Student Name: ....
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Question 1

For the circuit shown in Fig. 1, all the elements have unit values and the initial conditions are $i_2(0^-) = i_4(0^-) = v_1(0^-) = v_3(0^-) = 1$ **and** $i_5(0^-) = 0$**.**

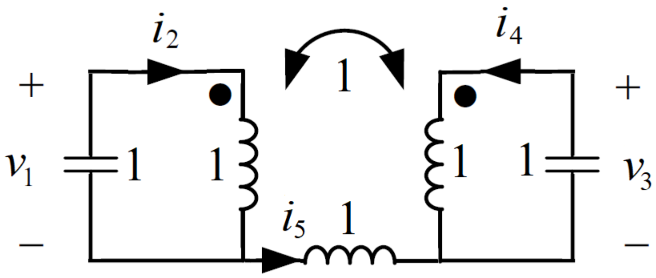

Figure 1: A coupled LTI circuit.

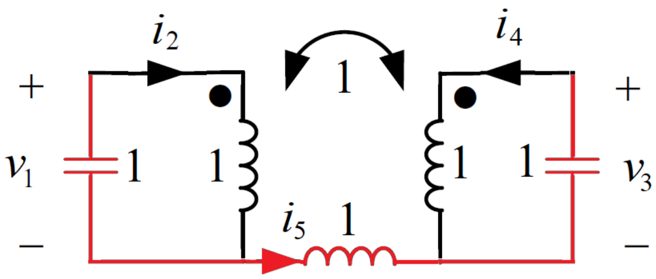

Figure 2: Network tree for the LTI circuit of Fig. 1.

(a) Find the state equations.

Question 1 continued on next page. . .

Considering the tree highlighted in red at Fig. 2,

$$\begin{cases} v_1'(t) + i_2(t) = 0 \\ v_3'(t) + i_4(t) = 0 \\ i_2'(t) + i_4'(t) - v_1(t) = 0 \\ i_4'(t) + i_2'(t) - v_3(t) = 0 \\ i_5(t) = 0 \end{cases}$$

Clearly,

$$v_1(t) = v_3(t) = i_2'(t) + i_4'(t)$$

and since $v_1(0^-) = v_3(0^-) = 1$,

$$i_2(t) = i_4(t) = -v_1'(t)$$

So, we have only two state equations as

$$\begin{cases} v_1'(t) = -i_2(t) \\ 2i_2'(t) = v_1(t) \end{cases} \Rightarrow \frac{d}{dt}\begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}$$

(b) Solve the state equation.

$$\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{A}\boldsymbol{X}(t), \quad \boldsymbol{A} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad \boldsymbol{X}(t) = \begin{bmatrix} v_1(t) \\ i_2(t) \end{bmatrix}$$

$$e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & 1 \\ -\frac{1}{2} & s \end{bmatrix}^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s}{s^2+\frac{1}{2}} & \frac{-1}{s^2+\frac{1}{2}} \\ \frac{\frac{1}{2}}{s^2+\frac{1}{2}} & \frac{s}{s^2+\frac{1}{2}} \end{bmatrix}\right\}$$

$$\boldsymbol{X}(t) = e^{\boldsymbol{A}t}\boldsymbol{X}_0 = \begin{bmatrix} \cos(\frac{1}{\sqrt{2}}t) & -\sqrt{2}\sin(\frac{1}{\sqrt{2}}t) \\ \frac{1}{\sqrt{2}}\sin(\frac{1}{\sqrt{2}}t) & \cos(\frac{1}{\sqrt{2}}t) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{cases} v_1(t) = \cos(\frac{1}{\sqrt{2}}t) - \sqrt{2}\sin(\frac{1}{\sqrt{2}}t) \\ i_2(t) = \frac{1}{\sqrt{2}}\sin(\frac{1}{\sqrt{2}}t) + \cos(\frac{1}{\sqrt{2}}t) \end{cases}$$

(c) Find the natural frequencies of $i_5(t)$.

Clearly, $i_5(t) = 0$ and therefore, it has no natural frequency.

(d) Find the natural frequencies of the circuit.

Question 1 continued on next page...

To obtain the natural frequencies, we should analyze the circuit for an arbitrary set of initial conditions. Even for a general set of conditions,

$$\begin{cases} v_1'(t) + i_2(t) = 0 \\ v_3'(t) + i_4(t) = 0 \\ i_2'(t) + i_4'(t) - v_1(t) = 0 \\ i_4'(t) + i_2'(t) - v_3(t) = 0 \\ i_5(t) = 0 \end{cases} \Rightarrow \begin{cases} sV_1(s) - v_1(0^-) + I_2(s) = 0 \\ sV_3(s) - v_3(0^-) + I_4(s) = 0 \\ sI_2(s) - i_2(0^-) + sI_4(s) - i_4(0^-) - V_1(s) = 0 \\ sI_4(s) - i_4(0^-) + sI_2(s) - i_2(0^-) - V_3(s) = 0 \\ I_5(s) = 0 \end{cases}$$

So, $I_5(s) = 0$, $V_1(s) = V_3(s)$, and $I_4(s) = I_2(s) + v_3(0^-) - v_1(0^-)$. Note that $i_2(t)$ and $i_4(t)$ differ at least by $[v_3(0^-) - v_1(0^-)]\delta(t)$ and therefore, they still have the same natural frequencies. As a results, we have two independent state variables and two natural frequencies. The two natural frequencies can be found from the state equations in part (a) as

$$\Delta(s) = \text{det}[s\boldsymbol{I} - \boldsymbol{A}] = \begin{vmatrix} s & 1 \\ -\frac{1}{2} & s \end{vmatrix} = s^2 + \frac{1}{2} = 0 \Rightarrow s = \pm j\frac{1}{\sqrt{2}}$$