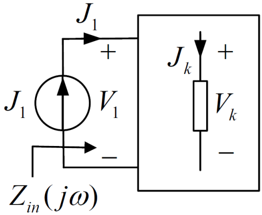


Figure: Driving-point impedance of a passive RLCMT network. Coupled inductors are replaced with their passive equivalent circuits. Tranformers do not consume power.

- Complex power conservation: $-0.5V_1J_1^* + 0.5\sum_{k=2}^{b} V_kJ_k^* = 0$
- Complex power conservation:
  $0.5Z_{in}(j\omega)|J_1|^2 = 0.5\sum_R R_k|J_k|^2 + 0.5j\omega\sum_L L_k|J_k|^2 - 0.5j\omega^{-1}\sum_C C_k^{-1}|J_k|^2$
- Driving-point impedance:
  $Z_{in}(j\omega) = \frac{\sum_R R_k|J_k|^2}{|J_1|^2} + j\frac{\sum_L \omega L_k|J_k|^2 - \sum_C \omega^{-1}C_k^{-1}|J_k|^2}{|J_1|^2} = \Re\{Z_{in}(j\omega)\} + j\Im\{Z_{in}(j\omega)\}$
- Passivity condition: $\Re\{Z_{in}(j\omega)\} \geq 0, \quad \Im\{Z_{in}(j\omega)\} \in \mathbb{R}, \quad |\angle Z_{in}(j\omega)| \leq \pi/2$

Driving-point Impedance

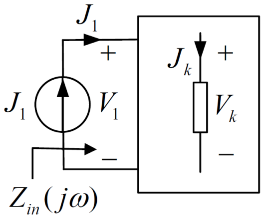


Figure: Driving-point impedance of a passive RLCMT network. Coupled inductors are replaced with their passive equivalent circuits. Tranformers do not consume power.

- Average dissipated power: $P_{av_k} = 0.5R_k|J_k|^2$
- Average stored magnetic energy: $\bar{\mathcal{E}}_{L_k} = 0.25L_k|J_k|^2$
- Average stored electrical energy: $\bar{\mathcal{E}}_{C_k} = 0.25C_k|V_k|^2 = 0.25C_k^{-1}\omega^{-2}|J_k|^2$
- Complex power conservation:
  $0.5Z_{in}(j\omega)|J_1|^2 = 0.5\sum_R R_k|J_k|^2 + 0.5j\omega\sum_L L_k|J_k|^2 - 0.5j\omega^{-1}\sum_C C_k^{-1}|J_k|^2$
- Complex power conservation:
  $S = \sum_R P_{av_k} + 2j\omega(\sum_L \bar{\mathcal{E}}_{L_k} - \sum_C \bar{\mathcal{E}}_{C_k}) = P_{av} + 2j\omega(\bar{\mathcal{E}}_L - \bar{\mathcal{E}}_C)$
- Driving-point impedance: $Z_{in}(j\omega) = \frac{2P_{av}+4j\omega(\bar{\mathcal{E}}_L-\bar{\mathcal{E}}_C)}{|J_1|^2}$

Driving-point Impedance

Circuit Type	$\Re\{Z_{in}(j\omega)\}$	$\Im\{Z_{in}(j\omega)\}$	$\angle Z_{in}(j\omega)$
RT	$\Re\{Z_{in}(j\omega)\} \geq 0$	$\Im\{Z_{in}(j\omega)\} = 0$	$\angle Z_{in}(j\omega) = 0$
RLMT	$\Re\{Z_{in}(j\omega)\} \geq 0$	$\Im\{Z_{in}(j\omega)\} \geq 0$	$0 \leq \angle Z_{in}(j\omega) \leq \pi/2$
RCT	$\Re\{Z_{in}(j\omega)\} \geq 0$	$\Im\{Z_{in}(j\omega)\} \leq 0$	$-\pi/2 \leq \angle Z_{in}(j\omega) \leq 0$
RLCMT	$\Re\{Z_{in}(j\omega)\} \geq 0$	$\Im\{Z_{in}(j\omega)\} \in \mathbb{R}$	$-\pi/2 \leq \angle Z_{in}(j\omega) \leq \pi/2$
LCMT	$\Re\{Z_{in}(j\omega)\} = 0$	$\Im\{Z_{in}(j\omega)\} \in \mathbb{R}$	$\angle Z_{in}(j\omega) = \pm\pi/2$
LMT	$\Re\{Z_{in}(j\omega)\} = 0$	$\Im\{Z_{in}(j\omega)\} \geq 0$	$\angle Z_{in}(j\omega) = \pi/2$
CT	$\Re\{Z_{in}(j\omega)\} = 0$	$\Im\{Z_{in}(j\omega)\} \leq 0$	$\angle Z_{in}(j\omega) = -\pi/2$

Table: Driving-point impedance of passive networks.

The End