MATHEMATICAL QUESTIONS

Question 1

Find the equivalent resistance of the ladder network in Fig. 1.

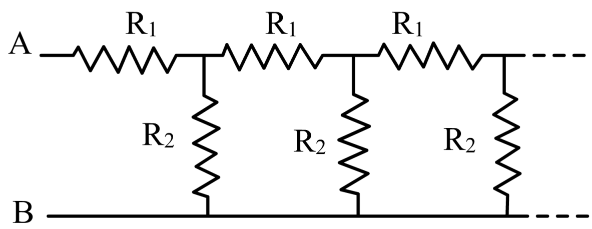


Figure 1: Ladder resistor network.

Let R be the equivalent resistor seen from the terminal point A and B. Since the ladder has infinite length, the same equivalent resistor R is seen from each vertical resistor R_2. Therefore, $R = R_1 + R_2||R = R_1 + R_2R/(R_2 + R)$. This is a second-order equation, whose acceptable solution is

$$R = \frac{R_1 + \sqrt{R_1^2 + 4R_1R_2}}{2}$$

.

Question 2

How are Δ and T resistor networks in Fig. 2 equivalent? (Hint: If two circuits are equivalent, the terminal voltages and currents must be equal.)

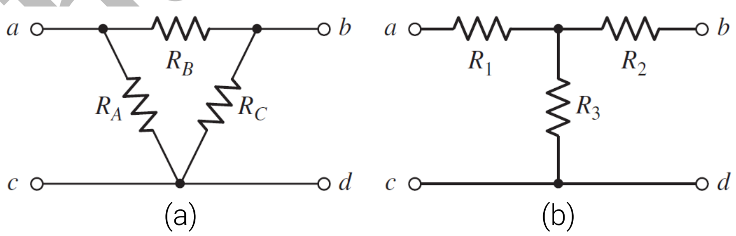
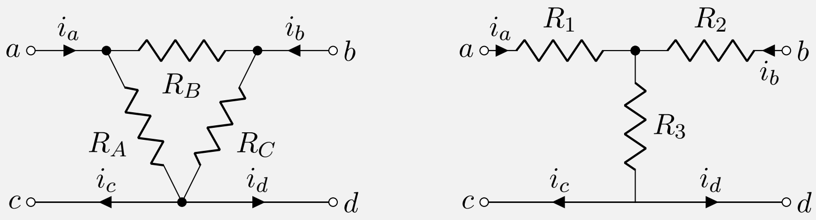


Figure 2: Two well-known equivalent resistor circuits. (a) Δ network. (b) T network.



The networks should behave the same for any values of i_a, i_b, i_c, i_d and v_a, v_b, v_c, v_d. Especially, when $i_b = 0$, then $v_c - v_a = i_a R_1 + i_a R_3 = i_a R_A || (R_B + R_C)$, which results in $R_1 + R_3 = R_A || (R_B + R_C)$. Similarly, if $i_a = 0$, $R_2 + R_3 = R_C || (R_B + R_A)$, and if $i_c + i_d = 0$, $R_1 + R_2 = R_B || (R_C + R_A)$. These equations lead to

$$R_1 = \frac{R_A R_B}{R_A + R_B + R_C}$$

,

$$R_2 = \frac{R_B R_C}{R_A + R_B + R_C}$$

, and

$$R_3 = \frac{R_A R_C}{R_A + R_B + R_C}$$

. Further,

$$R_A = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2}$$

,

$$R_B = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3}$$

, and

$$R_C = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}$$

.

Question 3

Determine the Thevenin equivalent seen by $-j10\ \Omega$ impedance of Fig. 3 and use this to compute V_1.

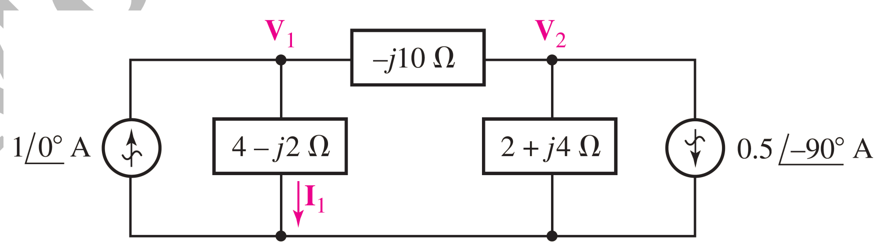


Figure 3: A circuit for which Thevenin equivalent seen by $-j10\ \Omega$ impedance is desired.

Question 3 continued on next page. . .

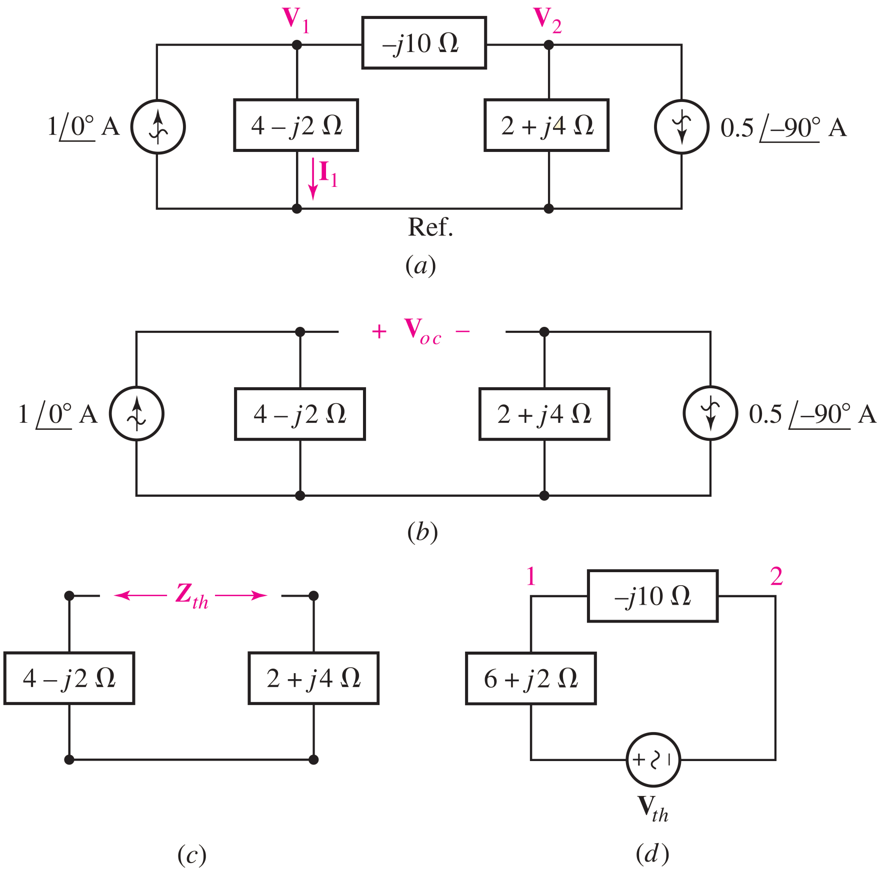


Figure 4: (a) The Thevenin equivalent seen by the $-j10\,\Omega$ impedance is desired. (b) V_{oc} is defined. (c) Z_{th} is defined. (d) The circuit is redrawn using the Thevenin equivalent.

The open-circuit voltage, defined in Fig. 4(b), is

$$V_{oc} = (1\angle 0^\circ)(4 - j2) - (-0.5\angle -90^\circ)(2 + j4) = 4 - j2 + 2 - j1 = 6 - j3\text{ V}$$

. The impedance of the inactive circuit of Fig. 4(c) as viewed from the load terminals is simply the sum of the two remaining impedances. Hence,

$$Z_{th} = 6 + j2\,\Omega$$

. When we reconnect the circuit as in Fig. 4(d), the current directed from node 1 toward node 2 through the $-j10\,\Omega$ load is

$$I_{12} = \frac{6 - j3}{6 + j2 - j10} = 0.6 + j0.3\text{ A}$$

. We now know the current flowing through the $-j10\,\Omega$ impedance of Fig. 4(a). Note that we are unable to compute V_1 using the circuit of Fig. 4(d) as the reference node no longer exists. Returning to the original circuit, then, and subtracting the $0.6 + j0.3$ A current from the left source current, the downward current through the $4 - j2\,\Omega$ branch is found

$$I_1 = 1 - 0.6 - j0.3 = 0.4 - j03\text{ A}$$

Question 3 continued on next page. . .

and, thus,

$$V_1 = (4 - j2)(0.4 - j03) = 1 - j2 \text{ V}$$

.

Question 4

Household electrical voltages are typically quoted as 220 V in Iran. However, these values do not represent the peak ac voltage. Rather, they represent what is known as the root mean square of the voltage, defined as

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t)dt}$$

where $T = \frac{1}{f}$ is the period of the waveform, V_m is the peak voltage, and $\omega = 2\pi f$ is the waveform angular frequency, where $f = 50$ Hz in Iran.

(a) Perform the indicated integration, and show that for a sinusoidal voltage $V_{rms} = \frac{V_m}{\sqrt{2}}$.

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t)dt} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \frac{1+\cos(2\omega t)}{2}dt} = \sqrt{\frac{1}{T}V_m^2 \frac{T+0}{2}} = \frac{V_m}{\sqrt{2}}$$

(b) Compute the peak voltages corresponding to the rms voltage 220 V.

$$V_m = 220\sqrt{2} = 311.13 \text{ V}$$

Question 5

Consider the circuit shown in Fig. 5, where V_{ref} is provided by a regulated voltage source. Show that the circuit can act like a current source and find the constant current I_s flowing to the resistive load R_L.

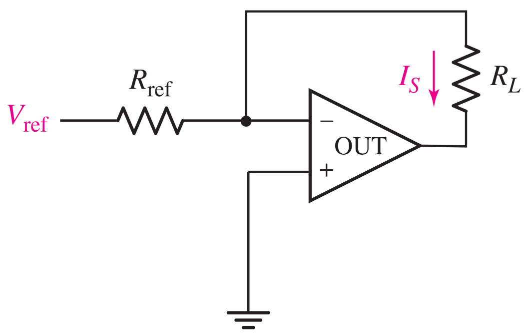


Figure 5: An Op Amp-based current source.

Question 5 continued on next page...

The voltage at the inputs of the Op Amp is 0. A KCL at the inverting leg of the Op Amp results in $\frac{V_{ref}}{R_{ref}} = I_S$. Clearly, the constant current I_S does not depend on R_L and flows through the load resistor R_L, regardless of its value.

Question 6

Find the differential equation relating $i_x(t)$ to $v_s(t)$ for the circuit displayed in Fig. 6 and obtain the corresponding impulse and step responses.

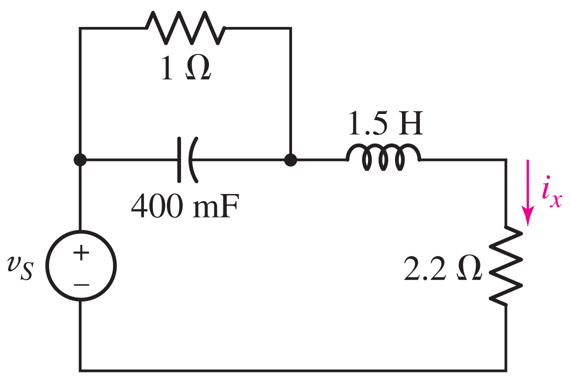


Figure 6: A circuit whose impulse and step responses are intended.

$$v_C(t) + 1.5i_x'(t) + 2.2i_x(t) = v_s(t), \quad i_x(t) = \frac{v_C(t)}{1} + 0.4v_C'(t) = v_C(t) + 0.4v_C'(t)$$

$$i_x(t) = (v_s(t) - 1.5i_x'(t) - 2.2i_x(t)) + 0.4(v_s(t) - 1.5i_x'(t) - 2.2i_x(t))'$$
$$i_x(t) = v_s(t) - 1.5i_x'(t) - 2.2i_x(t) + 0.4v_s'(t) - 0.6i_x''(t) - 0.88i_x'(t)$$
$$i_x''(t) + 3.97i_x'(t) + 5.33i_x(t) = 1.67v_s(t) + 0.67v_s'(t)$$

.
When the circuit is driven with $v_s(t) = \delta(t)$, the impulse response is the solution of

$$h''(t) + 3.97h'(t) + 5.33h(t) = 1.67\delta(t) + 0.67\delta'(t)$$

. The characteristic equation of the corresponding homogeneous differential equation is $s^2 + 3.97s + 5.33 = 0$ with the roots $s_1 = -1.985 + j1.179$ and $s_2 = -1.985 - j1.179$. After $t > 0$, the differential equation is identical with its homogeneous form

$$h''(t) + 3.97h'(t) + 5.33h(t) = 0$$

because $\delta(t) = \delta'(t) = 0$ for $t > 0$. Since, the characteristic equation has two complex conjugate roots, for $t > 0$, the solution is

$$h(t) = (B_1e^{-1.985t+j1.179t} + B_2e^{-1.985t-j1.179t})u(t)$$

Question 6 continued on next page...

or equivalently,

$$h(t) = e^{-1.985t}(A_1 \cos(1.179t) + A_2 \sin(1.179t))u(t)$$

. At $t = 0$, this solution should make both sides of the equation balanced. So,

$$h''(t) + 3.97h'(t) + 5.33h(t) = 1.67\delta(t) + 0.67\delta'(t)$$

. We have,

$$\begin{aligned} h'(t) = \Big[&-1.985e^{-1.985t}\left(A_1 \cos\left(1.179t\right) + A_2 \sin\left(1.179t\right)\right) \\ &+e^{-1.985t}\left(-1.179A_1 \sin\left(1.179t\right) + 1.179A_2 \cos\left(1.179t\right)\right)\Big]u(t) + A_1\delta(t) \\ h''(t) = \Big[&2.5504A_1e^{-1.985t}\cos\left(1.179t\right) + 4.681A_1e^{-1.985t}\sin\left(1.179t\right) \\ &+2.5504A_2e^{-1.985t}\sin\left(1.179t\right) - 4.681A_2e^{-1.985t}\cos\left(1.179t\right)\Big]u(t) \\ &+(-1.985A_1 + 1.179A_2)\delta(t) + A_1\delta'(t) \end{aligned}$$

Substituting the derivatives in the differential equation,

$$(-1.985A_1 + 1.179A_2)\delta(t) + A_1\delta'(t) + 3.97A_1\delta(t) = 1.67\delta(t) + 0.67\delta'(t)$$

Therefore,

$$\Rightarrow \begin{cases} A_1 = 0.67 \\ 1.985A_1 + 1.179A_2 = 1.67 \end{cases} \quad \Rightarrow \begin{cases} A_1 = 0.67 \\ A_2 = 0.29 \end{cases}$$

. Finally,

$$h(t) = e^{-1.985t}(0.67\cos(1.179t) + 0.29\sin(1.179t))u(t)$$

. Further,

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau = \left[0.0402e^{-1.985t}\sin\left(1.179t\right) - 0.3137e^{-1.985t}\cos\left(1.179t\right) + 0.3137\right]u(t)$$

.

Question 7

Consider a series RL circuit driven with the voltage source $v(t)$, where the loop current $i(t)$ should be calculated.

(a) *Find the zero-input response if the initial current is $i(0) = I_0$.*

$$Li'(t) + Ri(t) = 0, \quad i(0) = I_0$$

, which is a homogeneous first-order equation with the solution

$$i(t) = I_0 e^{-\frac{R}{L}t}$$

Question 7 continued on next page...

(b) Find the step response.

$$Li'(t) + Ri(t) = u(t), \quad i(0) = 0$$

, which is a non-homogeneous first-order equation with the solution

$$s(t) = i(t) = \frac{1}{R}(1 - e^{-\frac{R}{L}t})u(t)$$

(c) Find the impulse response.

$$h(t) = s'(t) = \frac{1}{R}(1 - e^{-\frac{R}{L}t})\delta(t) + \frac{1}{L}e^{-\frac{R}{L}t}u(t) = \frac{1}{L}e^{-\frac{R}{L}t}u(t)$$

(d) Find the zero-state response if $v(t) = V_0 e^{-t}u(t)$.

At first, let evaluate $e^{-\alpha t}u(t) * e^{-\beta t}u(t)$. Assuming $\alpha \neq \beta$,

$$e^{-\alpha t}u(t) * e^{-\beta t}u(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau)d\tau = e^{-\beta t}u(t)\int_0^t e^{(\beta-\alpha)\tau}d\tau$$

$$= u(t)e^{-\beta t}\frac{1}{\beta-\alpha}e^{(\beta-\alpha)\tau}\Big|_0^t = u(t)\frac{1}{\beta-\alpha}e^{-\beta t}(e^{(\beta-\alpha)t} - 1) = \frac{1}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t})u(t)$$

. For $\alpha = \beta$,

$$e^{-\alpha t}u(t) * e^{-\alpha t}u(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau)e^{-\alpha(t-\tau)}u(t-\tau)d\tau = e^{-\alpha t}u(t)\int_0^t d\tau = te^{-\alpha t}u(t)$$

Now,

$$i(t) = V_0 e^{-t}u(t) * \frac{1}{L}e^{-\frac{R}{L}t}u(t) = \frac{V_0}{L}e^{-t}u(t) * e^{-\frac{R}{L}t}u(t)$$

. If $R \neq L$,

$$i(t) = \frac{V_0}{L}\frac{1}{\frac{R}{L} - 1}(e^{-t} - e^{-\frac{R}{L}t})u(t) = \frac{V_0}{R-L}(e^{-t} - e^{-\frac{R}{L}t})u(t)$$

. When $R = L$,

$$i(t) = \frac{V_0}{L}te^{-t}u(t)$$

.

(e) Find the complete response if $v(t) = V_0 e^{-t}u(t)$ and $i(0) = I_0$.

If $R \neq L$,

$$i(t) = I_0 e^{-\frac{R}{L}t} + \frac{V_0}{R-L}(e^{-t} - e^{-\frac{R}{L}t})u(t)$$

. When $R = L$,

$$i(t) = I_0 e^{-t} + \frac{V_0}{L} t e^{-t} u(t)$$

.

(f) Find the complete response if $v(t) = V_0 \cos(\omega t + \theta)u(t)$ and $i(0) = I_0$. How does the complete response relate to the sinusoidal steady state response?

Question 7 continued on next page...

The zero-input, zero-state impulse, and zero-state step responses do not change by altering the input voltage. So, when the input is $v(t) = V_0 \cos(\omega t+\theta)$, the zero-state response equals

$$v(t) * h(t) = V_0 \cos(\omega t + \theta)u(t) * \frac{1}{L}e^{-\frac{R}{L}t}u(t) = \frac{V_0}{L}[\frac{e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}}{2}]u(t) * e^{-\frac{R}{L}t}u(t)$$

$$= \frac{V_0}{2L}\big[e^{j\theta}e^{j\omega t}u(t) * e^{-\frac{R}{L}t}u(t) + e^{-j\theta}e^{-j\omega t}u(t) * e^{-\frac{R}{L}t}u(t)\big]$$

$$= \frac{V_0}{2L}\big[e^{j\theta}\frac{1}{\frac{R}{L}+j\omega}(e^{j\omega t} - e^{-\frac{R}{L}t})u(t) + e^{-j\theta}\frac{1}{\frac{R}{L}-j\omega}(e^{-j\omega t} - e^{-\frac{R}{L}t})u(t)\big]$$

$$= \frac{V_0}{2L}u(t)\big[\frac{1}{\frac{R}{L}+j\omega}e^{j(\omega t+\theta)} + \frac{1}{\frac{R}{L}-j\omega}e^{-j(\omega t+\theta)}\big] - \frac{V_0}{2L}u(t)e^{-\frac{R}{L}t}\big[\frac{1}{\frac{R}{L}+j\omega}e^{j\theta} + \frac{1}{\frac{R}{L}-j\omega}e^{-j\theta}\big]$$

$$= \frac{V_0}{2L}u(t) \times 2\text{Re}\{\frac{1}{\frac{R}{L}+j\omega}e^{j(\omega t+\theta)}\} - \frac{V_0}{2L}u(t)e^{-\frac{R}{L}t} \times 2\text{Re}\{\frac{1}{\frac{R}{L}+j\omega}e^{j\theta}\}$$

$$= \frac{V_0}{L}u(t)\times\text{Re}\{\frac{1}{\sqrt{\frac{R^2}{L^2}+\omega^2}}e^{j(\omega t+\theta-\tan^{-1}(\frac{\omega L}{R}))}\} - \frac{V_0}{L}u(t)e^{-\frac{R}{L}t}\times\text{Re}\{\frac{1}{\sqrt{\frac{R^2}{L^2}+\omega^2}}e^{j(\theta-\tan^{-1}(\frac{\omega L}{R}))}\}$$

$$= \frac{V_0}{L}u(t)\frac{1}{\sqrt{\frac{R^2}{L^2}+\omega^2}}\cos(\omega t+\theta-\tan^{-1}(\frac{\omega L}{R})) - \frac{V_0}{L}u(t)e^{-\frac{R}{L}t}\frac{1}{\sqrt{\frac{R^2}{L^2}+\omega^2}}\cos(\theta-\tan^{-1}(\frac{\omega L}{R}))$$

$$= \frac{V_0}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t+\theta-\tan^{-1}(\frac{\omega L}{R}))u(t) - \frac{V_0}{\sqrt{R^2+\omega^2L^2}}e^{-\frac{R}{L}t}\cos(\theta-\tan^{-1}(\frac{\omega L}{R}))u(t)$$

Finally, the complete solution equals

$$i(t) = I_0e^{-\frac{R}{L}t} +$$

$$\frac{V_0}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t+\theta-\tan^{-1}(\frac{\omega L}{R}))u(t) - \frac{V_0}{\sqrt{R^2+\omega^2L^2}}e^{-\frac{R}{L}t}\cos(\theta-\tan^{-1}(\frac{\omega L}{R}))u(t)$$

. When the transients die at $t \to \infty$, the complete response is equal to

$$i(t) = \frac{V_0}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t+\theta-\tan^{-1}(\frac{\omega L}{R}))u(t)$$

, which is exactly the steady state sinusoidal current. In a better presentation,

$$i(t) = |\frac{V_0e^{j\theta}}{R+j\omega L}|\cos(\omega t + \angle[\frac{V_0e^{j\theta}}{R+j\omega L}])u(t)$$

, where $\frac{V_0e^{j\theta}}{R+j\omega L}$ is the phasor of the current.

SOFTWARE QUESTIONS

BONUS QUESTIONS

Question 8

Return your answers by filling the LaTeX template of the assignment.